Research Article

Hermite-Hadamard-Fejér Type Inequalities for Quasi-Geometrically Convex Functions via Fractional Integrals

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Some Hermite-Hadamard-Fejér type integral inequalities for quasi-geometrically convex functions in fractional integral forms have been obtained.

1. Introduction

Let $f : I \subseteq \mathbb{R} \rightarrow \mathbb{R}$ be a convex function defined on interval I of real numbers and $a, b \in I$ with $a < b$. The inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{2} \quad (1)$$

is well known in the literature as Hermite-Hadamard's inequality [1].

The most well-known inequalities related to the integral mean of a convex function f are the Hermite-Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities.

In [2], Fejér established the following Fejér inequality which is the weighted generalization of Hermite-Hadamard inequality (1).

Theorem 1. Let $f : [a, b] \rightarrow \mathbb{R}$ be convex function. Then, the inequality

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b g(x) dx &\leq \int_a^b f(x) g(x) dx \\ &\leq \frac{f(a) + f(b)}{2} \int_a^b g(x) dx \end{aligned} \quad (2)$$

holds, where $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable, and symmetric to $(a+b)/2$.

For some results which generalize, improve, and extend inequalities (1) and (2), see [3–6].

Definition 2 (see [7, 8]). A function $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if

$$f(x^t y^{1-t}) \leq t f(x) + (1-t) f(y) \quad (3)$$

for all $x, y \in I$ and $t \in [0, 1]$.

Definition 3 (see [9]). A function $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ is said to be quasi-geometrically convex on I if

$$f(x^t y^{1-t}) \leq \sup\{f(x), f(y)\} \quad (4)$$

for all $x, y \in I$ and $t \in [0, 1]$.

In [10], Latif et al. established the following inequality which is the weighted generalization of Hermite-Hadamard inequality for GA-convex functions as follows.

Theorem 4. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a GA-convex function and $a, b \in I$ with $a < b$. Let $g : [a, b] \rightarrow [0, \infty)$ be continuous positive mapping and geometrically symmetric to $\sqrt{ab}$. Then,

$$\begin{aligned} f(\sqrt{ab}) \int_a^b \frac{g(x)}{x} dx &\leq \int_a^b \frac{f(x) g(x)}{x} dx \\ &\leq \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} dx. \end{aligned} \quad (5)$$

We will now give definitions of the right-hand side and left-hand side Hadamard fractional integrals which are used throughout this paper.

Definition 5 (see [11]). Let $f \in L[a, b]$. The right-hand side and left-hand side Hadamard fractional integrals $J_{a+}^\alpha f$ and $J_{b-}^\alpha f$ of order $\alpha > 0$ with $b > a \geq 0$ are defined by

$$\begin{aligned} J_{a+}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_a^x \left(\ln \frac{x}{t}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad x > a, \\ J_{b-}^\alpha f(x) &= \frac{1}{\Gamma(\alpha)} \int_x^b \left(\ln \frac{t}{x}\right)^{\alpha-1} f(t) \frac{dt}{t}, \quad x < b, \end{aligned} \quad (6)$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt$.

Because of the wide application of Hermite-Hadamard type inequalities and fractional integrals, many researchers extend their studies to Hermite-Hadamard type inequalities involving fractional integrals not limited to integer integrals. Recently, more and more Hermite-Hadamard inequalities involving fractional integrals have been obtained for different classes of functions; see [9, 12–15].

In [9], İşcan represented Hermite-Hadamard's inequalities for GA-convex functions in fractional integral forms as follows.

Theorem 6. Let $f : I \subseteq (0, \infty) \rightarrow \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If f is a GA-convex function on $[a, b]$, then the following inequalities for fractional integrals hold:

$$\begin{aligned} f(\sqrt{ab}) &\leq \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \\ &\leq \frac{f(a) + f(b)}{2} \end{aligned} \quad (7)$$

with $\alpha > 0$.

In [16], the authors represented Hermite-Hadamard-Fejér inequalities for GA-convex functions in fractional integral forms as follows.

Theorem 7. Let $f : [a, b] \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a GA-convex function with $a < b$ and $f \in L[a, b]$. If $g : [a, b] \rightarrow \mathbb{R}$ is nonnegative, integrable, and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:

$$\begin{aligned} f(\sqrt{ab}) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] \\ \leq [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \\ \leq \frac{f(a) + f(b)}{2} [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] \end{aligned} \quad (8)$$

with $\alpha > 0$.

