Research Article

Properties of ϕ -Primal Graded Ideals

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Let R be a commutative graded ring with unity $1 \neq 0$. A proper graded ideal of R is a graded ideal I of R such that $I \neq R$. Let $\phi : \mathfrak{I}(R) \rightarrow \mathfrak{I}(R) \cup \{\emptyset\}$ be any function, where $\mathfrak{I}(R)$ denotes the set of all proper graded ideals of R . A homogeneous element $a \in R$ is ϕ -prime to I if $ra \in I - \phi(I)$ where r is a homogeneous element in R ; then $r \in I$. An element $a = \sum_{g \in G} a_g \in R$ is ϕ -prime to I if at least one component a_g of a is ϕ -prime to I . Therefore, $a = \sum_{g \in G} a_g \in R$ is not ϕ -prime to I if each component a_g of a is not ϕ -prime to I . We denote by $\nu_\phi(I)$ the set of all elements in R that are not ϕ -prime to I . We define I to be ϕ -primal if the set $P = \nu_\phi(I) + \phi(I)$ (if $\phi \neq \phi_0$) or $P = \nu_\phi(I)$ (if $\phi = \phi_0$) forms a graded ideal of R . In the work by Jaber, 2016, the author studied the generalization of primal superideals over a commutative super-ring R with unity. In this paper we generalize the work by Jaber, 2016, to the graded case and we study more properties about this generalization.

1. Introduction

In [1] the author studied the generalization of primal superideals over a commutative super-ring R with unity. In this paper we generalize this work to the graded case and we study more properties about this generalization. For example in Section 4 we study the properties of ϕ - P/J -primal graded ideals and in Section 5 we study the properties of ϕ_S -primal graded ideals of R_S .

Let G be an abelian group and let R be any commutative ring with unity; then R is called a G -graded ring (for short graded ring), if $R = \bigoplus_{g \in G} R_g$, such that if $a, b \in G$, then $R_a R_b \subseteq R_{ab}$. Let $h(R) = \bigcup_{g \in G} R_g$; then $h(R)$ is the set of homogeneous elements in R and $1 \in R_e$, where e is the identity element in G . By a proper graded ideal of R we mean a graded ideal I of R such that $I \neq R$.

We define a proper graded ideal P of R to be *prime* if $ab \in P$ implies that $a \in P$ or $b \in P$, where $a, b \in h(R)$. Let I be a proper graded ideal of R , an element $a \in h(R)$ is called *prime to I* if $ar \in I$, where $r \in h(R)$; then $r \in I$. If I is a proper graded ideal of R and $\rho(I)$ is the set of homogeneous elements of R that are not prime to I , then we define I to be *primal* if the set $P = \bigoplus_{g \in G} (\rho(I) \cap R_g)$ forms a graded ideal of R . In this case we say that I is a P -primal graded ideal of R . Moreover, if I is

a P -primal graded ideal of R , then it is easy to check that P is a prime graded ideal of R .

Throughout, R will be a commutative graded ring with unity. We will denote the set of all proper graded ideals of R by $\mathfrak{I}(R)$. If I and J are in $\mathfrak{I}(R)$, then the set $\{r \in R : rJ \subseteq I\}$ is a graded ideal of R which is denoted by $(I : J)$. Let $\phi : \mathfrak{I}(R) \rightarrow \mathfrak{I}(R) \cup \{\emptyset\}$ be any function and let $I \in \mathfrak{I}(R)$; we say that I is a ϕ -prime if whenever $x, y \in h(R)$, with $xy \in I - \phi(I)$, then $x \in I$ or $y \in I$. Since $I - \phi(I) = I - [\phi(I) \cap I]$, there is no loss of generality to assume that $\phi(I) \subseteq I$ for every proper graded ideal I of R .

Let $\phi : \mathfrak{I}(R) \rightarrow \mathfrak{I}(R) \cup \{\emptyset\}$ be any function. In this paper we always assume that, for any $I, J \in \mathfrak{I}(R)$, $\phi(J) \subseteq \phi(I)$ if $J \subseteq I$.

Given two functions $\psi_1, \psi_2 : \mathfrak{I}(R) \rightarrow \mathfrak{I}(R) \cup \{\emptyset\}$, we define $\psi_1 \subseteq \psi_2$ if $\psi_1(I) \subseteq \psi_2(I)$ for each $I \in \mathfrak{I}(R)$.

Let $\phi : \mathfrak{I}(R) \rightarrow \mathfrak{I}(R) \cup \{\emptyset\}$ be any function; then an element $a \in h(R)$ is ϕ -prime to I , if whenever $ra \in I - \phi(I)$, where $r \in h(R)$, then $r \in I$. That is, $a \in h(R)$ is ϕ -prime to I if

$$h((I : a)) - h((\phi(I) : a)) \subseteq h(I). \quad (1)$$

An element $a = \sum_{g \in G} a_g \in R$ is ϕ -prime to I if at least one component a_g of a is ϕ -prime to I . Therefore, $a = \sum_{g \in G} a_g \in R$ is not ϕ -prime to I if each component a_g of a is not ϕ -prime

to I . Denote by $\nu_\phi(I)$ the set of all elements in R that are not ϕ -prime to I . Note that if $a \in h(R)$ is not ϕ -prime to I , then $a \in \nu_\phi(I)$.

We define I to be ϕ -primal if the set $P = \nu_\phi(I) + \phi(I)$ (if $\phi \neq \phi_0$) or $P = \nu_\phi(I)$ (if $\phi = \phi_0$) forms a graded ideal of R . In this case we say that I is a ϕ - P -primal graded ideal of R , and P is the adjoint graded ideal of I .

In the next example we give some famous functions $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ and their corresponding ϕ -primal graded ideals.

Example 1.

