We introduce the notion of \((\lambda, \mu)\)-product of \(L\)-subsets. We give a necessary and sufficient condition for \((\lambda, \mu)\)-subgroup of a product of groups to be \((\lambda, \mu)\)-product of \((\lambda, \mu)\)-subgroups.

1. Introduction

Starting from 1980, by the concept of quasi-coincidence of a fuzzy point with a fuzzy set given by Pu and Liu [1], the generalized subalgebraic structures of algebraic structures have been investigated again. Using the concept mentioned above, Bhakta and Das [2, 3] gave the definition of \((\alpha, \beta)\)-fuzzy subgroup, where \(\alpha, \beta\) are any two of \([\epsilon, q, \epsilon \lor q, \epsilon \land q]\) with \(\alpha \neq \epsilon \land q\). The \((\epsilon, \epsilon \lor q)\)-fuzzy subgroup is an important and useful generalization of fuzzy subgroups that were laid by Rosenfeld in [4]. After this, many other researchers used the idea of the generalized fuzzy sets that give several characterization results in different branches of algebra (see [5–10]). In recent years, many researchers make generalizations which are referred to as \((\lambda, \mu)\)-fuzzy substructures and \((\epsilon_\lambda, \epsilon_\land q_\mu)\)-fuzzy substructures on this topic (see [11–15]).

Identifying the subgroups of a Cartesian product of groups plays an essential role in studying group theory. Many important results on characterization of Cartesian product of subgroups, fuzzy subgroups, and TL-fuzzy subgroups exist in literature. Chon obtained a necessary and sufficient condition for a fuzzy subgroup of a Cartesian product of groups to be product of fuzzy subgroups under minimum operation [16]. Later, some necessary and sufficient conditions for a TL-subgroup of a Cartesian product of groups to be a \(T\)-product of TL-subgroups were given by Yamak et al. [17]. A subgroup of a Cartesian product of groups is characterized by subgroups in the same study. Consequently, it seems to be interesting to extend this study to generalized \(L\)-subgroups. In this paper, we introduce the notion of the \((\lambda, \mu)\)-product of \(L\)-subsets and investigate some properties of the \((\lambda, \mu)\)-product of \(L\)-subgroups. Also, we give a necessary and sufficient condition for \((\lambda, \mu)\)-subgroup of a Cartesian product of groups to be a product of \((\lambda, \mu)\)-\(L\)-subgroups.

2. Preliminaries

In this section, we start by giving some known definitions and notations. Throughout this paper, unless otherwise stated, \(G\) always stands for any given group with a multiplicative binary operation, an identity \(e\) and \(L\) denote a complete lattice with top and bottom elements 1, 0, respectively.

An \(L\)-subset of \(X\) is any function from \(X\) into \(L\), which is introduced by Goguen [18] as a generalization of the notion of Zadeh's fuzzy subset [19]. The class of \(L\)-subsets of \(X\) will be denoted by \(F(X, L)\). In particular, if \(L = [0, 1]\), then it is appropriate to replace \(L\)-subset with fuzzy subset. In this case the set of all fuzzy subsets of \(X\) is denoted by \(\mathcal{F}(X)\).

We define a partial ordering on the set \(\mathcal{F}(X, L)\).

Definition 1 (see [20]). An \(L\)-subset of \(G\) is called an \(L\)-subgroup of \(G\) if, for all \(x, y \in G\), the following conditions hold:

\[(G1) \ A(x) \land A(y) \leq A(xy).
\]

\[(G2) \ A(x) \leq A(x^{-1}).\]

In particular, when \(L = [0, 1]\), an \(L\)-subgroup of \(G\) is referred to as a fuzzy subgroup of \(G\).
Definition 2 (see [12]). Let \( \lambda, \mu \in L \) and \( \lambda < \mu \). Let \( A \) be an \( L \)-subset of \( G \). \( A \) is called \((\lambda, \mu)\)-\( L \)-subgroup of \( G \) if, for all \( x, y \in G \), the following conditions hold:

(i) \( A(xy) \lor \lambda \geq A(x) \land A(y) \land \mu. \)

(ii) \( A(x^{-1}) \lor \lambda \geq A(x) \land \mu. \)

Denote by \( FS(\lambda, \mu, G, L) \) the set of all \((\lambda, \mu)\)-\( L \)-subgroups of \( G \). When \( L = [0, 1] \), its counterpart is written as \( FS(\lambda, \mu, G) \).

Unless otherwise stated, \( L \) always represents any given distributive lattice.

