Multiple-Term Refinements of Young Type Inequalities

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Recently, a multiple-term refinement of Young’s inequality has been proved. In this paper, we show its reverse refinement. Moreover, we will present multiple-term refinements of Young’s inequality involving Kantorovich constants. Finally, we will apply scalar inequalities to operators.

1. Introduction

The classical Young inequality states that if \( a, b > 0 \) and \( 0 \leq V \leq 1 \), then
\[
(1 - V) a + V b \geq a^{1 - V} b^V.
\]
(1)

For \( 0 \leq V \leq 1 \), we define three functions \( r_0(V) \), \( r_1(V) \), and \( R_0(V) \) by
\[
\begin{align*}
r_0(V) &= \min \{ V, 1 - V \}, \\
r_1(V) &= \min \{ 2r_0(V), 1 - 2r_0(V) \}, \\
R_0(V) &= 1 - r_0(V).
\end{align*}
\]
(2)

In [1] and the references there, the following improvements of Young inequality and its reverse are discussed:
\[
(1 - V) a + V b \geq a^{1 - V} b^V + r_0(V) (\sqrt{a} - \sqrt{b})^2,
\]
(3)

\[
(1 - V) a + V b \leq a^{1 - V} b^V + (1 - r_0(V)) (\sqrt{a} - \sqrt{b})^2 + r_1(V)
\]
\[
\cdot \left( (\sqrt{a} - \sqrt{ab})^2 \chi_{(0,1/2)}(V) + (\sqrt{b} - \sqrt{ab})^2 \chi_{(1/2,1)}(V) \right),
\]

(4)

where \( \chi_I \) is the characteristic function defined by
\[
\chi_I(x) = \begin{cases} 
1, & \text{if } x \in I \\
0, & \text{if } x \notin I.
\end{cases}
\]
(5)

Another form of Young type inequalities discussed in [1] is as follows:
\[
((1 - V) a + V b)^2 \geq (a^{1 - V} b^V)^2 + r_0^2(V) (a - b)^2 + r_1(V)
\]
\[
\cdot \left( (a - \sqrt{ab})^2 \chi_{(0,1/2)}(V) + (b - \sqrt{ab})^2 \chi_{(1/2,1)}(V) \right),
\]
(6)

\[
((1 - V) a + V b)^2 \leq (a^{1 - V} b^V)^2 + R_0^2(V) (a - b)^2 - r_1(V)
\]
\[
\cdot \left( (b - \sqrt{ab})^2 \chi_{(0,1/2)}(V) + (a - \sqrt{ab})^2 \chi_{(1/2,1)}(V) \right).
\]
Other types of improvements of the Young inequality is to use Kantorovich constants. Wu and Zhao [2] showed
\[(1 - \nu) a + \nu b \geq K_1 (a, b)^\gamma, a^{1-\nu} b^{\nu} + r_0 (\nu) \big((\sqrt{a} - \sqrt{b})^2, (1 - \nu) a + \nu b \leq K_1 (a, b)^\gamma, a^{1-\nu} b^{\nu} + R_0 (\nu) \big((\sqrt{a} - \sqrt{b})^2, \)
\[(7)
where
\[K_1 (a, b) = \frac{(\sqrt{a} + \sqrt{b})^2}{4\sqrt{ab}}. \tag{8}\]

Note that (7) improves (3), since \(K_1 (a, b) \geq 1\) for all \(a, b > 0\).

In [3], Liao and Wu improved (4) as follows:
\[(1 - \nu) a + \nu b \geq K_2 (a, b)^\gamma, a^{1-\nu} b^{\nu} + r_1 (\nu)
\cdot \left((a - \sqrt{ab}) \chi_{(0,1/2)} (\nu) + (b - \sqrt{ab}) \chi_{(1/2,1)} (\nu), (1 - \nu) a + \nu b \leq K_2 (a, b)^\gamma, a^{1-\nu} b^{\nu} + R_0 (\nu)
\cdot \left((a - \sqrt{ab})^2 - r_1 (\nu)
\cdot \left((b - \sqrt{ab})^2 \chi_{(0,1/2)} (\nu) + (a - \sqrt{ab})^2 \chi_{(1/2,1)} (\nu), \right) \right), \tag{9}\]
where
\[K_2 (a, b) = \frac{(\sqrt{a} + \sqrt{b})^2}{4\sqrt{ab}}, \tag{10}\]
\[r_2 (\nu) = \min \{2r_1 (\nu), 1 - 2r_1 (\nu)\}. \]

The constants of the form \((m + M)^2/4mnM\) are called Kantorovich constants.

Throughout the paper, we will use the following functions.

**Definition 1.** One defines the sequence \(\{r_n (\nu)\}\) of functions on \([0, 1]\) as follows:
\[r_0 (\nu) = \min \{\nu, 1 - \nu\}, \tag{11}\]
\[r_n (\nu) = \min \{2r_{n-1} (\nu), 1 - 2r_{n-1} (\nu)\}, \]
for \(n \in \mathbb{N}.

**Definition 2.** For \(l, k \in \mathbb{N}\) and \(a, b > 0\), we define the functions
\[g_{l,k} (a, b) \text{ by}\]
\[g_{l,k} (a, b) = \left(\frac{a^{1-l/(k-1)/2} b^{k/(k-1)/2} - a^{1-k/(k-1)/2} b^{l/(k-1)/2}}{2}\right)^2, \tag{12}\]
for \(\nu \in [0, 1]\) and an integer \(N > 0\). As we will see (Lemma 4), \(r_1 (k/2) = 0\) for any integer \(k\) with \(0 < k < 2\). Thus the interval of the characteristic function in \(\alpha_N (\nu)\) or \(\beta_N (\nu)\) can include boundary points. For example, \(\chi_{(k-1)/2, k/2)}\) may be replaced by \(\chi_{(k-1)/2, k/2]}\) or \(\chi_{(k-1)/2, k/2)}\).

