Hindawi
Journal of Mathematics
Volume 2017, Article ID 3070236, 6 pages
https://doi.org/10.1155/2017/3070236

Research Article

Inverse Commutativity Conditions for Second-Order Linear Time-Varying Systems

Mehmet Emir Koksal

Department of Mathematics, Ondokuz Mayis University, Atakum, 55139 Samsun, Turkey

Correspondence should be addressed to Mehmet Emir Koksal; emir_koksal@hotmail.com

Received 14 May 2017; Revised 24 June 2017; Accepted 27 June 2017; Published 27 August 2017

Academic Editor: Willi Freeden

The necessary and sufficient conditions where a second-order linear time-varying system A is commutative with another system B of the same type have been given in the literature for both zero initial states and nonzero initial states. These conditions are mainly expressed in terms of the coefficients of the differential equation describing system A. In this contribution, the inverse conditions expressed in terms of the coefficients of the differential equation describing system B have been derived and shown to be of the same form of the original equations appearing in the literature.

1. Introduction

As one of the main fields of applied mathematics, differential equations arise in acoustics, electromagnetics, electrodynamics, fluid dynamics, wave motion, wave distribution, and many other sciences and branches of engineering. There is a tremendous amount of work on the theory and techniques for solving differential equations and on their applications [1–4]. Particularly, they are used as a powerful tool for modelling, analyzing, and solving real engineering problems and for discussing the results turned up at the end of analyzing for resolution of naturel problems. For example, they are used in system and control theory, which is an interdisciplinary branch of electric-electronics engineering and applied mathematics that deal with the behavior of dynamical systems with inputs and how their behavior is modified by different combinations such as cascade and feedback connections [5–8]. When the cascade connection in system design is considered, the commutativity concept places an important role to improve different system performances [9–11].

As shown in Figure 1, when two linear time-varying systems A and B described by linear time-varying differential equations are connected in cascade so that the output of one appears as the input of the other, it is said that systems A and B are connected in cascade. If the order of connection does not affect the input-output relation of the combined system AB or BA, it is said that systems A and B are commutative.

Although the first paper about the commutativity appeared in 1977 [12] which had introduced the commutativity concept for the first time and studied the commutativity of the first-order continuous-time linear time-varying systems, this paper is important for proving that a time-varying system can be commutative with another time-varying system only; further very few classes of systems can be commutative. In particular, commutativity conditions for relaxed second-order systems first appeared in 1982 [13]. In 1984 [14] and 1985 [15], commutativity conditions for third- and fourth-order continuous-time linear time-varying systems were studied, respectively. The content of the published but undistributed work [15] can be found in journal paper [16] which presents an exhaustive study on the commutativity of continuous-time linear time-varying systems. That paper is the first tutorial paper in the literature.

During the period from 1989 to 2011, no publication about commutativity had appeared in the literature. In 2011, the second basic journal publication [17] appeared. In this paper, commutativity in case of nonzero initial conditions, commutativity of Euler's systems, new results about effects of commutativity, reduction of disturbance by change of connection order in a chain structure of subsystems, and the

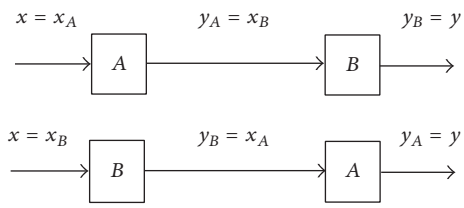

Figure 1: Cascade connection of the differential systems A and B.

most important explicit commutativity conditions for fifth-order systems were studied.

This work is directly focused on the commutativity of second-order linear time-varying systems with zero initial conditions. Section 2 summarizes the necessary and sufficient conditions of commutativity of such systems as appearing in the literature and then presents the inverse conditions. Section 3 covers an example and its simulation results. Finally, Section 4 includes conclusions.