Lemma 8 (see [16]). Let $f : [a, b] \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on (a, b) with $a < b$ and $f' \in L[a, b]$.

If $g : [a, b] \rightarrow \mathbb{R}$ is integrable and geometrically symmetric with respect to $\sqrt{ab}$, then the following equality for fractional integrals holds:

$$\begin{aligned} \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) \\ + J_{b-}^\alpha (fg)(a)] = \frac{1}{\Gamma(\alpha)} \int_a^b \left[\int_a^t \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right. \\ \left. - \int_t^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right] f'(t) dt \end{aligned} \quad (9)$$

with $\alpha > 0$.

Lemma 9 (see [15, 17]). For $0 < \alpha \leq 1$ and $0 \leq a < b$, one has

$$|a^\alpha - b^\alpha| \leq (b - a)^\alpha. \quad (10)$$

In [9], İşcan introduced the quasi-geometrically convex functions and presented some Hermite-Hadamard type integral inequalities for quasi-geometrically convex functions via fractional integrals. In [18], İşcan obtained generalized Hermite-Hadamard type integral inequalities for quasi-geometrically convex functions. In [19], İşcan et al. showed some Hermite-Hadamard and Simpson type inequalities for differentiable quasi-geometrically convex functions.

In this paper, it is the first time Hermite-Hadamard-Fejér type integral inequality for quasi-geometrically convex function has been studied. Obtained results are new and generalize the Hermite-Hadamard type integral inequality for quasi-geometrically convex functions.

2. Main Results

Throughout this section, let $\|g\|_\infty = \sup_{t \in [a, b]} |g(x)|$, for the continuous function $g : [a, b] \rightarrow \mathbb{R}$.

Theorem 10. Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$. If $|f'|$ is quasi-geometrically convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequality for fractional integrals holds:

$$\begin{aligned} \left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] \right. \\ \left. - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \leq \frac{\|g\|_\infty \ln^{\alpha+1}(b/a)}{\Gamma(\alpha+1)} \\ \cdot C_1(\alpha) \sup \{|f'(a)|, |f'(b)|\}, \end{aligned} \quad (11)$$

where

$$C_1(\alpha) = \int_0^{1/2} [(1-u)^\alpha - u^\alpha] [a^{1-u} b^u + a^u b^{1-u}] du \quad (12)$$

with $\alpha > 0$.

Proof. From Lemma 8 we have

$$\begin{aligned} & \left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] - [J_{a+}^{\alpha} (fg)(b) \right. \\ & \quad \left. + J_{b-}^{\alpha} (fg)(a)] \right| \leq \frac{1}{\Gamma(\alpha)} \int_a^b \left| \int_a^t \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right. \\ & \quad \left. - \int_t^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| |f'(t)| dt. \end{aligned} \quad (13)$$

Setting $t = a^{1-u}b^u$ and $dt = a^{1-u}b^u \ln(b/a)du$ gives

$$\begin{aligned} & \left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] - [J_{a+}^{\alpha} (fg)(b) \right. \\ & \quad \left. + J_{b-}^{\alpha} (fg)(a)] \right| \leq \frac{1}{\Gamma(\alpha)} \\ & \quad \cdot \int_0^1 \left| \int_a^{a^{1-u}b^u} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right. \\ & \quad \left. - \int_{a^{1-u}b^u}^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| |f'(a^{1-u}b^u)| a^{1-u}b^u \\ & \quad \cdot \ln \left(\frac{b}{a} \right) du. \end{aligned} \quad (14)$$