We can express \(r_0 (\nu)\) and \(r_1 (\nu)\) as multipart functions as follows:
\[r_0 (\nu) = \begin{cases} \nu, & 0 \leq \nu \leq \frac{1}{2}, \\ 1 - \nu, & \frac{1}{2} < \nu \leq 1. \end{cases} \]
For any $l \geq 0$, we can formulate $r_l(v)$ explicitly.

**Lemma 4.** Let $l \geq 0$ and $1 \leq k \leq 2^l$ be integers. If $(k-1)/2^l \leq v \leq k/2^l$, then

$$
r_l(v) = \begin{cases} 2v, & 0 \leq v \leq \frac{1}{4} \\ 2 - 2v, & 1 - \frac{1}{4} < v \leq \frac{1}{2} \\ 2v - 1, & \frac{1}{2} < v \leq \frac{3}{4} \\ 2 - 2v, & \frac{3}{4} < v \leq 1. \end{cases}
$$

(15)

**Proof.** We prove it by induction on $l$. The case $l = 0$ is obvious. Assume that $(k-1)/2^{l+1} \leq v \leq (k-1)/2^{l+1} + 1/2^{l+2}$. If $k = 2m - 1$ is odd, then $(m - 1)/2^l \leq v \leq (m/2 + 1/2^{l+2})$, and $r_l(v) = 2v - m + 1$ by induction. Since $v \leq (2k-1)/2^{l+2}$, one has

$$
r_l(v) = 2v - \frac{k-1}{2} \leq \frac{2k-1}{4} - \frac{k-1}{2} = \frac{1}{4},
$$

$$
r_{l+1}(v) = \min\{2r_l(v), 1 - 2r_l(v)\} = 2r_l(v)
$$

(17)

If $k = 2m$ is even, then $(2m-1)/2^{l+1} \leq v \leq m/2^l - 1/2^{l+2} < m/2^l$ and $r_l(v) = m - 2v$ by induction. Since $v \leq (2k-1)/2^{l+2}$, we have

$$
r_l(v) = \frac{k-1}{2} - 2v \geq \frac{k-2}{4} - \frac{k-1}{2} = \frac{1}{4},
$$

$$
r_{l+1}(v) = 1 - 2r_l(v) = 2^{l+1}v - 1.
$$

(18)

Using the same argument, we can show that if $(k-1)/2^{l+1} + 1/2^{l+2} < v \leq k/2^{l+1}$, then $r_{l+1}(v) = k - 2^{l+1}v$. We omit the detailed proof. \qed

**Lemma 5.** For a positive integer $N$, $\alpha_N(v)$ is the linear interpolation of $\mu(v)$ at $v = k/2^N$ for $k = 0, 1, \ldots, 2^N$.

**Proof.** Since $r_l(v)$ is a line segment on each interval $[(k-1)/2^{l+1}, k/2^{l+1})$ for $1 \leq k \leq 2^l$, $\alpha_N(v)$ is a line segment on $[(k-1)/2^N, k/2^N)$ for $1 \leq k \leq 2^N$. Thus it suffices to show

$$
\alpha_N\left(\frac{k}{2^N}\right) = \mu\left(\frac{k}{2^N}\right), \quad 1 \leq k < 2^N.
$$

(19)

Note that since $\alpha_N(v) = \mu(v) = 0$ at $v = 0, 1$, (19) holds for $k = 0, 2^N$. We will prove (19) by induction on $N$. Since $r_0(v) = r_0(v)(\sqrt{a} - \sqrt{b})^2$, one has

$$
\alpha_1\left(\frac{1}{2}\right) = \frac{1}{2} (\sqrt{a} - \sqrt{b})^2 = \mu\left(\frac{1}{2}\right).
$$

(20)

Assume that (19) holds and $1 \leq k < 2^{N+1}$. If $k = 2m$ is even, then

$$
\alpha_{N+1}\left(\frac{k}{2^{N+1}}\right) = \alpha_{N+1}\left(\frac{m}{2^N}\right) = \alpha_N\left(\frac{m}{2^N}\right)
$$

$$
+ r_N\left(\frac{m}{2^N}\right) \sum_{k=1}^{2^N} g_{N,k}(a, b) \chi_{(k-1)/2^N,k/2^N}\left(\frac{m}{2^N}\right)
$$

(21)

$$
= \alpha_N\left(\frac{m}{2^N}\right), \quad \text{since } r_N\left(\frac{m}{2^N}\right) = 0
$$

$$
= \mu\left(\frac{k}{2^{N+1}}\right) \quad \text{(by induction)}.
$$

If $k = 2m - 1$ is odd, then

$$
\alpha_{N+1}\left(\frac{k}{2^{N+1}}\right) = \alpha_N\left(\frac{m}{2^N} - 1\right) = \alpha_N\left(\frac{m}{2^N} - \frac{1}{2}\right)
$$

$$
+ r_N\left(\frac{m}{2^N} - 1\right) \sum_{k=1}^{2^N} g_{N,k}(a, b) \chi_{(k-1)/2^N,(k-1)/2^N}\left(\frac{m}{2^N} - 1\right)
$$

Since $\alpha_N(v)$ is the line segment joining $((m-1)/2^N, \mu((m-1)/2^N))$ and $(m/2^N, \mu(m/2^N))$ on $[(m-1)/2^N, m/2^N]$, we have

$$
\alpha_N\left(\frac{m}{2^N} - 1\right) = \frac{1}{2} (\sqrt{m-1} - \sqrt{m}) + \frac{1}{2} (\mu\left(\frac{m}{2^N}\right) - \mu\left(\frac{m}{2^N} - 1\right)).
$$

(22)

By Lemma 4, $r_N(m/2^N - 1/2^{N+1}) = 1/2$. Noting that $a^{1-v}b^v = (1-v)a + vb - \mu(v)$, we can write $g_{N,m}(a, b)$ by

$$
g_{N,m}(a, b) = a^{1-(m-1)/2^N} b^{(m-1)/2^N} + a^{1-m/2^N} b^{m/2^N} - 2a^{1-(m-1)/2^{N+1}} b^{(m-1)/2^{N+1}}
$$

$$
= -\mu\left(\frac{m-1}{2^N}\right) - \mu\left(\frac{m}{2^N}\right)
$$

$$
+ 2\mu\left(\frac{m-1}{2^{N+1}}\right).
$$

(24)

Thus, from (22), we deduce that

$$
\alpha_{N+1}\left(\frac{k}{2^{N+1}}\right) = \mu\left(\frac{m-1}{2^{N+1}}\right) = \mu\left(\frac{k}{2^{N+1}}\right).
$$

(25)

\qed

**Remark 6.** Since $\mu(v)$ is concave, $\mu(v) \geq \alpha_N(v)$ by Lemma 5, which proves Theorem 3 in a much simpler way, where the original proof is done by mathematical induction on $N$.