2. Inverse Conditions of Commutativity

Let A be a second-order linear time-varying system described by

$$A:$$

$$\begin{aligned} a_2(t)\,\ddot{y}_A(t) + a_1(t)\,\dot{y}_A(t) + a_0(t)\,y_A(t) &= x_A(t); \quad \forall t \geq t_0, \\ y_A(t_0) &= 0, \\ \dot{y}_A(t_0) &= 0, \end{aligned} \tag{1a}$$

where $x_A(\cdot)$ is the input; $y_A(\cdot)$ is the output functions of the system; $a_i(t)$'s, $i = 2, 1, 0$, are the time-varying coefficients. They are all defined for $[t_0, \infty) \to R$. The (double) dot on the top represents the (second) first-order derivative with respect to time $t \in R$, where $t \geq t_0$, t_0 being the initial time. Since A is of second order, $a_2(t) \not\equiv 0$; further, for the unique solvability of (1a) for the output $y_A(\cdot)$, it is sufficient that $x_A(\cdot)$, $a_i(\cdot) \in P[t_0, \infty)$, which is the set of piecewise continuous mappings $[t_0, \infty) \to R$ [2].

Let B be another second-order linear time-varying system defined by

$$B:$$

$$\begin{aligned} b_2(t)\,\ddot{y}_B(t) + b_1(t)\,\dot{y}_B(t) + b_0(t)\,y_B(t) &= x_B(t); \quad \forall t \geq t_0, \\ y_B(t_0) &= 0, \\ \dot{y}_B(t_0) &= 0, \end{aligned} \tag{1b}$$

where $x_B(\cdot)$, $y_B(\cdot)$, $b_i(t)$'s are defined in a similar manner as for system A and $b_2(t) \not\equiv 0$.

For the commutativity of systems A and B, it is necessary and sufficient that the coefficients of B must be expressible in terms of those of A by the matrix equation

$$\begin{bmatrix} b_2(t) \\ b_1(t) \\ b_0(t) \end{bmatrix} = \begin{bmatrix} a_2(t) & 0 & 0 \\ a_1(t) & a_2^{0,5}(t) & 0 \\ a_0(t) & a_2^{-0,5}(t)\,\dfrac{2a_1(t) - \dot{a}_2(t)}{4} & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}, \tag{2a}$$

where $c_2 \neq 0, c_1, c_0$ are arbitrary constants; further, the coefficients of A must satisfy the following equation for the general values of $t \geq t_0$:

$$-a_2^{0.5}\frac{d}{dt}\left[a_0 - \frac{4a_1^2 + 3\dot{a}_2^2 - 8a_1\dot{a}_2 + 8\dot{a}_1 a_2 - 4a_2\ddot{a}_2}{16a_2}\right]c_1 \equiv 0. \tag{2b}$$

Defining

$$f_A(t) = a_2^{-0.5}(t)\,\frac{2a_1(t) - \dot{a}_2(t)}{4}, \tag{3}$$

it can be shown by routine mathematical operations that the necessary and sufficient conditions in (2a) and (2b) can be rewritten as

$$\begin{bmatrix} b_2(t) \\ b_1(t) \\ b_0(t) \end{bmatrix} = \begin{bmatrix} a_2(t) & 0 & 0 \\ a_1(t) & a_2^{0.5}(t) & 0 \\ a_0(t) & f_A(t) & 1 \end{bmatrix} \begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix}, \tag{4a}$$

$$-a_2^{0.5}\frac{d}{dt}\left[A_0(t)\right]c_1 \equiv 0, \tag{4b}$$

where

$$A_0(t) = a_0(t) - f_A^2(t) - a_2^{0.5}(t)\,\dot{f}_A(t). \tag{4c}$$

If $c_1 = 0$, (2b) and (4b) are automatically satisfied and hence they are redundant. But if $c_1 \neq 0$, these equations, together with the information $a_2(t) \not\equiv 0$, are replaced by (4c) with $A_0(t)$ being constant, that is, time invariant.

