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*Research Article*

# Some Hermite-Hadamard-Fejér Type Integral Inequalities for Differentiable $\eta$-Convex Functions with Applications

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Using some results about generalized Hermite-Hadamard-Fejér type inequalities related to $\eta$-convex functions, we give some examples and applications for trapezoid and midpoint type inequalities for differentiable $\eta$-convex functions.

## 1. Introduction and Preliminaries

The celebrated Hermite-Hadamard-Fejér inequality (simply Fejér inequality) for convex functions has been proved in [1] as the following.

**Theorem 1.** *Let $f : [a,b] \to \mathbb{R}$ be a convex function. Then*

$$f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx \le \int_a^b f(x)\,g(x)\,dx \le \frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx, \tag{1}$$

*where $g : [a,b] \to \mathbb{R}^+ = [0,+\infty)$ is integrable and symmetric about $x = (a+b)/2$ ($g(x) = g(a+b-x)$, $\forall x \in [a,b]$).*

If in (1) we consider $g \equiv 1$, then we obtain Hermite-Hadamard inequality.

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f(x)\,dx \le \frac{f(a)+f(b)}{2}. \tag{2}$$

For various types of (1) and more results related to generalized Hermite-Hadamard-Fejér inequality, see [2–8] and references therein.

On the other hand, the concept of $\eta$-convex functions, firstly named by $\varphi$-convex functions, as generalization of convex functions has been introduced in [9].

Let $I$ be an interval in real line $\mathbb{R}$. Consider $\eta : A \times A \to B$ for appropriate $A, B \subseteq \mathbb{R}$.

*Definition 2* (see [9, 10]). A function $f : I \to \mathbb{R}$ is called convex with respect to $\eta$ (briefly $\eta$-convex), if

$$f(tx + (1-t)y) \le f(y) + t\eta(f(x), f(y)), \tag{3}$$

for all $x, y \in I$ and $t \in [0,1]$.

In fact, the above definition geometrically says that if a function is $\eta$-convex on $I$, then its graph between any $x, y \in I$ is on or under the path starting from $(y, f(y))$ and ending at $(x, f(y) + \eta(f(x), f(y)))$. If $f(x)$ should be the end point of the path for every $x, y \in I$, then we have $\eta(x,y) = x - y$ and the function reduces to a convex one. Note that by taking $x = y$ in (1) we get $t\eta(f(x), f(x)) \ge 0$ for any $x \in I$ and $t \in [0,1]$ which implies that

$$\eta(f(x), f(x)) \ge 0 \tag{4}$$

for any $x \in I$. Also if we take $t = 1$ in (1) we get

$$f(x) - f(y) \le \eta(f(x), f(y)) \tag{5}$$

for any $x, y \in I$.

There are simple examples about $\eta$-convexity of a function.

*Example 3* (see [9, 10]). (1) Consider a function $f : \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} -x, & x \geq 0; \\ x, & x < 0, \end{cases} \tag{6}$$

and define a bifunction $\eta$ as $\eta(x, y) = -x - y$, for all $x, y \in \mathbb{R}^- = (-\infty, 0]$. It is not hard to check that $f$ is a $\eta$-convex function but not a convex one.

(2) Define the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ as

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1; \\ 1, & x > 1, \end{cases} \tag{7}$$

and a bifunction $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\eta(x, y) = \begin{cases} x + y, & x \leq y; \\ 2(x + y), & x > y. \end{cases} \tag{8}$$

Then $f$ is $\eta$-convex but is not convex.

The following characterization of $\eta$-convexity holds [11].

**Theorem 4.** *A function $f : I \to \mathbb{R}$ is $\eta$-convex if and only if, for any $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$,*

$$\det \begin{pmatrix} 1 & x_1 & \eta(f(x_1), f(x_3)) \\ 1 & x_2 & f(x_2) - f(x_3) \\ 1 & x_3 & 0 \end{pmatrix} \geq 0, \qquad f(x_1) \leq f(x_3) + \eta(f(x_1), f(x_3)). \tag{9}$$

The following result is of importance.

**Theorem 5** (see [9, 11]). *Suppose that $f : I \to \mathbb{R}$ is a $\eta$-convex function and $\eta$ is bounded from above on $f(I) \times f(I)$. Then $f$ satisfies a Lipschitz condition on any closed interval $[a, b]$ contained in the interior $I^\circ$ of $I$. Hence, $f$ is absolutely continuous on $[a, b]$ and continuous on $I^\circ$.*

*Note.* As a consequence of Theorem 5, a $\eta$-convex function $f : [a, b] \to \mathbb{R}$ where $\eta$ is bounded from above on $f([a, b]) \times f([a, b])$ is integrable. For more results about $\eta$-convex functions, see [9–12].