- $\phi_0: \phi_0(I) = \emptyset, \forall I \in \mathfrak{S}(R)$ primal graded ideal
- $\phi_0: \phi_0(I) = \{0\}, \forall I \in \mathfrak{S}(R)$ weakly primal graded ideal
- $\phi_2: \phi_2(I) = I^2, \forall I \in \mathfrak{S}(R)$ almost primal graded ideal
- $\phi_n: \phi_n(I) = I^n, \forall I \in \mathfrak{S}(R)$ n -almost primal graded ideal
- $\phi_\omega: \phi_\omega(I) = \bigcap_{n=1}^{\infty} I^n, \forall I \in \mathfrak{S}(R)$ ω -primal graded ideal

Observe that $\phi_0 \leq \phi_0 \leq \phi_\omega \leq \dots \leq \phi_{n+1} \leq \phi_n \leq \dots \leq \phi_2$.

In this paper we study various properties of ϕ -primal graded ideals. Some of these properties for the non-graded case have been studied by Atani and Darani in [2, 3].

2. ϕ -Primal Graded Ideals

The next example shows that $\nu_\phi(I)$ need not to be a graded ideal of R (see this example also in [4]).

Example 2. Let $R = \mathbb{Z}_{24} + u\mathbb{Z}_{24}$, where $u^2 = 0$, be a commutative $\mathbb{Z}_2$ -graded ring and assume that $\phi = \phi_0$. Let $I = 8\mathbb{Z}_{24} + u\mathbb{Z}_{24}$. Then I is a graded ideal of R . Since $0 \neq \bar{2} \cdot \bar{4} \in I$ with $\bar{2}, \bar{4} \notin I$, then we get that $\bar{2}$ and $\bar{4}$ are not ϕ -prime to I , and hence $\bar{2}$ and $\bar{4}$ are in $\nu_\phi(I)$. Easy computations imply that $\bar{2} + \bar{4} = \bar{6}$ is ϕ -prime to I . Thus, $\bar{6} \notin \nu_\phi(I)$ which implies that $\nu_\phi(I)$ is not a graded ideal of R .

Next we give two examples of ϕ -primal $\mathbb{Z}$ -graded ideals of a given $\mathbb{Z}$ -graded ring R (where $\phi = \phi_0$).

Example 3. Let $R = \mathbb{Z}[x]$, $G = \mathbb{Z}$, and $|x| = 1$. Then $\mathbb{Z}[x] = \mathbb{Z} \oplus \mathbb{Z}x \oplus \mathbb{Z}x^2 \oplus \dots$, where $\mathbb{Z}[x]_0 = \mathbb{Z}$, $\mathbb{Z}[x]_1 = \mathbb{Z}x$, $\mathbb{Z}[x]_2 = \mathbb{Z}x^2$, and $\mathbb{Z}[x]_n = \{0\}$ for $n < 0$, is a $\mathbb{Z}$ -graded ring. Let $I = 4\mathbb{Z}[x]$ be a graded ideal in R , and let $\phi = \phi_0$. Then $\nu_\phi(I) = 2\mathbb{Z}[x]$ is a graded ideal of R ; hence I is ϕ -primal graded ideal of R .

Example 4. Let $R = \mathbb{Z}_4[x]$ and $|x| = 1$. Then R is $\mathbb{Z}$ -graded ring, where $\mathbb{Z}_4[x]_n = \bigoplus_{n=0}^{\infty} \mathbb{Z}_4 x^n$ and $\mathbb{Z}_4[x]_n = \{0\}$ for $n < 0$. Let $I = \{0\}$. Then, for $\phi = \phi_0$, by easy computations $\nu_\phi(I) = M[x]$ where $M = \{0, 2\}$. Hence $I = \{0\}$ is ϕ -primal graded ideal of R .

Lemma 5. Let I be a proper graded ideal of R , and let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be any function. Then $h(I) - h(\phi(I)) \subseteq \nu_\phi(I)$.

Proof. Let $a \in h(I) - h(\phi(I))$. Since I is proper, $1 \notin I$. Then $a = a \cdot 1 \in h(I) - h(\phi(I))$ implies that a is not ϕ -prime to I . Thus $a \in \nu_\phi(I)$. $\square$

We recall from Lemma 5 that $a = \sum_{g \in G} a_g \in \nu_\phi(I)$ if each component a_g of a is in $\nu_\phi(I)$. Thus, by Lemma 5, each element $a \in I$ can be written in the form $c + d$, where $c \in \nu_\phi(I)$ and $d \in \phi(I)$. Therefore, $I \subseteq \nu_\phi(I) + \phi(I)$.

Theorem 6. Let I be a proper graded ideal of R , and let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be any function. If the set $\nu_\phi(I)$ is an ideal of R , then it is ϕ -prime graded ideal of R .

Proof. Clearly $\nu_\phi(I)$ is a graded ideal of R . Now, let $a, b \in h(R)$ with $ab \in \nu_\phi(I) - \phi(\nu_\phi(I))$. Then ab is not ϕ -prime to I . So there exists $r \in h(R) - h(I)$ with $abr \in I - \phi(I)$. If $a \notin \nu_\phi(I)$, then a is ϕ -prime to I . So $abr \in I - \phi(I)$ implies that $br \in I$. Hence $br \in I - \phi(I)$, since if $br \in \phi(I)$, then $abr \in \phi(I)$, a contradiction. Thus, $b \in \nu_\phi(I)$, since $r \in h(R) - h(I)$. $\square$

Corollary 7. Let I be a proper graded ideal of R , and let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be any function. If the set $\nu_\phi(I)$ is an ideal of R , then

$$P = \begin{cases} \nu_\phi(I) + \phi(I) & \text{if } \phi \neq \phi_0 \\ \nu_\phi(I) & \text{if } \phi = \phi_0 \end{cases} \quad (2)$$

is ϕ -prime graded ideal of R .