**Lemma 7.** Let $\lambda(v)$ be the reflection of $\mu(v)$ about the point $(1/2, \mu(1/2))$; that is,

$$
\lambda(v) = 2\mu\left(\frac{1}{2}\right) - \mu(1-v),
$$

$$
= \left(\sqrt{a} - \sqrt{b}\right)^2 - wa - (1-v) b + a'b^{1-v},
$$

(26)

$$
= (1-v)a + vb + a'b^{1-v} - 2\sqrt{ab}.
$$

(27)
Then each of the following is true.

(1) \( \lambda(v) \geq \mu(v) \) for all \( 0 \leq v \leq 1 \).

(2) \( \beta_N(v) \) is the linear interpolation of \( \lambda(v) \) at \( v = k/2^N \) at \( k = 0, \ldots, 2^N \).

Proof. Let \( f(v) = \lambda(v) - \mu(v) \). Since

\[
f'(v) = a^v b^{1-v} \left( 1 - \left( \frac{a}{b} \right)^{1-2v} \right) \ln \frac{a}{b},
\]

one derives that \( f'(v) < 0 \) for \( v \in (0, 1/2) \), \( f'(v) > 0 \) for \( v \in (1/2, 1) \), and \( f'(1/2) = 0 \). Thus we have \( \lambda(v) - \mu(v) \geq \lambda(1/2) - \mu(1/2) = 0 \) for all \( 0 \leq v \leq 1 \). So part (1) is proved.

For the second part, it suffices to show that \( \beta_N(v) \) is the reflection of \( \alpha_N(v) \) about the point \( (1/2, \alpha_N(1/2)) \); that is,

\[
2\alpha_N \left( \frac{1}{2} \right) - \alpha_N (1 - v) = \beta_N (v).
\]  

(28)

Noting that \( r_l(v) = r_l(1-v) \) and \( g_l,2^l-k+1(a, b) = g_l(k, b, a) \), one gets that

\[
2\alpha_N \left( \frac{1}{2} \right) - \alpha_N (1 - v)
= 2\mu \left( \frac{1}{2} \right) - \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (1 - v)
= (\sqrt{a} - \sqrt{b})^2
\]

\[
\sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (v)
= \beta_N (v).
\]  

(29)

\[
(1 - v) a + vb
\geq a^{1-v} b^v + \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (v)
\]

\[
= a^{1-v} b^v + r_0(v) (\sqrt{a} - \sqrt{b})^2
\]

\[
+ \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (v).
\]

(30)

Theorem 8. For any integer \( N \geq 1 \) and \( 0 \leq v \leq 1 \), one has

\[
\alpha_N(v) \leq \mu(v) \leq \beta_N(v).
\]  

(31)

Corollary 9. For any integer \( N \geq 1 \) and \( 0 \leq v \leq 1 \), one has

\[
((1 - v) a + vb)^2
\geq (a^{1-v} b^v + r_0(v) (a - b)^2)
\]

\[
+ \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (v)\]

\[
((1 - v) a + vb)^2
\leq (a^{1-v} b^v + R_0(v) (a - b)^2)
\]

\[
- \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (v).
\]

(32)

Proof. Replacing \( a \) and \( b \) by their squares in Theorem 8, we obtain that

\[
(1 - v) a^2 + vb^2
\geq (a^{1-v} b^v + r_0(v) (a - b)^2)
\]

\[
+ \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (v),
\]

(33)

\[
(1 - v) a^2 + vb^2
\leq (a^{1-v} b^v + R_0(v) (a - b)^2)
\]

\[
- \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_l(k, b, a) \chi_{(k-1)/2^l,(k+1)/2^l} (v).
\]
Since \( r_0(v) - r_2^0(v) = R_0(v) - R_2^0(v) = v(1 - v) \) for all \( v \), we derive from the above that

\[
(1 - v)a^2 + vb^2 \\
\geq v(1 - v)(a - b)^2 + (a^{-1} - b^{-1})^2 + r_0^2(v)(a - b)^2 \\
+ \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^l g_{l,k}(a^2, b^2) X_{((k-1)/2N, k/2N)}(v),
\]

(34)

Thus (32) follows from the identity

\[
((1 - v)a + vb)^2 = (1 - v)a^2 + vb^2 \\
- v(1 - v)(a - b)^2.
\]

(35)

\[\square\]

3. Young Inequalities Involving Kantorovich Constants

In this section, we will discuss multiple-term improvements of Young inequality involving Kantorovich constants. For a nonnegative integer \( n \), we define \( K_n(a, b) \) by

\[
K_n(a, b) = \frac{(a^{1/n} + b^{1/n})^2}{4(ab)^{1/2n}}.
\]

(36)

Lemma 10. For \( 0 \leq v \leq 1 \), one has

\[
(1 - v)a + vb \geq K_0(a, b) s(v) a^{-1} - b^{v}.
\]

(37)

Proof. Replacing \( a^{-1} - b^{-1} \) by \( K_0(a, b) s(v) a^{-1} - b^{v} \), the inequality is equivalent to

\[
1 - v + vt \geq K_0(1, t) s(v) t^v,
\]

(38)

for \( t > 0 \). Taking the natural logarithm, it suffices to show that

\[
f(t) \equiv \ln(1 - v + vt) - r_0(v) \ln K_0(1, t) - v \ln t \geq 0.
\]