Motivated by the above works, in this paper we give some applications and examples for trapezoidal and midpoint type inequalities when the intended function is differentiable. Furthermore we consider integral quadrature formula and give an error estimate related to trapezoidal and midpoint formula. Furthermore some examples support our results.

The following two theorems have been proved in [13] which improve the right part and the lefty side of (1), respectively.

**Theorem 6** (see [13]). *Let $f : [a, b] \to \mathbb{R}$ be a $\eta$-convex function which $\eta$ is bounded from above on $f([a, b]) \times f([a, b])$. If $g : [a, b] \to \mathbb{R}^+$ is integrable on $[a, b]$, then we have inequalities*

$$\begin{aligned} \frac{1}{2}\int_a^b [f(x) + f(a+b-x)]\, g(x)\, dx &\leq \min\Big\{ f(b) \\ &\quad + \frac{1}{2}\eta(f(a), f(b)), f(a) + \frac{1}{2}\eta(f(b), f(a)) \Big\} \\ &\quad \cdot \int_a^b g(x)\, dx \leq \frac{1}{2}[f(a) + f(b)] \int_a^b g(x)\, dx \\ &\quad + \frac{1}{4}[\eta(f(a), f(b)) + \eta(f(b), f(a))] \\ &\quad \cdot \int_a^b g(x)\, dx. \end{aligned} \tag{10}$$

*If $g$ is symmetric on $[a, b]$, then from inequality (10) we get*

$$\begin{aligned} \int_a^b f(x)\, g(x)\, dx &\leq \min\Big\{ f(b) \\ &\quad + \frac{1}{2}\eta(f(a), f(b)), f(a) + \frac{1}{2}\eta(f(b), f(a)) \Big\} \\ &\quad \cdot \int_a^b g(x)\, dx \leq \frac{1}{2}[f(a) + f(b)] \int_a^b g(x)\, dx \\ &\quad + \frac{1}{4}[\eta(f(a), f(b)) + \eta(f(b), f(a))] \\ &\quad \cdot \int_a^b g(x)\, dx. \end{aligned} \tag{11}$$

**Theorem 7** (see [13]). *Let $f : [a, b] \to \mathbb{R}$ be a $\eta$-convex function with $\eta$ bounded from above on $f([a, b]) \times f([a, b])$. If $g : [a, b] \to \mathbb{R}^+$ is integrable on $[a, b]$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right) \int_a^b g(x)\, dx &\leq \int_a^b g(x) \min\Big\{ f(a+b-x) \\ &\quad + \frac{1}{2}\eta(f(x), f(a+b-x)), f(x) \\ &\quad + \frac{1}{2}\eta(f(a+b-x), f(x)) \Big\}\, dx \\ &\leq \min\Big\{ \int_a^b g(x)\, f(a+b-x)\, dx + \frac{1}{2} \\ &\quad \cdot \int_a^b g(x)\, \eta(f(x), f(a+b-x))\, dx, \\ &\quad \int_a^b g(x)\, f(x)\, dx + \frac{1}{2} \\ &\quad \cdot \int_a^b g(x)\, \eta(f(a+b-x), f(x))\, dx \Big\}. \end{aligned} \tag{12}$$

*Moreover, if $g$ is symmetric on $[a,b]$, then*

$$\begin{aligned} f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx &\le \int_a^b g(x)\min\Big\{f(a+b-x) \\ &+\frac{1}{2}\eta\big(f(x),f(a+b-x)\big),f(x) \\ &+\frac{1}{2}\eta\big(f(a+b-x),f(x)\big)\Big\}\,dx \le \int_a^b g(x) \\ &\cdot f(x)\,dx+\frac{1}{2}\int_a^b g(x)\,\eta\big(f(x),f(a+b \\ &-x)\big)\,dx. \end{aligned} \tag{13}$$

*Remark 8.* (a) Inequality (11) gives a refinement for the right side and inequality (13) gives a refinement for the left side of (1), respectively.