Proof. let $a, b \in h(R)$ with $ab \in P - \phi(P)$. Then $ab \notin \phi(I)$, since $\phi(I) \subseteq \phi(P)$. Therefore, $ab \in \nu_\phi(I)$ or $ab = c_g + d_g$, where $c_g \in (\nu_\phi(I))_g$ and $d_g \in (\phi(I))_g$.

If $ab \in \nu_\phi(I)$, then, by Theorem 6, $a \in \nu_\phi(I) \subseteq P$ or $b \in \nu_\phi(I) \subseteq P$. So, P is ϕ -prime graded ideal of R .

If $ab = c_g + d_g$, where $c_g \in (\nu_\phi(I))_g$ and $d_g \in (\phi(I))_g$, then there exists $r \in h(R) - h(I)$ with $c_g r \in I - \phi(I)$. So, $abr = (c_g + d_g)r \in I - \phi(I)$, since $d_g r \in \phi(I)$. Therefore, $ab \in \nu_\phi(I)$ and by Theorem 6, $a \in \nu_\phi(I) \subseteq P$ or $b \in \nu_\phi(I) \subseteq P$. $\square$

Definition 8. Let I be a proper graded ideal of R , and let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be any function. Then I is called ϕ - P -primal graded ideal of R , where $P = \nu_\phi(I) + \phi(I)$, if the set $\nu_\phi(I)$ forms an ideal of R . By the above corollary, P is always ϕ -prime graded ideal of R . In this case P is called the adjoint of I .

Proposition 9. Let I, P be proper graded ideals of R . Then the following statements are equivalent.

- (1) I is ϕ -primal graded ideal of R with the adjoint graded ideal P .
- (2) For $x \in h(R)$ with $x \notin h(P) - h(\phi(I))$ we have $h((I : x)) = h(I) \cup h((\phi(I) : x))$. If $x \in h(P) - h(\phi(I))$, then $h((I : x)) \supsetneq h(I) \cup h((\phi(I) : x))$.

Proof. (1) $\Rightarrow$ (2) If $x \in h(P) - h(\phi(I))$, then $x \in \nu_\phi(I)$. So there exists $r \in h(R) - h(I)$ with $rx \in I - \phi(I)$. Thus $r \in h((I : x))$ and $r \notin h(I) \cup h((\phi(I) : x))$. Since it is easy to see that $h((I : x)) \supseteq h(I) \cup h((\phi(I) : x))$, we have that $h((I : x)) \supsetneq h(I) \cup h((\phi(I) : x))$.

Now let $x \notin h(P) - h(\phi(I))$, where $x \in h(R)$. Then $x \notin \nu_\phi(I)$ and hence x is ϕ -prime to I . Let $r \in h((I : x))$. If $rx \notin \phi(I)$, then $r \in h(I)$. If $rx \in \phi(I)$, then $r \in h((\phi(I) : x))$. Hence

$$h((I : x)) \subseteq h(I) \cup h((\phi(I) : x)) \subseteq h((I : x)). \quad (3)$$

(2) $\Rightarrow$ (1) From part (2) we have $h(P) - h(\phi(I)) = \nu_\phi(I)$. Thus I is ϕ -primal graded ideal of R . $\square$

We say that P is ϕ -primal graded ideal of R if P itself is the adjoint of P . The next result shows that every ϕ -prime graded ideal of R is ϕ -primal.

Theorem 10. Every ϕ -prime graded ideal of R is ϕ -primal.

Proof. Let P be ϕ -prime graded ideal of R ; we show that P is ϕ - P -primal graded ideal of R . Thus we must prove that

$$P = \begin{cases} \nu_\phi(P) + \phi(P) & \text{if } \phi \neq \phi_0 \\ \nu_\phi(P) & \text{if } \phi = \phi_0. \end{cases} \quad (4)$$

If $P = \phi(P)$, then it is easy to check that $\nu_\phi(P) = \emptyset$; hence P is ϕ - P -primal graded ideal of R . Therefore, we may assume that $P \neq \phi(P)$. We show that $h(P) - h(\phi(P)) = \nu_\phi(P)$. Let $a \in h(P) - h(\phi(P))$. Then $a \cdot 1 \in P - \phi(P)$ with $1 \notin P$, so $a \in \nu_\phi(P)$. On the other hand let $a \notin h(P) - h(\phi(P))$. If $a \in h(\phi(P))$, then $ra \in \phi(P)$ for all $r \in h(R)$, so a is ϕ -prime to P and hence $a \notin \nu_\phi(P)$. If $a \notin h(\phi(P))$, then $a \notin P$, so for any $r_\alpha \in R_\alpha$ with $r_\alpha a \in P - \phi(P)$ we have that $r_\alpha \in P_\alpha$, since P is ϕ -prime. Thus a is ϕ -prime to P and hence $a \notin \nu_\phi(P)$. Therefore, $h(P) - h(\phi(P)) = \nu_\phi(P)$ which implies that P is ϕ - P -primal graded ideal of R . $\square$

Now we give an example of ϕ - P -primal graded ideal I of R such that I itself is not ϕ -prime.