3. Product of \((\lambda, \mu)\)-\( L \)-Subgroups

Definition 3 (see [16]). Let \( A_i \) be an \( L \)-subset of \( G_i \) for each \( i = 1, 2, \ldots, n \). Then product of \( A_i \) (\( i = 1, 2, \ldots, n \)) denoted by \( A_1 \times A_2 \times \cdots \times A_n \) is defined to be the \( L \)-subset of \( G_1 \times G_2 \times \cdots \times G_n \) that satisfies

\[
(A_1 \times A_2 \times \cdots \times A_n)(x_1, x_2, \ldots, x_n) = A_1(x_1) \land A_2(x_2) \land \cdots \land A_n(x_n).
\]

(1)

Example 4. We define the fuzzy subsets \( A \) and \( B \) of \( \mathbb{Z} \) and \( \mathbb{Z}_2 \), respectively, as follows:

\[
A(x) = \begin{cases} 
0.5, & x \in 2\mathbb{Z}, \\
0.3, & \text{otherwise,} 
\end{cases} \quad B(x) = \begin{cases} 
0.7, & x = \overline{0}, \\
0.2, & x = 1.
\end{cases}
\]

(2)

We obtain that

\[
A \times B = \begin{cases} 
0.5, & x \in 2\mathbb{Z} \times \{0\}, \\
0.3, & x \in \mathbb{Z} - 2\mathbb{Z} \times \{0\}, \\
0.2, & \text{otherwise.}
\end{cases}
\]

(3)

\[ A(x) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

(6)

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]

\[ A \times B = (A \times B)(x, y) = \begin{cases} 
0.8, & x = (\overline{0}, \overline{0}), \\
0.7, & x = (1, \overline{0}), \\
0.6, & x = (\overline{0}, 1), \\
0.5, & x = (1, 1).
\end{cases} \]
condition of Theorem 7, but there do not exist \( A_1, A_2 \) fuzzy subgroups of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) such that \( A = A_1 \times A_2 \).

In fact, suppose that there exist \( A_1, A_2 \in FS(0.0.5, \mathbb{Z}_2 \times \mathbb{Z}_2) \) such that \( A = A_1 \times A_2 \). Since \( A(\bar{0}, \bar{0}) = 0.7 \) and \( A(\bar{0}, \bar{1}) = 0.6 \), we have \( A_1(\bar{1}) \geq 0.7 \) and \( A_2(\bar{1}) \geq 0.6 \). Hence

\[
0.5 = A_1 \times A_2 (\bar{1}, \bar{1}) = A_1 (\bar{1}) \land A_2 (\bar{1}) \geq 0.7 \land 0.6
\]

(7)

a contradiction.

**Definition 9.** Let \( A_i \) be an \( L \)-subset of \( G_i \) for each \( i = 1, 2, \ldots, n \). Then \((\lambda, \mu)\)-product of \( A_i \) \((i = 1, 2, \ldots, n)\) denoted by \( A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n \) is defined to be an \( L \)-subset of \( G_1 \times G_2 \times \cdots \times G_n \) that satisfies

\[
\left( A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n \right) (x_1, x_2, \ldots, x_n)
= (A_1(x_1) \land A_2(x_2) \cdots \land A_n(x_n) \lor \land \lambda)
= \left( \bigwedge_{i=1}^{n} A_i (x_i) \land \mu \right) \lor \lambda.
\]

(8)

**Example 10.** We define the fuzzy subsets \( A \) and \( B \) of \( \mathbb{Z} \) and \( \mathbb{Z}_2 \), respectively, as in Example 4. Then \((0.4, 0.6)\)-product of \( A \) and \( B \) is as follows:

\[
A \times_{0.4} B (x) = \begin{cases} 0.5, & x \in 2 \mathbb{Z} \times \{0\}, \\ 0.4, & \text{otherwise}. \end{cases}
\]

(9)

**Lemma 11.** Let \( G_1, G_2, \ldots, G_n \) be groups. Then we have the following:

1. If \( A \) is \((\lambda, \mu)\)-subgroup of \( G_1 \times G_2 \times \cdots \times G_n \) and \( A_i(x) = A(e_1, e_2, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n) \) for \( i = 1, 2, \ldots, n \), then \( A_i \) is \((\lambda, \mu)\)-subgroup of \( G_i \) for all \( i = 1, 2, \ldots, n \).

2. If \( A_i \) is \((\lambda, \mu)\)-subgroup of \( G_i \) for all \( i = 1, 2, \ldots, n \), then \( A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n \) is \((\lambda, \mu)\)-subgroup of \( G_1 \times G_2 \times \cdots \times G_n \).