(b) If in Theorems 6 and 7 we consider $g(x)\equiv 1$ for all $x\in[a,b]$ (see Theorem 3.6 in [9]), then respectively, we have

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(x)\,dx &\le \min\Big\{f(a) \\ &+\frac{1}{2}\eta\big(f(b),f(a)\big),f(b)+\frac{1}{2}\eta\big(f(a),f(b)\big)\Big\} \\ &\le\left(\frac{f(a)+f(b)}{2}\right) \\ &+\left(\frac{\eta\big(f(a),f(b)\big)+\eta\big(f(b),f(a)\big)}{4}\right), \end{aligned} \tag{14}$$

which is a refinement for the right side of Hermite-Hadamard inequality related to $\eta$-convex functions, and

$$\begin{aligned} f\left(\frac{a+b}{2}\right)&-\frac{1}{2(b-a)}\int_a^b \eta\big(f(x),f(a+b-x)\big)\,dx \\ &\le (b-a)\min\Big\{f(a+b-x) \\ &+\frac{1}{2}\eta\big(f(x),f(a+b-x)\big),f(x) \\ &+\frac{1}{2}\eta\big(f(a+b-x),f(x)\big)\Big\}\,dx \le \frac{1}{b-a} \\ &\cdot\int_a^b f(x)\,dx, \end{aligned} \tag{15}$$

which is a refinement for the left side of Hermite-Hadamard inequality related to $\eta$-convex functions.

## 2. Main Results

As an application of Theorem 6, we can obtain some estimation results for the difference between the right and middle part of (2) and also for the difference between the left and middle part of (2), respectively.

The following identity for an absolutely continuous function $f:[a,b]\to\mathbb{R}$ holds (see [14]):

$$\begin{aligned} &\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt \\ &\quad=\int_a^b\left(t-\frac{a+b}{2}\right)f'(t)\,dt. \end{aligned} \tag{16}$$

**Theorem 9.** *Let $f:I^\circ\subset I\to\mathbb{R}$ be a differentiable function, $a,b\in I^\circ$, $a<b$. Suppose that the function $|f'|$ is a $\eta$-convex function and $\eta$ is bounded from above on $|f'|([a,b])\times|f'|([a,b])$, and then we have*

$$\begin{aligned} \left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| &\le \frac{1}{4}(b-a)^2 \\ &\times\min\Big\{\left|f'(b)\right|+\frac{1}{2}\eta\big(\left|f'(a)\right|,\left|f'(b)\right|\big),\left|f'(a)\right| \\ &+\frac{1}{2}\eta\big(\left|f'(b)\right|,\left|f'(a)\right|\big)\Big\} \le \frac{1}{8}(b-a)^2\times\Big[\left|f'(a)\right| \\ &+\left|f'(b)\right|+\frac{1}{2}\Big[\eta\big(\left|f'(a)\right|,\left|f'(b)\right|\big) \\ &+\eta\big(\left|f'(b)\right|,\left|f'(a)\right|\big)\Big]\Big]. \end{aligned} \tag{17}$$

*Proof.* From (16) we get

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt\right| \\ &\quad\le\int_a^b\left|t-\frac{a+b}{2}\right|\left|f'(t)\right|dt. \end{aligned} \tag{18}$$

Since $|f'|$ is a $\eta$-convex function and $g:[a,b]\to\mathbb{R}$ with $g(t):=|t-(a+b)/2|$ is symmetric, then by inequality (11) we have

$$\begin{aligned} \int_a^b\left|f'(t)\right|\left|t-\frac{a+b}{2}\right|dt &\le \min\Big\{\left|f'(b)\right| \\ &+\frac{1}{2}\eta\big(\left|f'(a)\right|,\left|f'(b)\right|\big),\left|f'(a)\right| \\ &+\frac{1}{2}\eta\big(\left|f'(b)\right|,\left|f'(a)\right|\big)\Big\}\times\int_a^b\left|t-\frac{a+b}{2}\right|dt \\ &\le\frac{1}{2}\Big[\left|f'(a)\right|+\left|f'(b)\right|\Big]\int_a^b\left|t-\frac{a+b}{2}\right|dt \\ &+\frac{1}{4}\Big[\eta\big(\left|f'(a)\right|,\left|f'(b)\right|\big)+\eta\big(\left|f'(b)\right|,\left|f'(a)\right|\big)\Big] \\ &\cdot\int_a^b\left|t-\frac{a+b}{2}\right|dt. \end{aligned} \tag{19}$$

A simple calculation shows that

$$\int_a^b\left|t-\frac{a+b}{2}\right|dt=\frac{1}{4}(b-a)^2; \tag{20}$$

then by (19) we get the desired result (17). □

Also there exists another identity for absolutely continuous functions $f : [a,b] \to \mathbb{R}$ as the following (see [15]).

$$(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt = \int_a^b H(t) f'(t)\,dt, \quad (21)$$

where $H : [a,b] \to \mathbb{R}$ is considered as

$$H(t) := \begin{cases} t-a, & t \in \left[a, \frac{a+b}{2}\right), \\ t-b, & t \in \left[\frac{a+b}{2}, b\right]. \end{cases} \quad (22)$$

So we can obtain a result for the difference between the left and middle part of (2).