Example 11. Let $\phi = \phi_0$ and let $R = \mathbb{Z}_8 + u\mathbb{Z}_8$ where $u^2 = 0$. Then R is a commutative $\mathbb{Z}_2$ -graded ring with unity. If $I = 4\mathbb{Z}_8 + u\mathbb{Z}_8$, then I is not a ϕ -prime graded ideal of R , since $\bar{2} \cdot \bar{2} \neq 0 \in I$, but $\bar{2} \notin I$. Let $P = 2\mathbb{Z}_8 + u\mathbb{Z}_8$; we show that I is ϕ - P -primal graded ideal of R . It is enough to show that $\nu(I) = h(P) - \{0\}$. Let $0 \neq \bar{a} \in h(P)$, if $\bar{a} \in 2\mathbb{Z}_8$; then $\bar{a} = 2k \in \mathbb{Z}_8$. If k is an odd number, then $0 \neq \bar{2}\bar{a} \in I$, but $\bar{2} \notin I$, and if k is an even number $0 \neq \bar{1}\bar{a} \in I$ with $\bar{1} \notin I$; hence $\bar{a} \in \nu(I)$. If $\bar{a} \in u\mathbb{Z}_8$, then $\bar{a} \in I \subseteq \nu(I)$. On the other hand, if $\bar{a} \in h(R) - h(P)$, then $\bar{a}$ is an odd number in $\mathbb{Z}_8$. If $0 \neq \bar{a}\bar{m} \in I$ for some $\bar{m} \in \mathbb{Z}_8$, then 4 divides am and so 4 divides m since $(4, a) = 1$; hence $\bar{m} \in I$. Thus I is ϕ - P -primal graded ideal of R .

Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function. We assume that, for any $I, J \in \mathfrak{F}(R)$, $\phi(I) \subseteq \phi(J)$ if $J \subseteq I$. Now we prove one of the main results in this section.

Theorem 12. Suppose that $\psi_1 \leq \psi_2$, where ψ_1 and ψ_2 are maps from $\mathfrak{F}(R)$ into $\mathfrak{F}(R) \cup \{\emptyset\}$, and let I be a ψ_2 - P -primal graded ideal of R , with $I_e I_\alpha \neq \psi_2(I)_\alpha$ for all $\alpha \in G$, where e is the identity element in G . If P is a prime graded ideal of R , then I is ψ_1 - P -primal.

Proof. Since I is ψ_2 - P -primal graded ideal of R , then

$$P = \begin{cases} \nu_{\psi_2}(I) + \psi_2(I) & \text{if } \psi_2 \neq \phi_0 \\ \nu_{\psi_2}(I) & \text{if } \psi_2 = \phi_0. \end{cases} \quad (5)$$

To show that I is ψ_1 - P -primal graded ideal of R we must prove that

$$P = \begin{cases} \nu_{\psi_1}(I) + \psi_1(I) & \text{if } \psi_2 \neq \phi_0 \\ \nu_{\psi_1}(I) & \text{if } \psi_1 = \phi_0. \end{cases} \quad (6)$$

If $\psi_2 = \phi_0$, then $\psi_1 = \psi_2$ and hence we have that $P = \nu_{\psi_1}(I)$ which implies that I is ψ_1 - P -primal graded ideal of R . Now we may assume that $\psi_2 \neq \phi_0$. Let $a \in h(\nu_{\psi_2}(I))$. Then there exists $r \in h(R) - h(I)$ with $rs \in I - \psi_2(I) \subseteq I - \psi_1(I)$, so $a \in h(\nu_{\psi_1}(I))$ which implies that

$$\nu_{\psi_2}(I) \subseteq \nu_{\psi_1}(I). \quad (7)$$

Now, let $a \in h(\psi_2(I))$. If $a \notin \psi_1(I)$, then $a \in I - \psi_1(I)$. So $1 \cdot a \in I - \psi_1(I)$ with $a \notin I$. Hence $a \in h(\nu_{\psi_1}(I))$. Therefore,

$$\psi_2(I) \subseteq \begin{cases} \nu_{\psi_1}(I) + \psi_1(I) & \text{if } \psi_2 \neq \phi_0 \\ \nu_{\psi_1}(I) & \text{if } \psi_1 = \phi_0. \end{cases} \quad (8)$$

From (7) and (8) we have that

$$P \subseteq \begin{cases} \nu_{\psi_1}(I) + \psi_1(I) & \text{if } \psi_2 \neq \phi_0 \\ \nu_{\psi_1}(I) & \text{if } \psi_1 = \phi_0. \end{cases} \quad (9)$$

Since $\psi_1(I) \subseteq \psi_2(I) \subseteq P$, by (9)

$$P = \begin{cases} \nu_{\psi_1}(I) + \psi_1(I) & \text{if } \psi_2 \neq \phi_0 \\ \nu_{\psi_1}(I) & \text{if } \psi_1 = \phi_0 \end{cases} \quad \text{if } \nu_{\psi_1}(I) \subseteq P. \quad (10)$$

Let $a \in (\nu_{\psi_1}(I))_\alpha$. Then there exists $r_\beta \in R_\beta - I_\beta$ with $ar_\beta \in I - \psi_1(I)$. If $ar_\beta \in I - \psi_2(I)$, then $a \in \nu_{\psi_2}(I) \subseteq P$. So we may assume that $ar_\beta \notin I - \psi_2(I)$; hence $ar_\beta \in \psi_2(I)$. First suppose that $aI_\beta \not\subseteq (\psi_2(I))_{\alpha\beta}$, say $as_\beta \in I_{\alpha\beta} - (\psi_2(I))_{\alpha\beta}$ with $s_\beta \in I_\beta$. Then $a(r_\beta + s_\beta) = ar_\beta + as_\beta \notin \psi_2(I)$ with $r_\beta + s_\beta \in R_\beta - I_\beta$. Hence $a \in \nu_{\psi_2}(I) \subseteq P$. Therefore, we may assume that $aI_\beta \subseteq (\psi_2(I))_{\alpha\beta}$.