(39)

A direct computation shows that

\[
(t + 1)(1 - v + vt) f'(t) = \left\{ \begin{array}{ll}
(1 - t^{-1})(v(1 - v)(t + 1) - r_0(v)(1 - v + vt)), & 0 \leq v \leq \frac{1}{2} \\
(1 - v)(2v - 1)(1 - t^{-1}), & \frac{1}{2} < v \leq 1.
\end{array} \right.
\]

(40)

Thus \( f(t) \geq f(1) = 0 \) for any \( t > 0 \). \[\square\]

Lemma 11. For a positive integer \( N \) and \( 0 \leq v \leq 1 \), define \( s_N(t) \) by

\[
s_N(t) = 1 - v + vt - \sum_{l=0}^{N-1} r_l(v) \\
\cdot \sum_{k=1}^{2^l} \left( \sqrt{t^{(k-1)/2N}} - \sqrt{t^{k/2N}} \right)^2 X_{((k-1)/2N, k/2N)}(v),
\]

for \( t > 0 \). Then one has

\[
s_N(t) = \sum_{k=1}^{2^N} \left( (k - 2^Nv) t^{(k-1)/2N} + (2^Nv - k + 1) t^{k/2N} \right) \\
\cdot X_{((k-1)/2N, k/2N)}(v).
\]

(42)

Proof. As mentioned in the previous section, since \( r_k(k/2^N) = 0 \) for \( 0 \leq k \leq 2^N \), \( X_{((k-1)/2N, k/2N)} \) in the definition of \( s_N(t) \) may be replaced by \( X_{((k-1)/2N, k/2N)} \). We prove (42) by induction on \( N \). We have

\[
s_1(t) = 1 - v + vt - r_0(v) \left( 1 - \sqrt{t} \right)^2 \\
= \left( 1 - 2v + 2v \sqrt{t} \right) X_{(0, 1/2)} + (2 - 2v) \sqrt{t} \left( 2v - 1 \right) X_{(1/2, 1)}.
\]

(43)

Suppose that (42) holds. Then

\[
s_{N+1}(t) = s_N(t) - r_0(v) \sum_{k=1}^{2^N} \left( \sqrt{t^{(k-1)/2N}} - \sqrt{t^{k/2N}} \right)^2 \\
\cdot X_{((k-1)/2N, k/2N)}(v) = \sum_{k=1}^{2^N} \left( (k - 2^N v) t^{(k-1)/2N} \\
+ (2^N v - k + 1) t^{k/2N} \\
- r_N(v) \left( \sqrt{t^{(k-1)/2N}} - \sqrt{t^{k/2N}} \right)^2 \right) \\
\cdot X_{((k-1)/2N, k/2N)}(v)
\]

(44)

\[
= \sum_{k=1}^{2^N} \left( \left( 2k - 2^{N+1}v - 1 \right) t^{(k-1)/2N} \\
+ \left( 2^{N+1} v - 2k + 2 \right) t^{(2k-1)/2N+1} \right) \\
\cdot X_{((k-1)/2N, (2k-1)/2N+1)}(v) + \sum_{k=1}^{2^N} \left( \left( 2k - 2^{N+1}v \right) t^{(2k-1)/2N+1} \\
+ \left( 2^{N+1} v - 2k + 1 \right) t^{k/2N} \right) X_{((2k-1)/2N+1, k/2N)}(v).
\]
Replacing $2k - 1$ in the first summation and $2k$ in the second summation, respectively, by $m, o$ one has

$$s_{N+1}(t) = \sum_{m=1 \text{ odd}}^{2^{N+1}} \left( (m - 2^Nv) t^{(m-1)/2^{N+1}} + (2^Nv - m + 1) t^{m/2^{N+1}} \right) X((m-1)/2^{N+1},m/2^{N+1}) (v)$$

$$+ \sum_{m=2 \text{ even}}^{2^{N+1}} \left( (m - 2^Nv) t^{(m-1)/2^{N+1}} + (2^Nv - m + 1) t^{m/2^{N+1}} \right) X((m-1)/2^{N+1},m/2^{N+1}) (v)$$

$$= \sum_{k=1}^{2^N} \left( (k - 2^Nv) t^{(k-1)/2^{N+1}} + (2^Nv - k + 1) t^{k/2^{N+1}} \right) X((k-1)/2^{N+1},k/2^{N+1}) (v).$$

Thus (42) holds for all positive integers $N$. □

The following shows a multiple-term refinement of Young inequality involving Kantorovich constants.

**Theorem 12.** For $a, b > 0$, $0 \leq v \leq 1$, and $N \in \mathbb{N}$, one has

$$1 - v + vt \geq K_N (a, b) r(v) a^{1-v} b^v$$

$$+ \sum_{l=0}^{N-1} \sum_{k=1}^{2^l} \left( t^{(k-1)/2^l} + t^{k/2^l} - 2t^{(2k-1)/2^l} \right) X((k-1)/2^l,k/2^l) (v).$$

Proof. Putting $t = a^{-1} b$, (46) can be rewritten as

$$1 - v + vt \geq K_N (1, t) r(v) t^v$$

$$+ \sum_{l=0}^{N-1} \sum_{k=1}^{2^l} \left( t^{(k-1)/2^l} + t^{k/2^l} - 2t^{(2k-1)/2^l} \right) X((k-1)/2^l,k/2^l) (v).$$

By Lemma II, the above can be expressed by

$$\sum_{k=1}^{2^N} \left( (k - 2^Nv) t^{(k-1)/2^N} + (2^Nv - k + 1) t^{k/2^N} \right) X((k-1)/2^N,k/2^N) (v) \geq K_N (1, t) r(v) t^v.$$ 