**Theorem 10.** *Let $f : I^\circ \subset I \to \mathbb{R}$ be a differentiable function, $a, b \in I^\circ$, $a < b$. Suppose that the function $|f'|$ is a $\eta$-convex function and $\eta$ is bounded from above on $|f'|([a,b]) \times |f'|([a,b])$, and then*

$$\begin{aligned} &\left|(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt\right| \le \frac{1}{4}(b-a)^2 \\ &\quad \times \min\left\{\left|f'(b)\right| + \frac{1}{2}\eta\left(\left|f'(a)\right|, \left|f'(b)\right|\right), \left|f'(a)\right| \right. \\ &\quad \left. + \frac{1}{2}\eta\left(\left|f'(b)\right|, \left|f'(a)\right|\right)\right\} \le \frac{1}{8}(b-a)^2 \times \left[\left|f'(a)\right| \right. \\ &\quad + \left|f'(b)\right| + \frac{1}{2}\left[\eta\left(\left|f'(a)\right|, \left|f'(b)\right|\right)\right. \\ &\quad \left.\left. + \eta\left(\left|f'(b)\right|, \left|f'(a)\right|\right)\right]\right]. \end{aligned} \quad (23)$$

*Proof.* From (21) we have

$$\begin{aligned} &\left|(b-a) f\left(\frac{a+b}{2}\right) - \int_a^b f(t)\,dt\right| \\ &\quad \le \int_a^b |H(t)| \left|f'(t)\right| dt. \end{aligned} \quad (24)$$

It follows that

$$|H(t)| = \begin{cases} t-a, & t \in \left[a, \frac{a+b}{2}\right), \\ b-t, & t \in \left[\frac{a+b}{2}, b\right], \end{cases} \quad (25)$$

which is a symmetric function on the interval $[a,b]$. Since $|f'|$ is a $\eta$-convex function and $g : [a,b] \to \mathbb{R}$ with $g(t) := |H(t)|$ is symmetric, then by inequality (11) we have

$$\begin{aligned} &\int_a^b \left|f'(t)\right| |H(t)|\,dt \le \min\left\{\left|f'(b)\right| \right. \\ &\quad + \frac{1}{2}\eta\left(\left|f'(a)\right|, \left|f'(b)\right|\right), \left|f'(a)\right| \\ &\quad \left. + \frac{1}{2}\eta\left(\left|f'(b)\right|, \left|f'(a)\right|\right)\right\} \times \int_a^b |H(t)|\,dt \\ &\quad \le \frac{1}{2}\left[\left|f'(a)\right| + \left|f'(b)\right|\right] \int_a^b |H(t)|\,dt \\ &\quad + \frac{1}{4}\left[\eta\left(\left|f'(a)\right|, \left|f'(b)\right|\right) + \eta\left(\left|f'(b)\right|, \left|f'(a)\right|\right)\right] \\ &\quad \cdot \int_a^b |H(t)|\,dt. \end{aligned} \quad (26)$$

Since

$$\int_a^b |H(t)|\,dt = \frac{1}{4}(b-a)^2, \quad (27)$$

then by (26) we get the desired result (23). □

*Example 11.* Consider the function $F : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$F(x) = \begin{cases} \frac{1}{2}x^2 + \frac{1}{2}, & 0 \le x \le 1; \\ x, & x > 1, \end{cases} \quad (28)$$

and a bifunction $\eta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ as

$$\eta(x,y) = \begin{cases} x+y, & x \le y; \\ 2(x+y), & x > y. \end{cases} \quad (29)$$

It is clear that the function

$$f(x) = \left|F'(x)\right| = \begin{cases} x, & 0 \le x \le 1; \\ 1, & x > 1, \end{cases} \quad (30)$$

is a $\eta$-convex function. From inequality (23) in Theorem 10, for $0 \le a < 1 < b \le 2$ with $a + b = 2$, we get

$$\begin{aligned} &\left|\int_a^b F(x)\,dx - F(1)(b-a)\right| \\ &\quad \le \frac{1}{4}(b-a)^2 \min\left\{1 + \frac{1}{2}\eta(a,1), a + \frac{1}{2}\eta(1,a)\right\}. \end{aligned} \quad (31)$$

Further calculations show that

$$\left|a^3 - 3b^2 - 3a + 6b - 1\right| \le \frac{3}{2}K(b-a)^2, \quad (32)$$

where

$$K = \begin{cases} 2a+1, & a \ge \frac{1}{3}; \\ \frac{a+3}{2}, & a < \frac{1}{3}. \end{cases} \quad (33)$$

Also with the same argument as above, from (17) in Theorem 9, for $0 \le a < 1 < b \le 2$ with $a + b = 2$, we can obtain the following inequality:

$$\left|\frac{F(a)+F(b)}{2}(b-a)-\int_a^b F(x)\,dx\right| \le \frac{1}{4}(b-a)^2 \min\left\{1+\frac{1}{2}\eta(a,1), a+\frac{1}{2}\eta(1,a)\right\}, \tag{34}$$

which implies that

$$\left|-5a^3+3a^2b-6ab-9a+12b^2+3b+2\right| \le 3K(b-a)^2, \tag{35}$$

where $K$ is defined as above.