Now suppose that $r_\beta I_e \not\subseteq (\psi_2(I))_\beta$. Then there exists $c \in I_e$ with $r_\beta c \in I_\beta - (\psi_2(I))_\beta$. Since $a^2 \in R_e$, then $(a^2 + c)r_\beta \in I_\beta - (\psi_2(I))_\beta$ with $r_\beta \notin I_\beta$. So $a^2 + c \in P_e$ but $c \in I_e \subseteq P_e$; therefore, $a^2 \in P$ and hence $a \in P$, since P is a prime graded ideal. So we may assume that $r_\beta I_e \subseteq (\psi_2(I))_\beta$. Since $(I_e I_\beta) \neq (\psi_2(I))_\beta$ there exists $\bar{a} \in I_e$ and $\bar{b} \in I_\beta$ with $\bar{a}\bar{b} \notin (\psi_2(I))_\beta$. Thus, $(a^2 + \bar{a})(r_\beta + \bar{b}) = a^2 r_\beta + a^2 \bar{b} + \bar{a} r_\beta + \bar{a} \bar{b} \notin \psi_2(I)$, so

$(a^2 + \bar{a})(r_\beta + \bar{b}) \in I - \psi_2(I)$ with $r_\beta + \bar{b} \in R_\beta - I_\beta$ which implies that $a^2 + \bar{a} \in (\nu_{\psi_2}(I))_e \subseteq P_e$. Hence $a^2 \in P_e \subseteq P$, and so $a \in P$, since P is a prime graded ideal of R . Therefore, $\nu_{\psi_1}(I) \subseteq P$, so

$$P = \begin{cases} \nu_{\psi_1}(I) + \psi_1(I) & \text{if } \psi_2 \neq \phi_0 \\ \nu_{\psi_1}(I) & \text{if } \psi_1 = \phi_0 \end{cases} \quad (11)$$

and hence I is ψ_1 - P -primal graded ideal of R . $\square$

3. Conditions on ϕ -Primal Graded Ideals

In this section we introduce some conditions under which ϕ -primal graded ideals are primal.

Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function. We have to recall that if I is ϕ - P -primal graded ideal of R , then

$$P = \begin{cases} \nu_\phi(I) + \phi(I) & \text{if } \phi \neq \phi_0 \\ \nu_\phi(I) & \text{if } \phi = \phi_0 \end{cases} \quad (12)$$

is ϕ -prime graded ideal of R .

Definition 13. Let r be a homogeneous element in R . Then $|r| = \alpha$ if $r \in R_\alpha$ for some $\alpha \in G$.

In the next theorem we provide some conditions under which ϕ -primal graded ideal is primal.

Theorem 14. Let R be a commutative graded ring with unity and let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function. Suppose that I is ϕ - P -primal graded ideal of R with $I_\gamma I_\delta \not\subseteq \phi(I)$ for each $\gamma, \delta \in G$. If P is a prime graded ideal of R , then I is P -primal.

Proof. Assume that a is a homogeneous element in P . Then $a \in \phi(I)$ or $a \in (\nu_\phi(I))_\alpha$ for some $\alpha \in G$ or $a = b_\beta + c_\beta$ where $b_\beta \in (\nu_\phi(I))_\beta$ and $c_\beta \in \phi(I)$ for some $\beta \in G$. If the first two cases hold, then a is not prime to I , since it is not ϕ -prime to I . In the last case, let d be a homogeneous element in R such that $d \notin I$ with $b_\beta d \in I - \phi(I)$. Then $ad = b_\beta d + c_\beta d \in I - \phi(I)$, because $ad \in \phi(I)$ implies that $b_\beta d \in \phi(I)$, since $c_\beta d \in \phi(I)$ which is a contradiction. Thus a is not ϕ -prime to I and hence a is not prime to I . Now assume that $b \in h(R)$ is not prime to I , so $rb \in I$ for some homogeneous element $r \in R - I$. If $rb \notin \phi(I)$, then b is not ϕ -prime to I , so $b \in P$. Thus assume that $rb \in \phi(I)$. Suppose that $|r| = \alpha$. First suppose that $bI_\alpha \not\subseteq \phi(I)$. Then, there exists $r' \in I_\alpha$ such that $br' \notin \phi(I)$. So $b(r + r') \in I - \phi(I)$, where $r + r'$ is a homogeneous element in $R - I$, implies that b is not ϕ -prime to I ; that is, $b \in P$. Therefore, we may assume that $bI_\alpha \subseteq \phi(I)$. Let $|b| = \beta$. If $rI_\beta \not\subseteq \phi(I)$, then $rc \notin \phi(I)$ for some $c \in I_\beta$. In this case $r(b + c) \in I - \phi(I)$ with $r \in R - I$; that is, $b + c \in P$, and hence $b \in P$, since $c \in I \subseteq P$. So we may assume that $rI_\beta \subseteq \phi(I)$. Since $I_\alpha I_\beta \not\subseteq \phi(I)$, there are $b' \in I_\alpha$ and $a' \in I_\beta$ with $b'a' \notin \phi(I)$. Now, $(b + a')(r + b') \in I - \phi(I)$, where $r + b'$ is a homogeneous element in $R - I$, implies that $b + a'$ is a homogeneous element in P . On the other hand $a' \in I \subseteq P$, so that $b \in P$. We have already shown that P is exactly the set of all elements of R that are not prime to I . Hence I is P -primal. $\square$

Let R and S be commutative G -graded rings, where G is an abelian group. It is easy to prove that the prime graded ideals of $R \times S$ have the form $P \times S$ or $R \times Q$ where P is a prime graded ideal of R and Q is a prime graded ideal of S . Also we have the following proposition about primal graded ideals of $R \times S$. We leave the easy proof for this proposition to the reader. For the trivial case (i.e., $R_g = S_g = \{0\}$ for all $g \neq e$) the next proposition is proved in [5, Lemma 13].

Proposition 15. Let R and S be commutative G -graded rings. If I is P -primal graded ideal of R and J is Q -primal graded ideal of S , then $I \times S$ (resp., $R \times J$) is $P \times S$ - (resp., $R \times Q$ -) primal graded ideal of $R \times S$.