Since $g : [a, b] \rightarrow \mathbb{R}$ is geometrically symmetric with respect to $\sqrt{ab}$, we write

$$\begin{aligned} & \int_{a^{1-u}b^u}^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \\ & = \int_a^{a^u b^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g\left(\frac{ab}{s}\right) \frac{ds}{s} \\ & = \int_a^{a^u b^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \end{aligned} \quad (15)$$

and then we have

$$\begin{aligned} & \left| \int_a^{a^{1-u}b^u} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right. \\ & \quad \left. - \int_{a^{1-u}b^u}^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| \\ & = \left| \int_{a^{1-u}b^u}^{a^u b^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| \end{aligned}$$

$$\begin{aligned} & \leq \begin{cases} \int_{a^{1-u}b^u}^{a^u b^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} |g(s)| \frac{ds}{s} & u \in \left[0, \frac{1}{2}\right] \\ \int_{a^u b^{1-u}}^b \left(\ln \frac{b}{s} \right)^{\alpha-1} |g(s)| \frac{ds}{s} & u \in \left[\frac{1}{2}, 1\right] \end{cases} \\ & \leq \|g\|_{\infty} \begin{cases} \int_{a^{1-u}b^u}^{a^u b^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} \frac{ds}{s} & u \in \left[0, \frac{1}{2}\right] \\ \int_{a^u b^{1-u}}^b \left(\ln \frac{b}{s} \right)^{\alpha-1} \frac{ds}{s} & u \in \left[\frac{1}{2}, 1\right] \end{cases} \\ & = \|g\|_{\infty} \frac{(\ln(b/a))^{\alpha}}{\alpha} \begin{cases} (1-u)^{\alpha} - u^{\alpha} & u \in \left[0, \frac{1}{2}\right] \\ u^{\alpha} - (1-u)^{\alpha} & u \in \left[\frac{1}{2}, 1\right] \end{cases}. \end{aligned} \quad (16)$$

Since $|f'|$ is quasi-geometrically convex on $[a, b]$, we know that, for $u \in [0, 1]$,

$$|f'(a^{1-u}b^u)| \leq \sup \{|f'(a)|, |f'(b)|\}, \quad (17)$$

and a combination of (14), (16), and (17) gives

$$\begin{aligned} & \left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] - [J_{a+}^{\alpha} (fg)(b) \right. \\ & \quad \left. + J_{b-}^{\alpha} (fg)(a)] \right| \leq \frac{1}{\Gamma(\alpha)} \\ & \quad \cdot \int_0^1 \left| \int_a^{a^{1-u}b^u} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right. \\ & \quad \left. - \int_{a^{1-u}b^u}^b \left(\ln \frac{s}{a} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| |f'(a^{1-u}b^u)| a^{1-u}b^u \\ & \quad \cdot \ln \left(\frac{b}{a} \right) du \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \left(\|g\|_{\infty} \right. \\ & \quad \cdot \frac{(\ln(b/a))^{\alpha}}{\alpha} [(1-u)^{\alpha} - u^{\alpha}] \\ & \quad \cdot (\sup \{|f'(a)|, |f'(b)|\}) a^{1-u}b^u \ln \left(\frac{b}{a} \right) du \\ & \quad + \frac{1}{\Gamma(\alpha)} \int_{1/2}^1 \left(\|g\|_{\infty} \frac{(\ln(b/a))^{\alpha}}{\alpha} [u^{\alpha} - (1-u)^{\alpha}] \right) \\ & \quad \cdot (\sup \{|f'(a)|, |f'(b)|\}) a^{1-u}b^u \ln \left(\frac{b}{a} \right) du \\ & \leq \frac{\ln^{\alpha+1}(b/a) \|g\|_{\infty}}{\Gamma(\alpha+1)} \left\{ \int_0^{1/2} [(1-u)^{\alpha} - u^{\alpha}] \right. \\ & \quad \cdot a^{1-u}b^u du + \int_{1/2}^1 [u^{\alpha} - (1-u)^{\alpha}] a^{1-u}b^u du \Big\} \\ & \quad \cdot \sup \{|f'(a)|, |f'(b)|\} = \frac{\ln^{\alpha+1}(b/a) \|g\|_{\infty}}{\Gamma(\alpha+1)} \end{aligned}$$

$$\begin{aligned} & \cdot \int_0^{1/2} [(1-u)^\alpha - u^\alpha] [a^{1-u}b^u + a^ub^{1-u}] du \\ & \cdot \sup \{|f'(a)|, |f'(b)|\}. \end{aligned} \quad (18)$$

This completes the proof. $\square$

Corollary 11. In Theorem 10, one can see the following.