Using the following identities for twice differentiable functions $f : [a,b] \to \mathbb{R}$, we can obtain some results similar to Theorems 9 and 10.

$$\frac{f(a)+f(b)}{2}(b-a)-\int_a^b f(t)\,dt = \frac{1}{2}\int_a^b (t-a)(b-t) f''(t)\,dt; \tag{36}$$

see [14];

$$\int_a^b f(t)\,dt-(b-a) f\left(\frac{a+b}{2}\right) = \frac{1}{2}\int_a^b G(t) f''(t)\,dt; \tag{37}$$

see [16], where

$$G(t) := \begin{cases} (t-a)^2, & t \in \left[a, \frac{a+b}{2}\right), \\ (t-b)^2, & t \in \left[\frac{a+b}{2}, b\right]; \end{cases} \tag{38}$$

then we have the following.

**Theorem 12.** *Let the function $f : I^\circ \subset I \to \mathbb{R}$ be twice differentiable, $a, b \in I^\circ$, $a < b$. Suppose that the function $|f''|$ is a $\eta$-convex function and $\eta$ is bounded from above on $|f''|([a,b]) \times |f''|([a,b])$, and then we have the inequalities*

$$\begin{aligned} &\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{12}(b-a)^2 \\ &\times \min\left\{\left|f''(b)\right|+\frac{1}{2}\eta\left(\left|f''(a)\right|,\left|f''(b)\right|\right), \left|f''(a)\right| \right. \\ &\left. +\frac{1}{2}\eta\left(\left|f''(b)\right|,\left|f''(a)\right|\right)\right\} \le \frac{1}{24}(b-a)^2 \\ &\times\left[\left|f''(a)\right|+\left|f''(b)\right|+\frac{1}{2}\left[\eta\left(\left|f''(a)\right|,\left|f''(b)\right|\right)\right.\right. \\ &\left.\left.+\eta\left(\left|f''(b)\right|,\left|f''(a)\right|\right)\right]\right]. \end{aligned} \tag{39}$$

*Proof.* Taking the modulus on (36) along with

$$\int_a^b (t-a)(b-t)\,dt = \frac{1}{6}(b-a)^3 \tag{40}$$

implies the expected result (39). □

**Theorem 13.** *With the assumptions of Theorem 12, we have the inequalities*

$$\begin{aligned} &\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \le \frac{1}{24}(b-a)^2 \\ &\times \min\left\{\left|f''(b)\right|+\frac{1}{2}\eta\left(\left|f''(a)\right|,\left|f''(b)\right|\right), \left|f''(a)\right| \right. \\ &\left. +\frac{1}{2}\eta\left(\left|f''(b)\right|,\left|f''(a)\right|\right)\right\} \le \frac{1}{48}(b-a)^2 \\ &\times\left[\left|f''(a)\right|+\left|f''(b)\right|+\frac{1}{2}\left[\eta\left(\left|f''(a)\right|,\left|f''(b)\right|\right)\right.\right. \\ &\left.\left.+\eta\left(\left|f''(b)\right|,\left|f''(a)\right|\right)\right]\right]. \end{aligned} \tag{41}$$

*Proof.* Taking the modulus on (37) along with

$$\int_a^b G(t)\,dt = \frac{1}{12}(b-a)^3, \tag{42}$$

implies the expected result (41). □

*Example 14.* Consider the function $F : \mathbb{R}^+ \to \mathbb{R}^+$ defined as

$$F(x) = \begin{cases} \frac{1}{6}x^3+\frac{1}{2}x-\frac{1}{6}, & 0 \le x \le 1; \\ \frac{1}{2}x^2, & x > 1 \end{cases} \tag{43}$$

and the bifunction $\eta$ defined in Example 11. It is easy to see that the function $f(x) = |F''|$ is a $\eta$-convex function. From inequality (41) in Theorem 13, for $0 \le a < 1 < b \le 2$ with $a + b = 2$, we get

$$\left|\int_a^b F(x)\,dx - F(1)(b-a)\right| \le \frac{1}{24}(b-a)^3 \min\left\{1+\frac{1}{2}\eta(a,1), a+\frac{1}{2}\eta(1,a)\right\}. \tag{44}$$

It follows that

$$\left|a^4-4b^3+6a^2+12b-16a+1\right| \le K(b-a)^3, \tag{45}$$

where $K$ is defined in Example 11.