Next, we generalize [6, Theorem 16] to the graded case. Then we use this generalization to prove Theorem 18.

Theorem 16. Let R_1 and R_2 be commutative G -graded rings with unities and let $\psi_i : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be functions. Let $\phi = \psi_1 \times \psi_2$. Then ϕ -primes of $R_1 \times R_2$ have exactly one of the following three types:

- (1) $I_1 \times I_2$ where I_i is a proper graded ideal of R_i with $\psi_i(I_i) = I_i$
- (2) $I_1 \times R_2$ where I_1 is ψ_1 -prime of R_1 which must be prime if $\psi_2(R_2) \neq R_2$
- (3) $R_1 \times I_2$ where I_2 is ψ_2 -prime of R_2 which must be prime if $\psi_1(R_1) \neq R_1$

Proof. We first note that a graded ideal of $R_1 \times R_2$ having one of these three types is ϕ -prime. Case (1) is clear since $\phi(I) = I$. If I_1 is a prime graded ideal of R_1 , certainly $I_1 \times R_2$ is prime and hence ϕ -prime. So suppose that I_1 is ψ_1 -prime and $\psi_2(R_2) = R_2$. Let a_1, b_1 be homogeneous elements in R_1 and let a_2, b_2 be homogeneous elements in R_2 with $(a_1 b_1, a_2 b_2) = (a_1, a_2)(b_1, b_2) \in I_1 \times R_2 - \psi_1(I_1) \times \psi_2(R_2) = I_1 \times R_2 - \psi_1(I_1) \times R_2 = (I_1 - \psi_1(I_1)) \times R_2$. Then $a_1 b_1 \in I_1 - \psi_1(I_1) \Rightarrow a_1 \in I_1$ or $b_1 \in I_1$, so $(a_1, a_2) \in I_1 \times R_2$ or $(b_1, b_2) \in I_1 \times R_2$. So, $I_1 \times R_2$ is ϕ -prime. The proof for case (3) is similar.

Next, suppose that $I_1 \times I_2$ is ϕ -prime. Let a, b be homogeneous elements in R_1 such that $ab \in I_1 - \psi_1(I_1)$. Then $(a, 0)(b, 0) = (ab, 0) \in I_1 \times I_2 - \phi(I_1 \times I_2)$, so $(a, 0) \in I_1 \times I_2$ or $(b, 0) \in I_1 \times I_2$; that is, $a \in I_1$ or $b \in I_1$. So I_1 is ψ_1 -prime. Likewise, I_2 is ψ_2 -prime. Suppose that $I_1 \times I_2 \neq \psi_1(I_1) \times \psi_2(I_2)$, say, $I_1 \neq \psi_1(I_1)$. Let p be a homogeneous element in R_1 such that $p \in I_1 - \psi_1(I_1)$ and let q be a homogeneous element in I_2 . Then $(p, 1)(1, q) = (p, q) \in I_1 \times I_2 - \phi(I_1 \times I_2)$. So $(1, q) \in I_1 \times I_2$ or $(p, 1) \in I_1 \times I_2$. Hence $I_1 = R_1$ or $I_2 = R_2$. Suppose that $I_2 = R_2$. So $I_1 \times R_2$ is ϕ -prime where I_1 is ψ_1 -prime. It remains to show that if $\psi_2(R_2) \neq R_2$, then I_1 is actually prime graded ideal of R_1 . Let a, b be homogeneous elements in R_1 such that $ab \in I_1$. Now $1 \in R_2$ is a homogeneous element in R_2 not in $\psi_2(R_2)$. Then $(a, 1)(b, 1) = (ab, 1) \in I_1 \times R_2 - \phi(I_1 \times R_2)$, so $(a, 1) \in I_1 \times R_2$ or $(b, 1) \in I_1 \times R_2$; that is, $a \in I_1$ or $b \in I_1$. $\square$

Let R_1, R_2 be commutative G -graded rings with unities and let $R = R_1 \times R_2$. Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be a function. In the next theorem we also provide some conditions under which ϕ -primal graded ideal of R is primal, but first we start with the following remark.

Remark 17. Let I be a proper graded ideal of a commutative graded ring R and let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be a function. If a homogeneous element $a \in R$ is not ϕ -prime to I , then there is a homogeneous element $r \in R - I$ such that $ar \in I - \phi(I) \subseteq I$ so a is not prime to I .

Theorem 18. Let R_1, R_2 be commutative G -graded rings with unities and let $R = R_1 \times R_2$. Let $\psi_i : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be functions with $\psi_i(R_i) \neq R_i$ for $i = 1, 2$. Let $\phi = \psi_1 \times \psi_2$. Assume that P is a graded ideal of R with $\phi(P) \neq P$. If I is a ϕ - P -primal graded ideal of R , then either $I = \phi(I)$ or I is P -primal.

Proof. Suppose $\phi(I) \neq I$. By Corollary 7, P is ϕ -prime graded ideal of R . Therefore, by Theorem 16, P has one of the following three cases.

Case 1. $P = P_1 \times P_2$, where P_i is a proper graded ideal of R_i with $\psi_i(P_i) = P_i$ for $i = 1, 2$. In this case $\phi(P) = P$, a contradiction.