It is naturally expected that if A and B are commutative, similar equations to (4a), (4b), and (4c) are valid when the coefficients of B are used. In fact, the first equation in (4a) is equivalent to

$$a_2(t) = \left(\frac{1}{c_2}\right)b_2(t). \tag{5a}$$

Using this equation in the second line of equation in (4a) and solving it for $a_1(t)$, we obtain

$$a_1(t) = \left(\frac{1}{c_2}\right)b_1(t) - \left(\frac{c_1}{c_2^{1.5}}\right)b_2^{0.5}(t). \tag{5b}$$

Using (5a) and (5b) in (3), we express $f_A(t)$ as

$$\begin{aligned} f_A(t) &= \left(\frac{b_2}{c_2}\right)^{-0,5} b\frac{(2/c_2)\,b_1 - \left(2c_1/c_2^{1.5}\right)b_2^{0.5} - (1/c_2)\,\dot{b}_2}{4} \\ &= \left(\frac{b_2}{c_2}\right)^{-0,5} \frac{\left(2b_1 - \dot{b}_2\right)/c_2 - \left(2c_1/c_2^{1.5}\right)b_2^{0.5}}{4} \\ &= \frac{1}{c_2^{0.5}}\frac{b_2^{-0.5}\left(2b_1 - \dot{b}_2\right)}{4} - \frac{c_1}{2c_2} = \frac{1}{c_2^{0.5}} f_B(t) - \frac{c_1}{2c_2}, \end{aligned} \tag{5c}$$

where

$$f_B(t) = b_2^{-0.5}(t)\,\frac{2b_1(t) - \dot{b}_2(t)}{4}. \tag{6}$$

Finally, substituting (5c) in the third line of equation of (4a) and solving it for $a_0(t)$, we obtain

$$\begin{aligned} a_0(t) &= \frac{1}{c_2}\left[b_0(t) - \frac{c_1}{c_2^{0.5}} f_B(t) + \frac{c_1^2}{2c_2} - c_0\right] \\ &= \left(\frac{1}{c_2}\right) b_0(t) + \left(-\frac{c_1}{c_2^{1.5}}\right) f_B(t) + \frac{c_1^2 - 2c_2c_0}{2c_2^2}. \end{aligned} \tag{7}$$

Hence, writing (5a), (5b), and (7) in matrix form and letting

$$\begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{c_2} \\ -\dfrac{c_1}{c_2^{1.5}} \\ \dfrac{c_1^2 - 2c_2c_0}{2c_2^2} \end{bmatrix} \tag{8a}$$

or equivalently (note $c_2 \neq 0$ so $k_2 \neq 0$)

$$\begin{bmatrix} c_2 \\ c_1 \\ c_0 \end{bmatrix} = \begin{bmatrix} \dfrac{1}{k_2} \\ -\dfrac{k_1}{k_2^{1.5}} \\ \dfrac{k_1^2 - 2k_2k_0}{2k_2^2} \end{bmatrix}, \tag{8b}$$

we obtain the first set of the inverse equations derived as

$$\begin{bmatrix} a_2(t) \\ a_1(t) \\ a_0(t) \end{bmatrix} = \begin{bmatrix} b_2(t) & 0 & 0 \\ b_1(t) & b_2^{0.5}(t) & 0 \\ b_0(t) & f_B(t) & 1 \end{bmatrix} \begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix}. \tag{9a}$$

To find the second inverse equation, we substitute $a_0(t)$ from (7), $f_A(t)$ from (5c), and $a_2(t)$ from (5a), all into (4c); and rearranging, we obtain

$$\begin{aligned} A_0(t) &= \frac{b_0}{c_2} - \frac{c_1}{c_2^{1.5}} f_B + \frac{c_1^2 - 2c_2c_0}{2c_2^2} - \left(\frac{f_B}{c_2^{0.5}} - \frac{c_1}{2c_2}\right)^2 \\ &\quad - \frac{b_2^{0.5}}{c_2^{0.5}}\left(\frac{\dot{f}_B}{c_2^{0.5}}\right) \\ &= \frac{1}{c_2}\left[b_0(t) - f_B^2(t) - b_2^{0.5}(t)\,\dot{f}_B(t)\right] \\ &\quad + \frac{c_1^2 - 4c_2c_0}{4c_2^2}. \end{aligned} \tag{$*$}$$