(1) If one takes $\alpha = 1$, one has the following Hermite-Hadamard-Fejér type inequality for quasi-geometrically convex function which is related to the right-hand side of (5):

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x)g(x)}{x} dx \right| \\ & \leq \frac{\|g\|_\infty \ln^2(b/a)}{2} C_1(1) \sup \{|f'(a)|, |f'(b)|\}. \end{aligned} \quad (19)$$

(2) If one takes $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function in fractional integral forms which is related to the right-hand side of (7):

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha + 1)}{2(\ln(b/a))^\alpha} [J_{a+}^\alpha f(b) + J_{b-}^\alpha f(a)] \right| \\ & \leq \frac{\ln(b/a)}{2} C_1(\alpha) \sup \{|f'(a)|, |f'(b)|\}. \end{aligned} \quad (20)$$

(3) If one takes $\alpha = 1$ and $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln(b/a)} \int_a^b \frac{f(x)}{x} dx \right| \\ & \leq \frac{\ln(b/a)}{2} C_1(1) \sup \{|f'(a)|, |f'(b)|\}. \end{aligned} \quad (21)$$

Theorem 12. Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$. If $|f'|^q$, $q \geq 1$, is quasi-geometrically convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:

$$\begin{aligned} & \left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] \right. \\ & \quad \left. - [J_{a+}^\alpha (fg)(b) + J_{b-}^\alpha (fg)(a)] \right| \\ & \leq \frac{\|g\|_\infty \ln^{\alpha+1}(b/a)}{\Gamma(\alpha + 1)} \left[\left(1 - \frac{1}{2^\alpha}\right) \left(\frac{2}{\alpha + 1}\right) \right]^{1-1/q} \\ & \quad \cdot \left[C_2^{1/q}(\alpha) [\sup \{|f'(a)|^q, |f'(b)|^q\}]^{1/q} \right], \end{aligned} \quad (22)$$

where

$$\begin{aligned} C_2(\alpha) = & \left[\int_0^{1/2} ((1-u)^\alpha - u^\alpha) \right. \\ & \cdot \left. \left[(a^{1-u}b^u)^q + (a^ub^{1-u})^q \right] du \right], \end{aligned} \quad (23)$$

with $\alpha > 0$.

Proof. Similar to the proof of Theorem 10, using Lemma 8, (14), (16), power mean inequality, and quasi-geometrically convexity of $|f'|^q$, we have

$$\begin{aligned} & \left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^\alpha g(b) + J_{b-}^\alpha g(a)] - [J_{a+}^\alpha (fg)(b) \right. \\ & \quad \left. + J_{b-}^\alpha (fg)(a)] \right| \leq \frac{1}{\Gamma(\alpha)} \int_0^1 \left| \int_{a^{1-u}b^u}^{a^ub^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| \\ & \quad \cdot |f'(a^{1-u}b^u)| a^{1-u}b^u \ln \left(\frac{b}{a} \right) du \\ & \leq \frac{\ln(b/a)}{\Gamma(\alpha)} \left[\int_0^1 \left| \int_{a^{1-u}b^u}^{a^ub^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| du \right]^{1-1/q} \\ & \quad \cdot \left[\int_0^1 \left| \int_{a^{1-u}b^u}^{a^ub^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| |f'(a^{1-u}b^u)|^q \right. \\ & \quad \cdot \left. (a^{1-u}b^u)^q du \right]^{1/q} \\ & \leq \frac{\|g\|_\infty^{1-1/q} \ln^{\alpha(1-1/q)+1}(b/a)}{\alpha^{1-1/q} \Gamma(\alpha)} \left[\int_0^{1/2} [(1-u)^\alpha - u^\alpha] du \right. \\ & \quad \left. + \int_{1/2}^1 [u^\alpha - (1-u)^\alpha] du \right]^{1-1/q} \\ & \quad \cdot \left[\int_0^1 \left| \int_{a^{1-u}b^u}^{a^ub^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} \frac{ds}{s} \right| \sup \{|f'(a)|^q, |f'(b)|^q\} \right. \\ & \quad \cdot \left. (a^{1-u}b^u)^q du \right]^{1/q} \\ & = \frac{\|g\|_\infty^{1-1/q} \ln^{\alpha(1-1/q)+1}(b/a)}{\alpha^{1-1/q} \Gamma(\alpha)} \left[\left(1 - \frac{1}{2^\alpha}\right) \left(\frac{2}{\alpha + 1}\right) \right]^{1-1/q} \\ & \quad \cdot [\sup \{|f'(a)|^q, |f'(b)|^q\}]^{1/q} \\ & \quad \cdot \left[\int_0^1 \left| \int_{a^{1-u}b^u}^{a^ub^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} \frac{ds}{s} \right| (a^{1-u}b^u)^q du \right]^{1/q} \\ & = \frac{\|g\|_\infty \ln^{\alpha+1}(b/a)}{\Gamma(\alpha + 1)} \left[\left(1 - \frac{1}{2^\alpha}\right) \left(\frac{2}{\alpha + 1}\right) \right]^{1-1/q} \\ & \quad \cdot [\sup \{|f'(a)|^q, |f'(b)|^q\}]^{1/q} \left[\int_0^1 |(1-u)^\alpha - u^\alpha| \right. \\ & \quad \cdot \left. (a^{1-u}b^u)^q du \right]^{1/q} = \frac{\|g\|_\infty \ln^{\alpha+1}(b/a)}{\Gamma(\alpha + 1)} \left[\left(1 - \frac{1}{2^\alpha}\right) \left(\frac{2}{\alpha + 1}\right) \right]^{1-1/q} \\ & \quad \cdot [\sup \{|f'(a)|^q, |f'(b)|^q\}]^{1/q} \left[\int_0^{1/2} ((1-u)^\alpha - u^\alpha) (a^{1-u}b^u)^q du \right. \\ & \quad \left. + \int_{1/2}^1 (u^\alpha - (1-u)^\alpha) (a^{1-u}b^u)^q du \right]^{1/q} \end{aligned}$$