Case 2. $P = P_1 \times R_2$ where P_1 is ψ_1 -prime graded ideal of R_1 . Since $\psi_2(R_2) \neq R_2$, by Theorem 16(2), P_1 is a prime graded ideal of R_1 and so P is a prime graded ideal of R . We show that $I_2 = R_2$. Since $I \neq \phi(I)$, there exists a homogeneous element (a, b) in $I - \phi(I)$. So $(a, 1)(1, b) = (a, b) \in I - \phi(I)$. If $(a, 1) \notin I$, then $(1, b)$ is not ϕ -prime to I ; hence $(1, b) \in P = P_1 \times R_2$, so $1 \in P_1$, a contradiction. Thus $(a, 1) \in I = I_1 \times I_2$; that is, $1 \in I_2$ and $I_2 = R_2$. Now we prove that I_1 is P_1 -primal graded ideal of R_1 . Let a_1 be a homogeneous element in P_1 . Then $(a_1, 0) \in P_1 \times R_2 = P$. If $(a_1, 0) \in \phi(I) = \psi_1(I_1) \times \psi_2(R_2)$, then $a_1 \in \psi_1(I_1) \subseteq I_1$ so a_1 is not prime to I_1 . Therefore, we may assume that $(a_1, 0) \in \gamma_\phi(I)$. In this case there exists a homogeneous element $(r_1, r_2) \in R - I$ such that $(a_1, 0)(r_1, r_2) \in I - \phi(I)$ so $a_1 r_1 \in I_1 - \psi_1(I_1)$ with $r_1 \in R_1 - I_1$, since $R - I = (R_1 - I_1) \times R_2$, implies that a_1 is not ψ_1 -prime to I_1 ; hence by Remark 17, a_1 is not prime to I_1 . Conversely, let b_1 be a homogeneous element in R_1 such that b_1 is not prime to I_1 . Then there exists a homogeneous element c_1 in $R_1 - I_1$ with $b_1 c_1 \in I_1$. Since $\psi_2(R_2) \neq R_2$, $(b_1, 1)(c_1, 1) = (b_1 c_1, 1) \in I_1 \times R_2 - (I_1 \times \psi_2(R_2)) \subseteq I - \phi(I)$ with $(c_1, 1) \in R - I$. Hence $(b_1, 1)$ is not ϕ -prime to I which implies that $(b_1, 1) \in P = P_1 \times R_2$ and so $b_1 \in P_1$. We have already shown that the set of homogeneous elements in P_1 consists exactly of the homogeneous elements of R_1 that are not prime to I_1 . Hence I_1 is P_1 -primal graded ideal of R_1 so by Proposition 15, I is P -primal graded ideal of R .

Case 3. $P = R_1 \times P_2$, where P_2 is ψ_2 -primal graded ideal of R_2 . The proof of Case 3 is similar to that of Case 2. $\square$

4. ϕ_J - P/J -Primal Graded Ideals

Let R be a commutative graded ring with unity and let J be a proper graded ideal of R . Let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be any function. As a generalization of [6], we define $\phi_J : \mathfrak{S}(R/J) \rightarrow \mathfrak{S}(R/J) \cup \{\emptyset\}$ by $\phi_J(I/J) = (\phi(I) + J)/J$ for every graded ideal $I \in \mathfrak{S}(R)$ with $J \subseteq I$ (and $\phi_J(I/J) = \emptyset$ if $\phi = \phi_\emptyset$).

We leave the trivial proof of the next lemma to the reader.

Lemma 19. Let R be a commutative graded ring with unity and let J be a proper graded ideal of R . Let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup$

$\{\emptyset\}$ be any function. If P is ϕ -prime graded ideal of R containing J , then P/J is ϕ_J -prime graded ideal of R/J .

Lemma 20. Let R be a commutative graded ring with unity and let J be a proper graded ideal of R , and let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be any function. Let P be a graded ideal of R containing J . If P/J is ϕ_J -prime graded ideal of R/J with $J \subseteq \phi(P)$, then P is ϕ -prime graded ideal of R .

Proof. Let a, b be homogeneous elements in R with $ab \in P - \phi(P)$. Then $ab \in J + P$ and $ab \notin J + \phi(P) = \phi(P)$. Thus, $ab \in (J + P) - (J + \phi(P))$, so $\bar{a}\bar{b} \in (J + P)/J - (J + \phi(P))/J$ which implies that $\bar{a}\bar{b} \in P/J - \phi_J(P/J)$; that is, $\bar{a} \in P/J$ or $\bar{b} \in P/J$ so $a \in P$ or $b \in P$. Therefore, P is ϕ -prime graded ideal of R . $\square$

In the next result and under the condition that $J \subseteq \phi(I)$ we prove that I is ϕ -primal graded ideal of R if and only if I/J is ϕ_J -primal graded ideal of R/J .

Theorem 21. Let R be a commutative graded ring with unity and let $\phi : \mathfrak{S}(R) \rightarrow \mathfrak{S}(R) \cup \{\emptyset\}$ be any function. Let I be a proper graded ideal of R , and let J be a graded ideal of R with $J \subseteq \phi(I)$. Then I is ϕ - P -primal graded ideal of R if and only if I/J is ϕ_J - P/J -primal graded ideal of R/J .

Proof. Suppose that I is ϕ - P -primal graded ideal of R with $J \subseteq I$. Then, by Corollary 7, P is ϕ -prime graded ideal of R containing J , so, by Lemma 19, P/J is ϕ_J -prime graded ideal of R/J . We show that I/J is ϕ_J - P/J -primal graded ideal of R/J . That is, we must prove that

$$\frac{P}{J} = \begin{cases} \nu_{\phi_J}\left(\frac{I}{J}\right) + \phi_J\left(\frac{I}{J}\right) & \text{if } \phi_J \neq \phi_\emptyset \\ \nu_{\phi_J}\left(\frac{I}{J}\right) & \text{if } \phi_J = \phi_\emptyset. \end{cases} \quad (13)$$

Let $\bar{a} \in h(P/J)$; then $a \in h(P)$ is not ϕ -prime to I . So there exists $r \in h(R) - h(I)$ with $ra \in I - \phi(I)$. If $a \in J + \phi(I) = \phi(I)$, then $\bar{a} \in \phi_J(I/J)$. So we may assume that $a \notin J + \phi(I) = \phi(I)$. Therefore, $\bar{r}\bar{a} \in I/J - (J + \phi(I))/J = I/J - \phi_J(I/J)$ and because $\bar{r} \notin I/J$ we get that $\bar{a} \in \nu_{\phi_J}(I/J)$.