Using this in (4b), eliminating a_2 by (5a) and c_2, c_1 by (8b), and finally multiplying by -1, we proceed with

$$\begin{aligned} -a_2^{0.5}\frac{d}{dt}\left[\frac{1}{c_2}\left(b_0 - f_B^2 - b_2^{0.5}\dot{f}_B\right) + \frac{c_1^2 - 4c_2c_0}{4c_2^2}\right] c_1 &= 0, \\ -\frac{b_2^{0.5}}{c_2^{0.5}}\frac{d}{dt}\left[\frac{1}{c_2}\left(b_0 - f_B^2 - b_2^{0.5}\dot{f}_B\right)\right] c_1 &= 0, \\ b_2^{0.5}\frac{d}{dt}\left[\left(b_0 - f_B^2 - b_2^{0.5}\dot{f}_B\right)\right]\left(\frac{-c_1}{c_2^{1.5}}\right) &= 0, \\ -b_2^{0.5}\frac{d}{dt}\left[B_0(t)\right] k_1 &\equiv 0, \end{aligned} \tag{9b}$$

where

$$B_0(t) = b_0(t) - f_B^2(t) - b_2^{0.5}(t)\,\dot{f}_B(t). \tag{9c}$$

Hence, (9a) and (9b) constitute the inverse necessary and sufficient conditions in terms of the coefficients of subsystem B. These equations are the duals of (4a) and (4b), respectively; similarly, (9c) is the dual of (4c).

Finally, using (9c) in the above obtained intermediate equation for $A_0(t)$, we write

$$A_0(t) = \frac{1}{c_2}B_0(t) + \frac{c_1^2 - 4c_2c_0}{4c_2^2}. \tag{10a}$$

Solving it for $B_0(t)$, we obtain

$$B_0(t) = c_2A_0(t) + c_0 - \frac{c_1^2}{4c_2}. \tag{10b}$$

Using (8a) and (8b), we obtain the duals of (10a) and (10b) as

$$B_0(t) = \frac{1}{k_2}A_0(t) + \frac{k_1^2 - 4k_2k_0}{4k_2^2}, \tag{11a}$$

$$A_0(t) = k_2B_0(t) + k_0 - \frac{k_1^2}{4k_2}, \tag{11b}$$

respectively.

We remark that $f_A(t)$ in (3) and $f_B(t)$ in (6) are similarly defined so they constitute also dual equations. We express the results we obtained by a theorem.

Theorem 1. *Let A and B be two second-order linear time-varying systems described by (1a) and (1b), respectively.*

The necessary and sufficient conditions where subsystem B is commutative with subsystem A are as follows:

(i) *The coefficients of B are expressed in terms of the coefficients of A as in (4a) where $c_2 \neq 0$, c_1, c_0 are some constants.*

(ii) *Further, the coefficients of A and c_1 satisfy (4b).*

Conversely, the necessary and sufficient conditions where subsystem A is commutative with subsystem B are as follows:

(iii) *The coefficients of A are expressed in terms of the coefficients of B as in (9a) where $k_2 \neq 0$, k_1, k_0 are some constants.*

(iv) *Further, the coefficients of B and k_1 satisfy (9b).*

Conditions (i) and (ii) together are equivalent to conditions (iii) and (iv) together. There exists a unique relation between the constants c_2, c_1, c_0 and k_2, k_1, k_0 which are expressed by the dual equations (8a) and (8b).