$$\begin{aligned}
&= \frac{\|g\|_{\infty} \ln^{\alpha+1}(b/a)}{\Gamma(\alpha+1)} \left[\left(1 - \frac{1}{2^{\alpha}}\right) \left(\frac{2}{\alpha+1}\right) \right]^{1-1/q} \\
&\cdot \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q} \left[\int_0^{1/2} ((1-u)^{\alpha} - u^{\alpha}) \right. \\
&\cdot \left. \left[(a^{1-u}b^u)^q + (a^ub^{1-u})^q \right] du \right]^{1/q}.
\end{aligned} \tag{24}$$

This completes the proof. $\square$

Corollary 13. In Theorem 12, one has the following.

(1) If one takes $\alpha = 1$, one has the following Hermite-Hadamard-Fejér type inequality for quasi-geometrically convex function which is related to the right-hand side of (5):

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x)g(x)}{x} dx \right| \\
&\leq \|g\|_{\infty} \ln^2\left(\frac{b}{a}\right) \left(\frac{1}{2}\right)^{2-1/q} \\
&\cdot \left[C_2^{1/q}(1) \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q} \right].
\end{aligned} \tag{25}$$

(2) If one takes $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function in fractional integral forms which is related to the right-hand side of (7):

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \right| \\
&\leq \frac{\ln(b/a)}{2} \left[\left(1 - \frac{1}{2^{\alpha}}\right) \left(\frac{2}{\alpha+1}\right) \right]^{1-1/q} \\
&\cdot \left[C_2^{1/q}(\alpha) \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q} \right].
\end{aligned} \tag{26}$$

(3) If one takes $\alpha = 1$ and $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function:

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln(b/a)} \int_a^b \frac{f(x)}{x} dx \right| \leq \ln\left(\frac{b}{a}\right) \\
&\cdot \left(\frac{1}{2}\right)^{2-1/q} \\
&\cdot \left[C_2^{1/q}(1) \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q} \right].
\end{aligned} \tag{27}$$

Theorem 14. Let $f : I \subseteq \mathbb{R}_+ \rightarrow \mathbb{R}$ be a differentiable mapping on I° and $f' \in L[a, b]$ with $a < b$. If $|f'|^q$, $q > 1$, is quasi-geometrically convex on $[a, b]$ and $g : [a, b] \rightarrow \mathbb{R}$ is continuous and geometrically symmetric with respect to $\sqrt{ab}$, then the following inequalities for fractional integrals hold:

$$\begin{aligned}
&(i) \\
&\left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] \right. \\
&\quad \left. - [J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a)] \right| \\
&\leq \frac{\|g\|_{\infty} \ln^{\alpha+1-1/q}(b/a)}{q^{1/q} \Gamma(\alpha+1)} \left[\frac{2}{\alpha p + 1} \left(1 - \frac{1}{2^{\alpha p}}\right) \right]^{1/p} \\
&\cdot (b^q - a^q)^{1/q} \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}
\end{aligned} \tag{28}$$

with $\alpha > 0$;