Thus it suffices to show that if $(k-1)/2^N < v \leq k/2^N$, then

$$(k - 2^Nv) t^{(k-1)/2^N} + (2^Nv - k + 1) t^{k/2^N} \geq K_N (1, t) r(v) t^v,$$

for $t > 0$. Replacing $t$ by $t^{2N}$ and letting $\mu = 2^N v$, the above is equivalent to

$$(k - \mu) t^{k-1} + (\mu - k + 1) t^k \geq K_0 (1, t) r(v) t^\mu.$$ 

Let $s = \min[k - \mu, \mu - k + 1]$. By Lemma 10, one gets that

$$(k - \mu) t^{k-1} + (\mu - k + 1) t^k \geq K_0 (t^{k-1}, t^k) t^{(k-1)(k-1)+(\mu-k+1)k} = K_0 (1, t) t^\mu.$$ 

Since

$$r_N(v) = \begin{cases} 
2^N v - k + 1, & \frac{k - 1}{2^N} \leq v \leq \frac{2k - 1}{2^N+1} \\
2^N v, & \frac{2k - 1}{2^N+1} < v \leq \frac{k}{2^N} 
\end{cases}$$

by Lemma 4, it follows that if $(k - 1)/2^N < v \leq k/2^N$, then

$s = \min[k - 2^N v, 2^N v - k + 1] = r_N(v)$ and therefore (50) holds.

Note that (46) can be written as

$$(1 - v) a + vb \geq K_N (a, b) r(v) a^{1-v} b^v$$

$$+ r_0 (v) (\sqrt{a} - \sqrt{b})^2$$

$$+ \sum_{l=1}^{N-1} \sum_{k=1}^{2^l} r_l (v) g_{lk} (a, b) X((k-1)/2^l,k/2^l) (v),$$

which gives the first inequalities of (7) and (9) with $N = 1$ and 2, respectively.

Now we consider a reverse inequality corresponding to Theorem 12. A given inequality of the form $(1 - v) a + vb \geq \xi(v, a, b)$ can be utilized to derive its reverse in many cases. For example, replacing $v$ by $1 - v$ in

$$(1 - v) a + vb \geq a^{1-v} b^v + r_0 (v) (\sqrt{a} - \sqrt{b})^2,$$

which is the first inequality of (3), we obtain

$$(1 - v) a + vb \leq a + b - a^{1-v} b^v - r_0 (v) (\sqrt{a} - \sqrt{b})^2$$

$$= 2 \sqrt{ab} - a^{1-v} b^v + R_0 (v) (\sqrt{a} - \sqrt{b})^2.$$

Since $2 \sqrt{ab} \leq a^{1-v} b^v + a^{1-v} b^v$, the above implies the second inequality of (3). Similarly, replacing $v$ by $1 - v$ in

$$(1 - v) a + vb \geq a^{1-v} b^v + r_0 (v) (\sqrt{a} - \sqrt{b})^2 + r_1 (v)$$

$$+ (\sqrt{a} - \sqrt{ab})^2 X_{(0,1/2)} (v),$$

and therefore (51) follows.
which is the first inequality of (4), we get

\[
(1 - \nu) a + \nu b \leq a + b - a^\nu b^{1-\nu} - r_0(\nu) \left( \sqrt[\nu]{a} - \sqrt[1-\nu]{b} \right)^2
- r_1(\nu) \left( \sqrt[\nu]{b} - \sqrt[1-\nu]{a} \right)^2 \chi_{(0,1/2)}(\nu)
+ \left( \sqrt[\nu]{a} - \sqrt[1-\nu]{b} \right)^2 \chi_{(1/2,1)}(\nu)
+ R_0(\nu) \left( \sqrt[\nu]{a} - \sqrt[1-\nu]{b} \right)^2 - r_1(\nu)
\cdot \left( \left( \sqrt[\nu]{a} - \sqrt[1-\nu]{b} \right)^2 \chi_{(1/2,1)}(\nu)\right)
+ \left( \sqrt[\nu]{b} - \sqrt[1-\nu]{a} \right)^2 \chi_{(0,1/2)}(\nu).
\]

(57)

Since \( 2 \sqrt{a} b \leq a^\nu b^{1-\nu} + a^{1-\nu} b^\nu \), the above implies the second inequality of (4). In the same way, the first inequality in Theorem 8 can be used to derive

\[
(1 - \nu) a + \nu b \leq 2 \sqrt{a} b - a^\nu b^{1-\nu} + R_0(\nu) \left( \sqrt[\nu]{a} - \sqrt[1-\nu]{b} \right)^2
- \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(b,a) \chi_{([k-1]/2^l, [k]/2^l)}(\nu),
\]

(58)

which is stronger than the second inequality in the theorem. Based on such an observation, we can show a reverse inequality corresponding to Theorem 12 as follows.

**Theorem 13.** For \( a, b > 0, 0 \leq \nu \leq 1, \) and \( N \in \mathbb{N} \), one has

\[
(1 - \nu) a + \nu b \leq 2 \sqrt{a} b - a^\nu b^{1-\nu} + R_0(\nu) \left( \sqrt[\nu]{a} - \sqrt[1-\nu]{b} \right)^2
- \sum_{l=1}^{N-1} r_l(\nu) \sum_{k=1}^{2^l} g_{l,k}(b,a) \chi_{([k-1]/2^l, [k]/2^l)}(\nu),
\]

(59)

where the last inequality results from

\[
2 \sqrt{a} b \leq K_N(a, b)^{\nu}(\nu) a^\nu b^{1-\nu} + K_N(a, b)^{-\nu}(\nu) a^{1-\nu} b^\nu.
\]

(60)

Note that the second inequalities of (7) and (9) follow from the above theorem with \( N = 1 \) and 2, respectively.

### 4. Operator Inequalities

From now on, we use uppercase letters for invertible positive operators on a Hilbert space and lowercase letters for real numbers. The following notations will be used:

(i) \( A \geq B \) (\( A > B \)) denotes that \( A - B \) is a positive (invertible positive) operator.

(ii) \( A \geq 0 \) (\( A > 0 \)) denotes that \( A \) is a positive (invertible positive) operator.