If $c_1 \neq 0$ which is equivalent to $k_1 \neq 0$ due to (8a) and (8b), the second and fourth conditions above are replaced, respectively, by the following:

(v) *$A_0(t)$ defined in (4c) is independent of time and equal to a constant.*

(vi) *$B_0(t)$ defined in (9c) is independent of time and equal to a constant.*

Conditions (i) and (v) together are equivalent to conditions (iii) and (vi) together. Further (v) and (vi) are equivalent conditions due to (10a), (10b), (11a), and (11b).

3. Example

Let the coefficients of A be $a_2 = t^4$ and $a_1 = t^3$. Then, by (3), $f_A(t) = -t/2$; hence $\dot{f}_A(t) = -1/2$. Substituting these in (4c) and choosing $A_0 = -1$, we find $a_0(t) = -1 - t^2/4$. Hence, A is described by

$$A\colon t^4 \ddot{y}_A(t) + t^3 \dot{y}_A(t) - \left(1 + \frac{t^2}{4}\right) y_A(t) = x_A(t). \tag{12a}$$

Since $a_0(t)$ is chosen so that (4c) is satisfied with a constant A_0, the above system has second-order pairs computed by (4a) where c_1 can be chosen different from zero due to condition (v) of Theorem 1. In fact, choosing $c_2 = 4$, $c_1 = -3$, and $c_0 = 1$, the following commutative pair B of A is obtained:

$$\begin{bmatrix} b_2 \\ b_1 \\ b_0 \end{bmatrix} = \begin{bmatrix} t^4 & 0 & 0 \\ t^3 & t^2 & 0 \\ -\left(1 + \frac{t^2}{4}\right) & -\frac{t}{2} & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4t^4 \\ 4t^3 - 3t^2 \\ -t^2 + 1.5t - 3 \end{bmatrix},$$

$$B\colon 4t^4 \ddot{y}_B(t) + \left(4t^3 - 3t^2\right) \dot{y}_B(t) - \left(t^2 - 1.5t + 3\right) y_B(t) = x_B(t). \tag{12b}$$

Using (6)

$$f_B(t) = \frac{1}{2t^2} \frac{2\left(4t^3 - 3t^2\right) - 16t^3}{4} = -t - 0.75, \tag{13}$$

where $\dot{f}_B(t) = -1$. Evaluating (9c), we find

$$B_0 = -t^2 + 1{,}5t - 3 - (-t - 0{,}75)^2 - 2t^2(-1) = -\frac{57}{16}. \tag{14}$$

Or directly from (10b)

$$B_0 = -1(4) + 1 - \frac{9}{16} = -3 - \frac{9}{16} = -\frac{57}{16}, \tag{15}$$

which is the same result in (14). The constants k_2, k_1, k_0 are found from (8a) as

$$\begin{bmatrix} k_2 \\ k_1 \\ k_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ -\frac{(-3)}{8} \\ \frac{[9 - 2(4)]}{2(16)} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{1}{32} \end{bmatrix}. \tag{16}$$

So that the coefficients of A can be inversely computed by using (9a):

$$\begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = \begin{bmatrix} 4t^4 & 0 & 0 \\ 4t^3 - 3t^2 & 2t^2 & 0 \\ -t^2 + 15t - 3 & -t - 0.75 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{4} \\ \frac{3}{8} \\ \frac{1}{32} \end{bmatrix} = \begin{bmatrix} t^4 \\ t^3 - \frac{3}{4}t^2 + \frac{3}{4}t^2 \\ \frac{t^2}{4} + \frac{1{,}5t}{4} - \frac{3}{4} - \frac{3t}{8} - \frac{9}{32} + \frac{1}{32} \end{bmatrix} = \begin{bmatrix} t^4 \\ t^3 \\ -\frac{t^2}{4} - 1 \end{bmatrix}. \tag{17}$$

Simulation results for AB and BA for inputs $x_1(t) = -6.5 + 10\sin(8\pi t)$ and $x_2(t) = 2.5e^{-t}$ are shown for $0.5 \leq t \leq 4$ in Figure 2. For both inputs, AB and BA give the same response ($AB1 = BA1$ and $AB2 = BA2$), respectively. If the coefficient $a_0(t)$ of A is changed to $-1 - 1.25t^2$, then (4c) is not constant any more since $A_0(t) = -1 - t^2$; so, A will not have a second-order time-varying commutative pair unless $c_1 = 0$.