(ii)

$$\begin{aligned}
&\left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] \right. \\
&\quad \left. - [J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a)] \right| \\
&\leq \frac{\|g\|_{\infty} \ln^{\alpha+1-1/q}(b/a)}{q^{1/q} \Gamma(\alpha+1)} \left[\frac{1}{\alpha p + 1} \right]^{1/p} (b^q - a^q)^{1/q} \\
&\cdot \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}
\end{aligned} \tag{29}$$

for $0 < \alpha \leq 1$, where $1/p + 1/q = 1$.

Proof. (i) Using Lemma 8, (14), (16), Hölder's inequality, and quasi-geometrically convexity of $|f'|^q$, we have

$$\begin{aligned}
&\left| \left(\frac{f(a) + f(b)}{2} \right) [J_{a+}^{\alpha} g(b) + J_{b-}^{\alpha} g(a)] \right. \\
&\quad \left. - [J_{a+}^{\alpha} (fg)(b) + J_{b-}^{\alpha} (fg)(a)] \right| \leq \frac{1}{\Gamma(\alpha)} \\
&\cdot \int_0^1 \left| \int_{a^{1-u}b^u}^{a^ub^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right| |f'(a^{1-u}b^u)| a^{1-u}b^u \\
&\cdot \ln\left(\frac{b}{a}\right) du \\
&\leq \frac{\ln(b/a)}{\Gamma(\alpha)} \left[\int_0^1 \left| \int_{a^{1-u}b^u}^{a^ub^{1-u}} \left(\ln \frac{b}{s} \right)^{\alpha-1} g(s) \frac{ds}{s} \right|^p du \right]^{1/p} \\
&\cdot \left[\int_0^1 |f'(a^{1-u}b^u)|^q (a^{1-u}b^u)^q du \right]^{1/q} \\
&\leq \frac{\|g\|_{\infty} \ln^{\alpha+1}(b/a)}{\Gamma(\alpha+1)} \left[\int_0^{1/2} [(1-u)^{\alpha} - u^{\alpha}]^p du \right. \\
&\quad \left. + \int_{1/2}^1 [u^{\alpha} - (1-u)^{\alpha}]^p du \right]^{1/p} \\
&\cdot \left[\int_0^1 \sup \{ |f'(a)|^q, |f'(b)|^q \} (a^{1-u}b^u)^q du \right]^{1/q}
\end{aligned}$$

$$\begin{aligned}
&= \frac{\|g\|_{\infty} \ln^{\alpha+1-1/q}(b/a)}{q^{1/q} \Gamma(\alpha+1)} \left[\int_0^{1/2} [(1-u)^{\alpha} - u^{\alpha}]^p du \right. \\
&\quad \left. + \int_{1/2}^1 [u^{\alpha} - (1-u)^{\alpha}]^p du \right]^{1/p} (b^q - a^q)^{1/q} \\
&\quad \cdot \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q} \\
&\leq \frac{\|g\|_{\infty} \ln^{\alpha+1-1/q}(b/a)}{q^{1/q} \Gamma(\alpha+1)} \left[\int_0^{1/2} (1-u)^{\alpha p} - u^{\alpha p} du \right. \\
&\quad \left. + \int_{1/2}^1 u^{\alpha p} - (1-u)^{\alpha p} du \right]^{1/p} (b^q - a^q)^{1/q} \\
&\quad \cdot \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q} \\
&= \frac{\|g\|_{\infty} \ln^{\alpha+1-1/q}(b/a)}{q^{1/q} \Gamma(\alpha+1)} \left[\frac{2}{\alpha p + 1} \left(1 - \frac{1}{2^{\alpha p}} \right) \right]^{1/p} (b^q \\
&\quad - a^q)^{1/q} \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}.
\end{aligned} \tag{30}$$

Here, we use

$$[(1-t)^{\alpha} - t^{\alpha}]^p \leq (1-t)^{\alpha p} - t^{\alpha p} \tag{31}$$

for $t \in [0, 1/2]$ and

$$[t^{\alpha} - (1-t)^{\alpha}]^p \leq t^{\alpha p} - (1-t)^{\alpha p} \tag{32}$$

for $t \in [1/2, 1]$, which follows from

$$(A - B)^q \leq A^q - B^q, \tag{33}$$

for any $A \geq B \geq 0$ and $q \geq 1$. Hence, inequality (28) is proved.