In [13], we estimated the difference of the left and middle section of (1) as the following.

**Theorem 15** (see [13]). *Suppose that $f : I^\circ \subset I \to \mathbb{R}$ is a differentiable mapping, $a, b \in I^\circ$, $a < b$. If $g : [a,b] \to \mathbb{R}^+$ is a continuous mapping symmetric about $(a+b)/2$ and $|f'|$ is*

*a $\eta$-convex mapping on $[a,b]$ with $\eta$ bounded from above on $f([a,b])\times f([a,b])$, then*

$$\left|\int_a^b f(x)\,g(x)\,dx - f\left(\frac{a+b}{2}\right)\int_a^b g(x)\,dx\right| \le \frac{1}{(b-a)}\int_{(a+b)/2}^b \left[(x-a)^2-(b-x)^2\right] g(x)\,K\,dx, \tag{46}$$

*where*

$$K = \min\left\{\left|f'(b)\right| + \frac{\eta\left(\left|f'(a)\right|,\left|f'(b)\right|\right)}{2}, \left|f'(a)\right| + \frac{\eta\left(\left|f'(b)\right|,\left|f'(a)\right|\right)}{2}\right\}. \tag{47}$$

**Theorem 16** (see [10]). *Suppose that $f: I^\circ \subset I \to \mathbb{R}$ is a differentiable mapping, $a,b\in I^\circ$, $a<b$. If $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric about $(a+b)/2$ and $|f'|$ is a $\eta$-convex function with $\eta$ bounded from above on $f([a,b])\times f([a,b])$, then*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)\,dx - \int_a^b f(x)\,g(x)\,dx\right| \le \frac{(b-a)}{4}\left[2\left|f'(b)\right| + \eta\left(\left|f'(a)\right|,\left|f'(b)\right|\right)\right] \cdot \int_0^1\int_{((1+t)/2)a+((1-t)/2)b}^{((1-t)/2)a+((1+t)/2)b} g(u)\,du\,dt. \tag{48}$$

If in Theorem 16 we set $\eta(x,y)=x-y$ and $g\equiv 1$, then we have the following result.

Using of Theorems 15 and 16 can result in some numerical inequalities. This fact is shown in the following example.

*Example 17.* Consider all assumptions of Example 11 and $g(x)=(1-x)^2$ $(x\ge 0)$ which is symmetric to $x=1$. From inequality (46) in Theorem 15, for $0\le a<1<b\le 2$ with $a+b=2$, we get

$$\left|\int_a^b F(x)(x-1)^2\,dx - F(1)\int_a^b (x-1)^2\,dx\right| \le \frac{K}{(b-a)}\int_1^b \left[2x(b-a)+\left(a^2-b^2\right)\right](x-1)^2\,dx, \tag{49}$$

where

$$K = \min\left\{2a+1, \frac{a+3}{2}\right\}. \tag{50}$$

Further calculations in (49) show that

$$\left|\frac{1}{5}\left(b^5-a^5\right) - \frac{1}{2}\left(b^4-a^4\right) + \left(b^2-a^2\right) - (b-a)\right| \le K(b-1)^4, \tag{51}$$

where

$$K = \begin{cases} 2a+1, & a\ge \dfrac{1}{3};\\ \dfrac{a+3}{2}, & a<\dfrac{1}{3}.\end{cases} \tag{52}$$

As an application of Theorems 15 and 16, we give an error estimate for trapezoidal and midpoint formula that are generalization of Proposition 4.1 in [17] and Proposition 4.1 in [18], respectively.

Suppose that $d$ is a partition $a=x_0<x_1<\cdots<x_{n-1}<x_n=b$ of interval $[a,b]$. Consider quadrature formula

$$\int_a^b f(x)\,g(x)\,dx = T_j(f,g,d) + E_j(f,g,d), \quad j=1,2, \tag{53}$$

where

$$T_1(f,g,d) = \sum_{i=0}^{n-1}\frac{f(x_i)+f(x_{i+1})}{2}\int_{x_i}^{x_{i+1}} g(x)\,dx,$$
$$T_2(f,g,d) = \sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)\int_{x_i}^{x_{i+1}} g(x)\,dx \tag{54}$$

and $E_j(f,g,d)$ is the approximation error for trapezoidal and midpoint formula.