For \( A, B > 0 \) and \( 0 \leq \nu \leq 1 \), the \( \nu \)-arithmetic and \( \nu \)-geometric means of \( A \) and \( B \) are defined, respectively, by

\[
A\nu B = (1 - \nu) A + \nu B,
\]

\[
A^\nu B = A^{1/2} \left( A^{-1/2} B A^{-1/2} \right)^\nu A^{1/2}.
\]

(62)

In the case \( \nu = 1/2 \), we will omit the \( \nu \)-value in them. For example, \( A \nu B \) denotes \( A_{1/2} B \).

The operator version of (1) is well known as follows:

\[
AV_B \geq A\nu_B,
\]

(63)

for \( A \) and \( B \) positive invertible operators and \( 0 \leq \nu \leq 1 \) (see [4, 5] for more matrix Young inequalities). To show operator inequalities corresponding to their scalar versions, we will use the operator monotonicity of continuous functions; that is, if \( f \) is a real valued continuous function defined on the spectrum of a self-adjoint operator \( A \), then \( f(t) \geq 0 \) for every \( t \) in the spectrum of \( A \) implies that \( f(A) \) is a positive operator.
From now on, \( g_{l,k}(A,B) \) will denote the operator version of \( g_{l,k}(a,b) \) defined in Definition 2. That is,
\[
g_{l,k}(A,B) = A^*_k (k-1)/2 B + A^*_k (2k-1)/2^l B - 2 A^*_k (2k-1)/2^{l+1} B,
\]
for \( A, B > 0 \).

**Theorem 14.** Let \( A, B > 0 \) and \( 0 \leq v \leq 1 \). Then
\[
AV_B \geq A^*_v B + 2R_0(v) (AVB - A^*_v B) + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(A,B) \chi_{((k-1)/2^l,k/2^l)}(v),
\]
\[
AV_B \leq A^*_v B + 2R_0(v) (AVB - A^*_v B) - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(B,A) \chi_{((k-1)/2^l,k/2^l)}(v).
\]

**Proof.** For any \( x > 0 \), we have
\[
(1 - v) + vx \geq x^v + r_0(v) \left( 1 - \sqrt{x} \right)^2 + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(1,x) \chi_{((k-1)/2^l,k/2^l)}(v),
\]
\[
(1 - v) + vx \leq x^v + R_0(v) \left( 1 - \sqrt{x} \right)^2 - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(x,1) \chi_{((k-1)/2^l,k/2^l)}(v),
\]
by Theorem 8. Thus, for any positive operator \( X \), we have
\[
(1 - v) I + vX \geq X^v + r_0(v) \left( I + X - 2 \sqrt{X} \right) + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} h_{l,k}(I,X) \chi_{((k-1)/2^l,k/2^l)}(v),
\]
\[
(1 - v) I + vX \leq X^v + R_0(v) \left( I + X - 2 \sqrt{X} \right) - \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} h_{l,k}(X,I) \chi_{((k-1)/2^l,k/2^l)}(v),
\]
by the operator monotonicity of continuous functions, where \( I \) is the identity operator. Note that since \( A^*_k B = B^*_{k-1} A \), we can express \( g_{l,k}(I,X) \) and \( g_{l,k}(X,I) \) by
\[
g_{l,k}(I,X) = I^*_k (k-1)/2 X + I^*_k (2k-1)/2^{l+1} X
\]
\[
= X^{(k-1)/2^l} + X^{(2k-1)/2^{l+1}},
\]
\[
g_{l,k}(X,I) = X^*_k (k-1)/2 I + X^*_k (2k-1)/2^{l+1} I
\]
\[
= I^*_1 (k-1)/2 X + I^*_1 (2k-1)/2^{l+1} X
\]
\[
- 2 I^*_1 (k-1)/2^{l+1} X
\]
\[
= X^{1-(k-1)/2^l} X^{1-(2k-1)/2^{l+1}}.
\]

Letting \( X = A^{-1/2} B A^{-1/2} \) and then multiplying all terms by \( A \) on both sides, (67) yields (65), where \( g_{l,k}(B,A) \) can be obtained as follows:
\[
A^{1/2} g_{l,k} \left( A^{-1/2} B A^{-1/2}, I \right) A^{1/2}
\]
\[
= A^*_1 (k-1)/2 B + A^*_1 (2k-1)/2^{l+1} B - 2 A^*_1 (2k-1)/2^{l+1} B
\]
\[
= B^*_k (k-1)/2 A + B^*_k (2k-1)/2^{l+1} A
\]
\[
= g_{l,k}(B,A).
\]

The following shows matrix inequalities corresponding to Corollary 9.

**Theorem 15.** Let \( A, B, X \) be \( n \times n \) complex matrices. If \( A \) and \( B \) are positive semidefinite, then
\[
\|(1 - v) AX + v XB\|_2^2 \geq \left\| A^{1-v} XB \right\|_2^2 + 2r_0(v) \left\| AX - XB \right\|_2^2 + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} h_{l,k}(A,B;X) \chi_{((k-1)/2^l,k/2^l)}(v),
\]
\[
\|(1 - v) AX + v XB\|_2^2 \leq \left\| A^{1-v} XB \right\|_2^2 + 2R_0(v) \left\| AX - XB \right\|_2^2 + \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} h_{l,k}(X,A;X) \chi_{((k-1)/2^l,k/2^l)}(v),
\]
where \( \| \cdot \|_2 \) denotes the Frobenius norm and
\[
h_{l,k}(A,B;X) = \left\| A^{1-(k-1)/2^l} \left( X B^{(k-1)/2^l} A - A^{1-k/2^l} X B^{k/2^l} \right) \right\|_2^2.
\]