(ii) Inequality (29) is easily proved using (30) and Lemma 9. $\square$

Corollary 15. In Theorem 14, one can see the following.

(i) In (28), consider the following:

- (1) If one takes $\alpha = 1$, one has the following Hermite-Hadamard-Fejér type inequality for quasi-geometrically convex function which is related to the right-hand side of (5):

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \right| \\
&\leq \frac{\|g\|_{\infty} \ln^{2-1/q}(b/a)}{2q^{1/q}} \left[\frac{2}{p+1} \left(1 - \frac{1}{2^p} \right) \right]^{1/p} \\
&\quad \cdot (b^q - a^q)^{1/q} \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}.
\end{aligned} \tag{34}$$

- (2) If one takes $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function in fractional integral forms which is related to the right-hand side of (7):

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \right| \\
&\leq \frac{\ln^{1-1/q}(b/a)}{2q^{1/q}} \left[\frac{2}{\alpha p + 1} \left(1 - \frac{1}{2^{\alpha p}} \right) \right]^{1/p} (b^q \\
&\quad - a^q)^{1/q} \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}.
\end{aligned} \tag{35}$$

- (3) If one takes $\alpha = 1$ and $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function:

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln(b/a)} \int_a^b \frac{f(x)}{x} dx \right| \\
&\leq \frac{\ln^{1-1/q}(b/a)}{2q^{1/q}} \left[\frac{2}{p+1} \left(1 - \frac{1}{2^p} \right) \right]^{1/p} (b^q - a^q)^{1/q} \\
&\quad \cdot \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}.
\end{aligned} \tag{36}$$

(ii) In (29), consider the following:

- (1) If one takes $\alpha = 1$, one has the following Hermite-Hadamard-Fejér type inequality for quasi-geometrically convex function which is related to the right-hand side of (5):

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} \int_a^b \frac{g(x)}{x} dx - \int_a^b \frac{f(x) g(x)}{x} dx \right| \\
&\leq \frac{\|g\|_{\infty} \ln^{2-1/q}(b/a)}{2q^{1/q}} \left[\frac{1}{p+1} \right]^{1/p} (b^q - a^q)^{1/q} \\
&\quad \cdot \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}.
\end{aligned} \tag{37}$$

- (2) If one takes $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function in fractional integral forms which is related to the right-hand side of (7):

$$\begin{aligned}
&\left| \frac{f(a) + f(b)}{2} - \frac{\Gamma(\alpha+1)}{2(\ln(b/a))^{\alpha}} [J_{a+}^{\alpha} f(b) + J_{b-}^{\alpha} f(a)] \right| \\
&\leq \frac{\ln^{1-1/q}(b/a)}{2q^{1/q}} \left[\frac{1}{\alpha p + 1} \right]^{1/p} (b^q - a^q)^{1/q} \\
&\quad \cdot \left[\sup \{ |f'(a)|^q, |f'(b)|^q \} \right]^{1/q}.
\end{aligned} \tag{38}$$

- (3) If one takes $\alpha = 1$ and $g(x) = 1$, one has the following Hermite-Hadamard type inequality for quasi-geometrically convex function:

$$\begin{aligned} & \left| \frac{f(a) + f(b)}{2} - \frac{1}{\ln(b/a)} \int_a^b \frac{f(x)}{x} dx \right| \\ & \leq \frac{\ln^{1-1/q}(b/a)}{2q^{1/q}} \left[\frac{1}{p+1} \right]^{1/p} (b^q - a^q)^{1/q} \quad (39) \\ & \cdot \left[\sup \left\{ |f'(a)|^q, |f'(b)|^q \right\} \right]^{1/q}. \end{aligned}$$

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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