**Proof.** (1) Let \( x, y \in G_1 \). Since \( A \in FS(\lambda, \mu, G_1 \times G_2 \times \cdots, G_n, L) \), we have

\[
A_1 (x) \land A_1 (y) \land \mu
= A(e_1, e_2, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n)
\land A(e_1, e_2, \ldots, e_{i-1}, y, e_{i+1}, \ldots, e_n) \land \mu
\leq A(e_1, e_2, \ldots, e_{i-1}, x \cdot y, e_{i+1}, \ldots, e_n) \lor \lambda
= A_1 (x \cdot y) \lor \lambda.
\]

(10)

Similarly, we can show that \( A_i (x) \land \mu \leq A_i (x^{-1}) \lor \lambda \). Hence, \( A_i \in FS(\lambda, \mu, G_i, L) \) by Definition 2 for \( i = 1, 2, \ldots, n \).

(2) Let \((x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \in G_1 \times G_2 \times \cdots \times G_n \).

Then,

\[
A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n \left( (x_1, x_2, \ldots, x_n), (y_1, y_2, \ldots, y_n) \right)
\lor \lambda
= A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n (x_1, y_1, x_2, y_2, \ldots, x_n, y_n)
\lor \lambda
= \left( \bigwedge_{i=1}^{n} A_i (x_i) \land \mu \right) \lor \lambda = A(\bar{1}, x_1, y_1, x_2, \ldots, x_n, y_n)
\land \lambda
= A \left( (x_1, x_2, \ldots, x_n) \land \mu \right) \lor \lambda
\land \lambda
= A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n (x_1, x_2, \ldots, x_n) \land \mu \lor \lambda.
\]

(11)

Similarly, we can show that

\[
A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n \left( (x_1, x_2, \ldots, x_n) \right) \lor \lambda
\geq A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n (x_1, x_2, \ldots, x_n) \land \mu.
\]

(12)

Hence, \( A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n \) is \((\lambda, \mu)\)-subgroup of \( G_1 \times G_2 \times \cdots \times G_n \).

**Theorem 12.** Let \( G_1, G_2, \ldots, G_n \) be groups and \( A \in FS(\lambda, \mu, G_1 \times G_2 \times \cdots \times G_n, L) \). Then \( A(e_1, e_2, \ldots, e_{i-1}, x_i e_i, e_{i+1}, \ldots, e_n) \land A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \land \lambda = A(x_1, x_2, \ldots, x_n) \land \lambda \) for \( i = 1, 2, \ldots, n \) if and only if \( A \) is \( A_1 \times^\lambda \mu \cdots \times^\lambda \mu A_n \), where \( A_i(x) = A(e_1, e_2, \ldots, e_{i-1}, x, e_{i+1}, \ldots, e_n) \).

**Proof.** Suppose that \( A(e_1, e_2, \ldots, e_{i-1}, x_i, e_{i+1}, \ldots, e_n) \land A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \land \mu \lor \lambda = A(x_1, x_2, \ldots, e_i, x_{i+1}, \ldots, x_n) \lor \lambda \) for \( i = 1, 2, \ldots, n \). Now, for any \((x_1, x_2, \ldots, x_n) \in G_1 \times G_2 \times \cdots \times G_n \), we have

\[
A(x_1, x_2, \ldots, x_n) = A(x_1, e_2, \ldots, e_n)
\land A(x_2, x_3, \ldots, x_n) \land \mu \lor \lambda
= A_1 (x_1)
\land \left( (A(e_1, x_2, \ldots, x_n) \land A(e_2, x_3, \ldots, x_n) \land \mu) \lor \lambda \right).
\]
Where, $A = A_1 \wedge A_2 \wedge \cdots \wedge A_n$. Then, we obtain

$$A(x_1, x_2, \ldots, x_n) = A_1 \wedge A_2 \wedge \cdots \wedge A_n(x_1, x_2, \ldots, x_n).$$

(13)

Hence we obtain that $A = A_1 \wedge A_2 \wedge \cdots \wedge A_n$. Conversely, assume that $A = A_1 \wedge A_2 \wedge \cdots \wedge A_n$. Then, we obtain

$$A(x_1, x_2, \ldots, x_n) = A_1 \wedge A_2 \wedge \cdots \wedge A_n(x_1, x_2, \ldots, x_n).$$

(14)

On the other hand, $(A(x_1, x_2, \ldots, x_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu) \wedge \nu \leq A(x_1, x_2, \ldots, x_n) \wedge \nu$.

Since $A(x_1, x_2, \ldots, x_n) \geq \lambda$, $(A(x_1, x_2, \ldots, x_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu) \wedge \nu \leq A(x_1, x_2, \ldots, x_n) \wedge \nu$.

Hence, $A(x_1, x_2, \ldots, x_n) = (A(x_1, x_2, \ldots, x_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu) \wedge \nu$.