**Proof.** Since \( A \) and \( B \) are positive semidefinite, there exist unitary matrices \( U \) and \( V \) such that \( A = U \text{diag}(a_1, \ldots, a_n) U^* \) and \( B = V \text{diag}(b_1, \ldots, b_n) V^* \), where \( \text{diag}(c_1, \ldots, c_n) \) denotes the \( n \times n \) diagonal matrix whose \( i \)-th diagonal entry is \( c_i \). Letting \( Y = U^* X V = (y_{ij})_{n \times n} \), we can show the following directly:
\[
AX - XB = U \left( \left( (1 - v) a_i + v b_j \right) y_{ij} \right) V^*,
\]
\[
(1 - v) AX + v XB = U \left( \left( 1 - v a_i + v b_j \right) y_{ij} \right) V^*,
\[ A^{1-v}XB^v = U \left( a_i^{1-v}b_j^v y_{ij} \right) V^*, \]

\[ A^{1-(k-1)/2}XB^{(k-1)/2} - A^{1-k/2}XB^{k/2} = U \left( a_i^{1-(k-1)/2}b_j^{(k-1)/2} - a_i^{1-k/2}b_j^{k/2} \right) y_{ij} V^*. \]

(72)

Since \( \| \cdot \|_2 \) is unitarily invariant, one obtains that

\[ \| (1-v)AX + vXB \|_2^2 \geq \| A^{1-v}XB^v \|_2^2 + r_0^2(v) \]

\[ \cdot \| AX - XB \|_2^2 + \sum_{l=1}^{N-1} r_l(v) \]

\[ + 2 \cdot \sum_{k=1}^{L} \sum_{l=1}^{T} \sum_{j=1}^{N} \sum_{i=1}^{N} g_{l,k} \left( a_i^*, b_j^* \right) |y_{ij}|^2 \chi_{\left( (l-1)/2, k/2 \right)}^2(v), \]

\[ \| (1-v)AX + vXB \|_2^2 \leq \| A^{1-v}XB^v \|_2^2 + R_0^2(v) \]

\[ \cdot \| AX - XB \|_2^2 - \sum_{l=1}^{N-1} r_l(v) \]

\[ - 2 \cdot \sum_{k=1}^{L} \sum_{l=1}^{T} \sum_{j=1}^{N} \sum_{i=1}^{N} g_{l,k} \left( b_j^*, a_i^* \right) |y_{ij}|^2 \chi_{\left( (l-1)/2, k/2 \right)}^2(v). \]

(74)

Thus, by (32), we have

\[ \| (1-v)AX + vXB \|_2^2 \geq \| A^{1-v}XB^v \|_2^2 + r_0^2(v) \]

\[ \cdot \| AX - XB \|_2^2 + \sum_{l=1}^{N-1} r_l(v) \]

\[ + 2 \cdot \sum_{k=1}^{L} \sum_{l=1}^{T} \sum_{j=1}^{N} \sum_{i=1}^{N} g_{l,k} \left( a_i^*, b_j^* \right) |y_{ij}|^2 \chi_{\left( (l-1)/2, k/2 \right)}^2(v), \]

\[ \| (1-v)AX + vXB \|_2^2 \leq \| A^{1-v}XB^v \|_2^2 + R_0^2(v) \]

\[ \cdot \| AX - XB \|_2^2 - \sum_{l=1}^{N-1} r_l(v) \]

\[ - 2 \cdot \sum_{k=1}^{L} \sum_{l=1}^{T} \sum_{j=1}^{N} \sum_{i=1}^{N} g_{l,k} \left( b_j^*, a_i^* \right) |y_{ij}|^2 \chi_{\left( (l-1)/2, k/2 \right)}^2(v). \]

(74)

By Theorem 12, we have

\[ 1 - v + vt \geq K_N(1,t)^{r_0(v)} t^v \]

\[ + \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{L} g_{l,k} \left( 1, t \right) \chi_{\left( (l-1)/2, k/2 \right)}(v), \]

(78)

for any \( t > 0 \). Since \( K_N(1,t) \) is an increasing function on \((1, \infty)\) and \( K_N(1,t)^{-1} = K_N(1,t) \) for \( t > 0 \), \( K_N(1,h) \leq K_N(1,t) \) for \( t \geq h \) or \( t \leq 1/h \). Thus, from (78), we have

\[ 1 - v + vt \geq K_N(1,h)^{r_0(v)} t^v \]

\[ + \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{L} g_{l,k} \left( 1, t \right) \chi_{\left( (l-1)/2, k/2 \right)}(v), \]

(79)

for \( t \geq h \) or \( t \leq 1/h \). Replacing \( t \) by \( X = A^{-1/2}BA^{-1/2} \) which satisfies \( hI \leq X \) or \( X \leq (1/h)I \), we get that

\[ (1-v)I + vX \]

\[ \geq K_N(1,h)^{r_0(v)} X^v \]

\[ + \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{L} g_{l,k} \left( I, X \right) \chi_{\left( (l-1)/2, k/2 \right)}(v). \]

(80)
Multiplying the inequality by \( A^{1/2} \) on both sides, we have

\[
AV_v B \geq K_N (1, h)^{r_N (v)} A^+_v B + \sum_{l=0}^{N-1} r_l (v)
\]

\[
\cdot \sum_{k=1}^{2^l} \left( A^+_{(k-1)/2} B + A^+_{k/2} B - 2A^+_{(2k-1)/2} B \right)
\cdot X_{(k-1)/2} (v).
\]

(81)

Since the zeroth term \((l = 0)\) in the summation is \( r_0(v) (A + B - 2A_B) \), we finish the proof of the first inequality of this theorem.