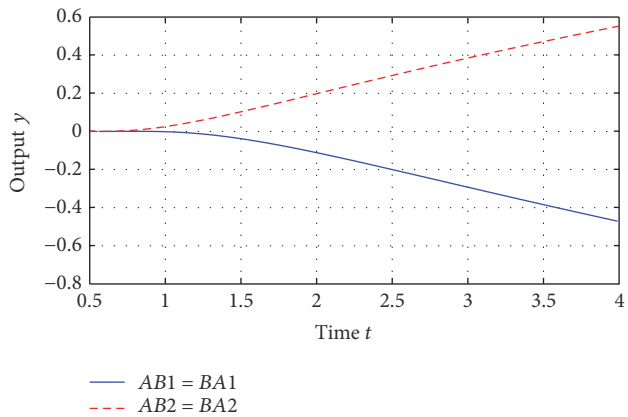

Figure 2: Responses of cascade connections AB and BA for sinusoidal ($AB1 = BA1$) and exponential ($AB2 = BA2$) inputs.

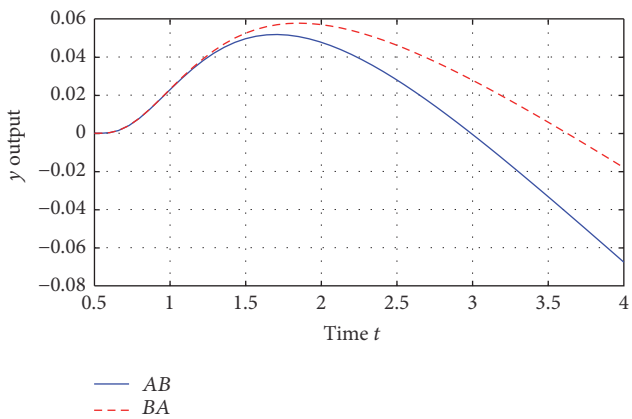

Figure 3: Responses of AB and BA for sinusoidal input when A and B do not satisfy commutativity conditions.

For $c_1 = -3 \neq 0$ and (2a) is spoiled between the coefficients of A and B both, A will not commute with B any more. This is verified by Figure 3 which is obtained by input $x_1(t) = -6.5 + 10\sin(8\pi t)$; it is seen obviously that the responses of AB and BA are not coincident at all. All simulations are performed with the initial and final times $t_0 = 0.5$ and $t = 4$, respectively, by Simulink program of MATLAB 2010a.

4. Conclusions

The commutativity conditions for a second-order linear time-varying system B commutative with another second-order linear time-varying system A is well known in the literature. In this paper, the inverse commutativity conditions are obtained. It is shown that the commutativity conditions for any two second-order linear time-varying systems A and B are in the same form whether they are expressed in terms of the coefficients of systems A or B. The results are illustrated by an example. Inverse commutativity conditions obtained are used in transitivity property of commutativity for time-varying systems, which is the subject of future work. Further, the problem of commutativity of switched systems [18], which are also linear time-varying systems, can be an interesting research subject which has not been studied before. Additionally, commutativity conditions could be studied for fractional order linear differential systems and even linear systems of some fractional difference equations [19–21].

Conflicts of Interest

The author declares that they have no conflicts of interest.

Acknowledgments

This study is supported by the Scientific and Technological Research Council of Turkey (TUBITAK) under the Project no. 115E952.

References

[1] G. M. Murphy, *Ordinary Differential Equations and Their Solutions*, D. Van Nostrand Co., Inc., London, UK, 1960.