Similarly, we get $(A(e_1, \cdots, e_i, x_{i+1}, \cdots, e_n) \wedge A(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \wedge \mu) \wedge \nu = A(x_1, \cdots, x_n)$ for $i = 2, 3, \ldots, n$.

Lemma 13. Let $G_1, G_2, \ldots, G_n$ be groups, let $e_1, e_2, \ldots, e_n$ be identities of $G_1, G_2, \ldots, G_n$, respectively, and $A \in FS(\mu, G_1 \times G_2 \times \cdots \times G_n)$. Then $A(e_1, e_2, \ldots, e_i, x, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu = A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$.

Proof. Now assume that $A(e_1, e_2, \ldots, e_i, x, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu = A(x_1, x_2, \ldots, x_n)$ for $i = 1, 2, \ldots, n$. Next, for any $(x_1, x_2, \ldots, x_n) \in G_1 \times G_2 \times \cdots \times G_n$, we have

$$A(x_1, x_2, \ldots, x_n) = A(e_1, e_2, \ldots, e_i, x, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{i-1}, e_i, x_{i+1}, \ldots, x_n) \wedge \mu \leq A(e_1, e_2, \ldots, e_i, x, e_{i+1}, \ldots, e_n) \wedge \mu.$$

(16)
Definition 2 and Lemma 11, we obtain that
\[ A(x_1, x_2, \ldots, x_n) = A(x_1, x_2, \ldots, x_n) \]
\[ \wedge A(x_1, x_2, \ldots, x_n) \wedge \cdots \wedge A(x_1, x_2, \ldots, x_n) \]
\[ \leq A(e_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, e_n) \wedge \cdots \]
\[ \wedge A(e_1, e_2, \ldots, x_{n-1}, x_n) \wedge \mu \]
\[ \leq A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, e_n) \wedge \cdots \]
\[ \wedge A(e_1, e_2, \ldots, x_{n-1}, x_n) \wedge \mu \]
\[ \vdots \]
\[ \leq A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu \]
\[ \leq A(x_1, x_2, \ldots, x_n) \].

Hence, \( A(x_1, x_2, \ldots, x_n) = A(x_1, e_2, \ldots, e_n) \wedge A(e_1, x_2, \ldots, x_n) \wedge \mu \). Similarly, we get \( A(e_1, e_2, \ldots, e_{1-i}, x_i, e_{i+1}, \ldots, e_n) \wedge A(x_1, x_2, \ldots, x_{n-1}, x_i, e_{i+1}, \ldots, e_n) = A(x_1, x_2, \ldots, x_n) \) for \( i = 2, 3, \ldots, n \).

As a consequence of Theorem 12 and Lemma 14, we have the following corollary.

**Corollary 15.** Let \( G_1, G_2, \ldots, G_n \) be groups and \( A \in FS(0, \mu, G_1 \times G_2 \times \cdots \times G_n, L) \). Then \( A(e_1, e_2, \ldots, e_{1-i}, x_i, e_{i+1}, \ldots, e_n) \wedge \mu \geq A(x_1, x_2, \ldots, x_n) \) for \( i = 1, 2, \ldots, k-1, k+1, \ldots, n \) if and only if \( A = A_1 \times A_2 \times \cdots \times A_n \), where \( A_i(x) = A(e_1, e_2, \ldots, e_{1-i}, x_i, e_{i+1}, \ldots, e_n) \).

The following example shows that Corollary 15 may not be true when \( \lambda \neq 0 \).

**Example 16.** Consider
\[ A(x) = \begin{cases} 
0.4, & x = (\overline{0}, \overline{0}) \\
0.3, & x = (\overline{1}, \overline{0}) \\
0.2, & x = (\bar{0}, \overline{1}) \\
0.1, & x = (\bar{1}, \overline{1}) 
\end{cases} \] (18)

\( A \) is \((0.2, 0.5)\)-fuzzy subgroup of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \) and satisfies the necessary condition of Corollary 15. But there is not any \( A_1 \) and \( A_2 \) \((0.2, 0.5)\)-fuzzy subgroup of \( \mathbb{Z}_2 \times \mathbb{Z}_2 \), which hold \( A = A_1 \otimes_{0.2} A_2 \).

**4. Conclusion**

In this study, we give a necessary and sufficient condition for \((0, \mu)\)-\( L \)-subgroup of a Cartesian product of groups to be a product of \((0, \mu)\)-\( L \)-subgroups. The results obtained are not valid for \( \lambda \neq 0 \), and a counterexample is provided.

**Conflict of Interests**

The authors declare that there is no conflict of interests regarding the publication of this paper.

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