We can prove the second one in the same way. Letting \( t = a^{-1} b \) in Theorem 13, we have

\[
1 - v + vt \leq 2 \sqrt{t} - K_N (1, t)^{r_t (v)} t^{1-v}
\]

\[
+ R_0 (v) \left( t + 1 - 2 \sqrt{t} \right)
\]

\[
- \sum_{l=0}^{N-1} r_l (v) \sum_{k=1}^{2^l} g_{l,k} (t, 1) X_{(k-1)/2} (v)
\]

\[
\leq K_N (1, t)^{-r_t (v)} t^v + R_0 (v) \left( t + 1 - 2 \sqrt{t} \right)
\]

\[
- \sum_{l=0}^{N-1} r_l (v) \sum_{k=1}^{2^l} g_{l,k} (t, 1) X_{(k-1)/2} (v),
\]

(82)

for any \( t > 0 \). In particular, if \( t = h \) or \( t \leq 1/h \), then

\[
1 - v + vt \leq 2 \sqrt{t} - K_N (1, h)^{r_h (v)} t^{1-v}
\]

\[
+ R_0 (v) \left( t + 1 - 2 \sqrt{t} \right)
\]

\[
- \sum_{l=0}^{N-1} r_l (v) \sum_{k=1}^{2^l} g_{l,k} (t, 1) X_{(k-1)/2} (v)
\]

\[
\leq K_N (1, h)^{-r_h (v)} t^v + R_0 (v) \left( t + 1 - 2 \sqrt{t} \right)
\]

\[
- \sum_{l=0}^{N-1} r_l (v) \sum_{k=1}^{2^l} g_{l,k} (t, 1) X_{(k-1)/2} (v).
\]

(83)

Replacing \( t \) by \( X = A^{-1/2} BA^{-1/2} \), one has

\[
(1 - v) I + vX
\]

\[
\leq 2 \sqrt{X} - K_N (1, h)^{r_h (v)} X^{1-v}
\]

\[
+ R_0 (v) \left( X + I - 2 \sqrt{X} \right)
\]

\[
- \sum_{l=1}^{N-1} r_l (v) \sum_{k=1}^{2^l} g_{l,k} (X, I) X_{(k-1)/2} (v)
\]

\[
\leq K_N (1, h)^{-r_h (v)} X^v + R_0 (v) \left( X + I - 2 \sqrt{X} \right)
\]

\[
- \sum_{l=1}^{N-1} r_l (v) \sum_{k=1}^{2^l} g_{l,k} (X, I) X_{(k-1)/2} (v).
\]

(84)

Finally, multiplying each term by \( A^{1/2} \) on both sides, we get that

\[
AV_v B \leq 2A_B - K_N (1, h)^{r_h (v)} A^+ B + R_0 (v) (A + B - 2A_B)
\]

\[
- 2A_B)
\]

\[
\cdot \sum_{k=1}^{2^l} \left( A^+_{(k-1)/2} B + A^+_{k/2} B - 2A^+_{(2k-1)/2} B \right)
\cdot X_{(k-1)/2} (v).
\]

(85)

Since \( A^+ = B \mu A \), the above shows the second inequality of this theorem.

\[\square\]

In the following, we consider special values of \( v \).

**Corollary 17.** Let \( A, B > 0 \) and \( 0 \leq v \leq 1 \). If \( v = k_0/2^N \) for some \( 1 \leq k_0 < 2^N \), then

\[
AV_v B \geq A^+_v B + \sum_{l=0}^{N-1} r_l (v)
\]

\[
\cdot \sum_{k=1}^{2^l} \left( A^+_{(k-1)/2} B + A^+_{k/2} B - 2A^+_{(2k-1)/2} B \right)
\cdot X_{(k-1)/2} (v).
\]

(86)
If $v = (2k_0 - 1)/2^{N+1}$ for some $1 \leq k_0 \leq 2^N$, then
\[
A \nabla_v B \geq \frac{1}{4} (A_+ v_{-2-N} B + A_+ v_{-2+N} B + 2A_+ B)
\]

\[
+ \sum_{l=0}^{N-1} r_l(v)
\]

\[
\cdot \sum_{k=1}^{2^l} \left( A_+ (k-1)/2^l B + A_+ k/2^l B - 2A_+ (2k-1)/2^{l+1} B \right)
\]

\[
\cdot \chi_{((k-1)/2^l,k/2^l)}(v),
\]

\[
A \nabla_v B \leq 2A_+ B - \frac{1}{4} \left( B_+ v_{+2-N} A + B_+ v_{+2-N} A + 2B_+ A \right)
\]

\[
+ R_0(v) (A + B - 2A_+ B) - \sum_{l=0}^{N-1} r_l(v)
\]

\[
\cdot \sum_{k=1}^{2^l} \left( B_+ (k-1)/2^l A + B_+ k/2^l A + B_+ (2k-1)/2^{l+1} A \right)
\]

\[
\cdot \chi_{((k-1)/2^l,k/2^l)}(v).
\]

Proof. If $v = k_0/2^N$, then $r_N(v) = 0$. Thus from (78) and (82), we have
\[
1 - v + vt \geq t^v + \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(1,t) \chi_{((k-1)/2^l,k/2^l)}(v),
\]

\[
1 - v + vt \leq 2\sqrt{t} - t^{1-v} + R_0(v) \left( t + 1 - 2\sqrt{t} \right)
\]

\[
- \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(t,1) \chi_{((k-1)/2^l,k/2^l)}(v)
\]

\[
\leq t^v + R_0(v) \left( t + 1 - 2\sqrt{t} \right)
\]

\[
- \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(t,1) \chi_{((k-1)/2^l,k/2^l)}(v).
\]

Using the same argument as in the proof of Theorem 1, it is easy to derive the desired operator inequalities from the above.

Meanwhile, if $v = (2k_0 - 1)/2^{N+1}$, then $r_N(v) = 1$. Thus, from (78) and the first inequality of (82), we have
\[
1 - v + vt \geq \frac{1}{4} \left( t^{1-v} + t^{v-1/2} + 2t^{v} \right)
\]

\[
+ \sum_{l=0}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(1,t) \chi_{((k-1)/2^l,k/2^l)}(v),
\]

\[
1 - v + vt \leq 2\sqrt{t} - \frac{1}{4} \left( t^{1-v} + t^{v-1/2} + 2t^{1-v} \right)
\]

\[
- \sum_{l=1}^{N-1} r_l(v) \sum_{k=1}^{2^l} g_{l,k}(t,1) \chi_{((k-1)/2^l,k/2^l)}(v).
\]

Letting $t = X = A^{-1/2} BA^{-1/2}$, we can obtain the desired operator inequalities.

\[\square\]

Competing Interests

The author declares that there is no competing interests regarding the publication of this paper.

References