**Theorem 18.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable function, $g:[a,b]\to\mathbb{R}^+$ is a continuous function and symmetric with respect to $(a+b)/2$, and $|f'|$ is a $\eta$-convex function where $\eta$ is bounded from above on $[a,b]$. Then*

$$\left|E_1(f,g,d)\right| \le \sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)}{4} \cdot\left[2\left|f'(x_{i+1})\right| + \eta\left(\left|f'(x_i)\right|,\left|f'(x_{i+1})\right|\right)\right] \times\int_0^1\int_{((1+t)/2)x_i+((1-t)/2)x_{i+1}}^{((1-t)/2)x_i+((1+t)/2)x_{i+1}} g(x)\,dx\,dt. \tag{55}$$

*Proof.* It is enough to apply Theorem 15 on the subinterval $[x_i,x_{i+1}]$ $(i=0,1,\ldots,n-1)$ of the partition $d$ for interval $[a,b]$ and to sum all achieved inequalities over $i$ and then use triangle inequality. □

**Theorem 19.** *Suppose that $f:[a,b]\to\mathbb{R}$ is a differentiable mapping, $g:[a,b]\to\mathbb{R}^+$ is a continuous mapping symmetric about $(a+b)/2$, and $|f'|$ is a $\eta$-convex mapping on $[a,b]$ with a bounded $\eta$ from above. Then*

$$\left|E_2(f,g,d)\right| \le \sum_{i=0}^{n-1}\frac{1}{(x_{i+1}-x_i)} \cdot\int_{(x_i+x_{i+1})/2}^{x_{i+1}}\left[(x-x_i)^2-(x_{i+1}-x)^2\right] g(x)\,K_i\,dx, \tag{56}$$

*where*

$$K_i = \min\left\{\left[\left|f'(x_{i+1})\right| + \frac{\eta\left(\left|f'(x_i)\right|, \left|f'(x_{i+1})\right|\right)}{2}\right], \left[\left|f'(x_i)\right| + \frac{\eta\left(\left|f'(x_{i+1})\right|, \left|f'(x_i)\right|\right)}{2}\right]\right\}, \tag{57}$$

*for* $i = 0, 1, \ldots, n-1$.

*Proof.* It is enough to apply Theorem 16 on the subinterval $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition $d$ for interval $[a, b]$ and to sum all achieved inequalities over $i$ and then use triangle inequality. □

*Example 20.* Suppose that $d$ is the partition $x_0 = 0 < x_1 = 1 < x_2 = 2$ of interval $[0, 2]$. Consider the functions $F, g$ and the bifunction $\eta$ as defined in Example 17 for $a = 0,\ b = 2$. So the approximation error related to the trapezoidal and midpoint formula can be calculated as the following. From (55) we get

$$\begin{aligned}
\left|E_1(F, g, d)\right| &\le \sum_{i=0}^{1} \frac{(x_{i+1} - x_i)}{4} \\
&\cdot \left[2\left|F'(x_{i+1})\right| + \eta\left(\left|F'(x_i)\right|, \left|F'(x_{i+1})\right|\right)\right] \\
&\times \int_0^1 \int_{((1+t)/2)x_i + ((1-t)/2)x_{i+1}}^{((1-t)/2)x_i + ((1+t)/2)x_{i+1}} (x-1)^2\, dx\, dt \\
&= \frac{1}{4}\left(2 + \eta(0,1)\right) \int_0^1 \int_{(1-t)/2}^{(1+t)/2} (x-1)^2\, dx\, dt \\
&+ \frac{1}{4}\left(2 + \eta(1,1)\right) \int_0^1 \int_{(3-t)/2}^{(3+t)/2} (x-1)^2\, dx\, dt \\
&= \left[\frac{1}{32} + \frac{1}{24}\right] \int_0^1 \left[(t-1)^3 + (t+1)^3\right] dt = \frac{49}{192}.
\end{aligned} \tag{58}$$

Also from (56) we have

$$\begin{aligned}
&\left|E_2(F, g, d)\right| \\
&\quad \le \int_{1/2}^{1} \left[(x)^2 - (1-x)^2\right](x-1)^2 K_0 dx \\
&\qquad + \int_{3/2}^{2} \left[(x-1)^2 - (2-x)^2\right](x-1)^2 K_1 dx,
\end{aligned} \tag{59}$$

where

$$\begin{aligned}
K_0 &= \min\left\{1 + \frac{\eta(0,1)}{2}, \frac{\eta(0,1)}{2}\right\}, \\
K_1 &= \min\left\{1 + \frac{\eta(1,1)}{2}, 1 + \frac{\eta(1,1)}{2}\right\}.
\end{aligned} \tag{60}$$

Then with some calculations we get

$$\left|E_2(F, g, d)\right| \le \frac{1}{96}K_0 + \frac{17}{96}K_1 = \frac{35}{192}. \tag{61}$$

Note that we can decrease the approximation error by choosing $n > 2$.