[2] A. D. Polyanin and V. F. Zaitsev, *Handbook of Exact Solutions for Ordinary Differential Equations*, CRC Press, London, UK, 2nd edition, 2003.

[3] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, Academic Press, New York, NY, USA, 7th edition, 2007.

[4] W. G. Kelley and A. C. Peterson, *The Theory of Differential Equations: Classical and Qualitative*, Springer, 2nd edition, 2010.

[5] K. Ogata, *Modern Control Engineering*, Prentice Hall, New York, NY, USA, 5th edition, 2010.

[6] A. G. J. Holt and K. M. Reineck, "The transfer function synthesis for a cascade connection network," *IEEE Transactions on Circuit Theory*, vol. 15, no. 2, pp. 162–165, 1968.

[7] M. R. Ainbund and I. P. Maslenkov, "Improving the characteristics of microchannel plates in cascade connection," *Instrumens and Experimental Techniques*, vol. 26, no. 3, pp. 650–652, 1983.

[8] I. Gohberg, M. A. Kaashoek, and A. C. Ran, "Partial pole and zero displacement by cascade connection," *SIAM Journal on Matrix Analysis and Applications*, vol. 10, no. 3, pp. 316–325, 1989.

[9] M. Koksal, "Effects of nonzero initial conditions on commutativity and those of commutativity on system sensitivity," in *Proceeding of the 2nd National Congress of Electrical Engineers*, vol. 2/2, pp. 570–573, Ankara, Turkey, 1987.

[10] M. Koksal, "Effects of nonzero initial conditions on the commutativity of linear time-varying systems," in *Proceedings of the International Conference on Modeling and Simulation*, vol. 1A, pp. 49–55, Istanbul, Turkey, June 1988.

[11] M. Koksal, "Effects of commutativity on system's sensitivity," in *Proceeding of the 6th International Symposium on Networks, Systems and Signal Processing*, pp. 61-62, Zagreb, Yugoslavia, June 1989.

[12] J. E. Marshall, "Commutativity of time-varying systems," *Electronics Letters*, vol. 13, no. 18, pp. 539-540, 1977.

[13] M. Koksal, "Commutativity of second order time varying systems," *International Journal of Control*, vol. 36, no. 3, pp. 541–544, 1982.

[14] M. Koksal, "General conditions for commutativity of time-varying systems," in *Proceedings of the International Conference*

on Telecommunication and Control, pp. 223–225, Halkidiki, Greece, 1984.

[15] M. Koksal, "A Survey on the Commutativity of Time-Varying Systems," METU, GEEE CAS-85/1, Gaziantep Engineering Faculty, 1985.

[16] M. Koksal, "Exhaustive study on the commutativity of time-varying systems," *International Journal of Control*, vol. 47, no. 5, pp. 1521–1537, 1988.

[17] M. Koksal and M. E. Koksal, "Commutativity of linear time-varying differential systems with nonzero initial conditions: a review and some new extensions," *Mathematical Problems in Engineering*, vol. 2011, Article ID 678575, 25 pages, 2011.

[18] X. Zhao, S. Yin, H. Li, and B. Niu, "Switching stabilization for a class of slowly switched systems," *IEEE Transactions on Automatic Control*, vol. 60, pp. 221–226, 2015.

[19] I. K. Dassios, D. I. Baleanu, and G. I. Kalogeropoulos, "On non-homogeneous singular systems of fractional nabla difference equations," *Applied Mathematics and Computation*, vol. 227, pp. 112–131, 2014.

[20] I. K. Dassios, "Optimal solutions for mon-consistent singular linear systems of fractional nabla difference equations," *Circuits, Systems, and Signal Processing*, vol. 34, no. 6, pp. 1769–1797, 2015.

[21] I. K. Dassios, "Stability and robustness of singular systems of fractional nabla difference equations," *Circuits, Systems, and Signal Processing*, vol. 36, no. 1, pp. 49–64, 2017.