## 3. Conclusions

The convexity of a function is the basis for many inequalities in mathematics. Note that, in new problems related to the convexity, generalized notions about convex functions are required to obtain applicable results. One of this generalizations is the notion of $\eta$-convex functions which can generalize many inequalities related to convex functions such as Hermite-Hadamard inequality, Fejér inequality, and trapezoid and midpoint type inequality.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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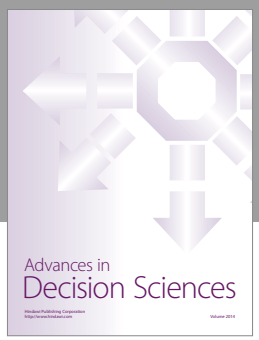

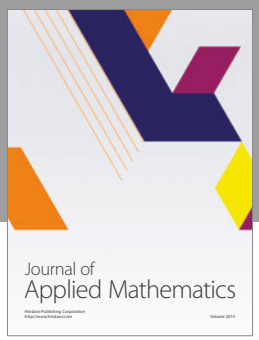

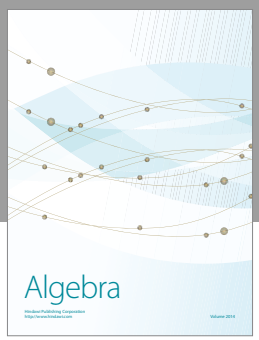

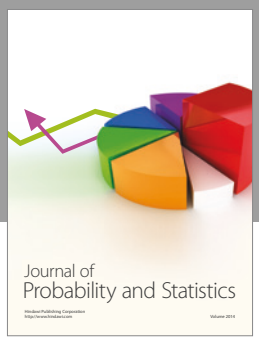

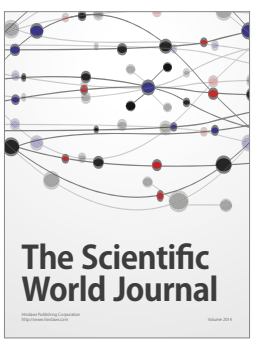

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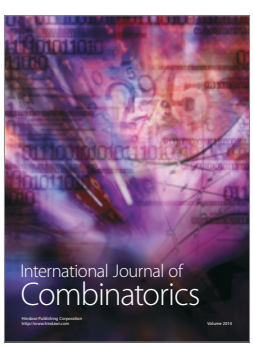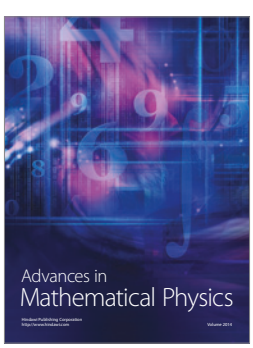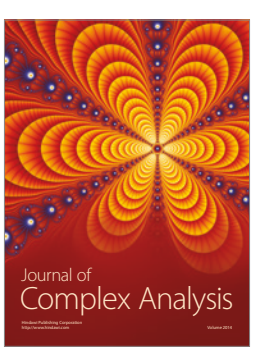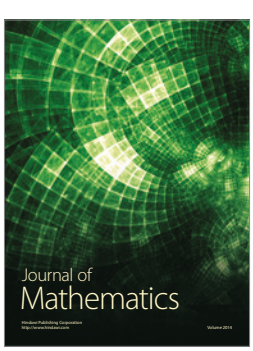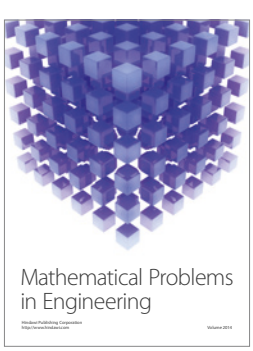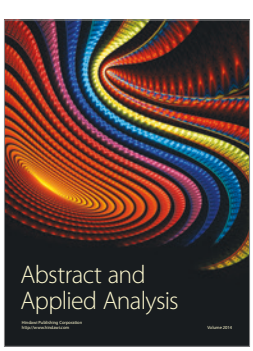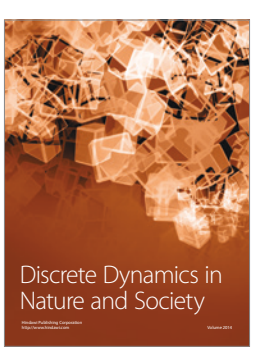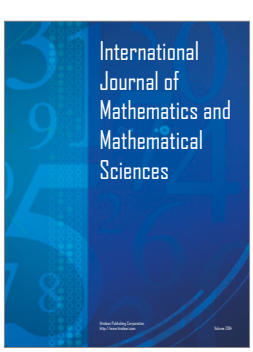
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