Now, assume that $\bar{b}$ is a homogeneous element in R/J such that $\bar{b} \in \nu_{\phi_J}(I/J)$. Then there exists a homogeneous element $\bar{r}$ in $R/J - I/J$ such that $\bar{r}\bar{b} \in I/J - \phi_J(I/J)$, so $rb \in I - \phi(I)$ with $r \notin I$. Thus, b is not ϕ -prime to I which implies that $b \in P$, and hence $\bar{b} \in P/J$. Therefore,

$$\frac{P}{J} = \begin{cases} \nu_{\phi_J}\left(\frac{I}{J}\right) + \phi_J\left(\frac{I}{J}\right) & \text{if } \phi_J \neq \phi_\emptyset \\ \nu_{\phi_J}\left(\frac{I}{J}\right) & \text{if } \phi_J = \phi_\emptyset \end{cases} \quad (14)$$

and so I/J is ϕ_J - P/J -primal graded ideal of R/J .

Conversely, suppose that I/J is ϕ_J - P/J -primal graded ideal of R/J with the adjoint graded ideal P/J . We show that I is ϕ - P -primal graded ideal of R . Now, by Corollary 7, P/J is ϕ_J -prime graded ideal of R/J with $J \subseteq P$, so, by Lemma 20, P

is ϕ -prime graded ideal of R . To finish the proof we need to show that

$$P = \begin{cases} \nu_\phi(I) + \phi(I) & \text{if } \phi \neq \phi_0 \\ \nu_\phi(I) & \text{if } \phi = \phi_0. \end{cases} \quad (15)$$

Clearly, $\phi(I) \subseteq I \subseteq P$. Let $a \in h(\nu_\phi(I))$; then there exists a homogeneous element $r \in R - I$ with $ra \in I - \phi(I)$. Since $ra \notin \phi(I) = J + \phi(I)$ we get that $\bar{r}\bar{a} \in I/J - (J + \phi(I))/J = I/J - \phi_J(I/J)$ and $\bar{r} \notin I/J$. So, $\bar{a} \in P/J$ and hence $a \in P$.

Now, let $a \in h(P)$. Suppose that $\bar{a} \in I/J$. Then $a \in I$. If $a \in \phi(I)$, then we are done. Assume that $a \notin \phi(I)$. Then $a \in I - \phi(I)$ and, so, a is not ϕ -prime to I ; hence $a \in \nu_\phi(I)$. Therefore, we may assume that $\bar{a} \notin I/J$, so there exists $\bar{r} \in h(R/J) - h(I/J)$ with $\bar{r}\bar{a} \in I/J - \phi_J(I/J)$, and so $\bar{r}\bar{a} \notin (J + \phi(I))/J$; that is, $ra \notin J + \phi(I) = \phi(I)$. Therefore, $ra \in I - \phi(I)$ with $r \notin I$; that is, $a \in \nu_\phi(I)$. $\square$

By Theorem 21, we get the following result.

Corollary 22. *Let R be a commutative graded ring with unity, and let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function. let J be a graded ideal of R . Then there is one-to-one correspondence between ϕ - P^I -primal graded ideals I of R containing J with $J \subseteq \phi(I)$ and $\phi_J - P^I/J$ -primal graded ideals of R/J .*

5. ϕ_S -Primal Graded Ideals of R_S

Let R be a commutative G -graded ring (for short graded ring) with unity, let S be a multiplicatively closed subset of $h(R)$, and denote by R_S the ring of fractions $S^{-1}R$. We define a grading on R_S by setting

$$(R_S)_g = \left\{ \frac{a}{s} \in R_S : a, s \text{ are homogeneous with } (|a|)(|s|^{-1}) = g \right\}. \quad (16)$$

It is easy to see that R_S is G -graded ring (for short graded ring). Also, for G -graded ideal I of R , $I_S = S^{-1}I$ is G -graded ideal of R_S .

Consider the canonical homomorphism $\rho : R \rightarrow R_S$ which is defined by $r \mapsto r/1$ for all $r \in h(R)$. Then ρ is a homogenous graded homomorphism of degree 0.

Now let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, we define $\phi_S : \mathfrak{F}(R_S) \rightarrow \mathfrak{F}(R_S) \cup \{\emptyset\}$ by $\phi_S(J) = (\phi(\rho^{-1}(J)))_S$ for every $J \in \mathfrak{F}(R_S)$. Note that $\phi_S(J) \subseteq J$, since, for $J \in \mathfrak{F}(R_S)$, we have that $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(J)$ implies $\phi_S(J) \subseteq (\rho^{-1}(J))_S \subseteq J$.

Example 23. Let $R = \mathbb{Z}_6 + u\mathbb{Z}_6$ with $u^2 = 0$. Let $S = \{1, 2, 4\}$, $\phi = \phi_2$. Then S is a multiplicatively closed subset of $h(R)$. If $P = \{0\}$, then one can easily check that P is ϕ_2 - P -primal graded ideal of R . Moreover, $P_S = (\phi(P))_S = \{0\}$; hence $\phi_S(P_S) = P_S$, since $\rho^{-1}(P_S) = \{0, 3, 3u\}$, where $\rho : R \rightarrow R_S$ is the canonical homomorphism. Therefore, P_S is ϕ_S - P_S -primal graded ideal in R_S .

We start by proving the following properties about ϕ -prime graded ideals of R , where $\rho : R \rightarrow R_S$ is the canonical homomorphism and S is a multiplicatively closed subset of $h(R)$.

Lemma 24. *Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, and let I be ϕ -prime graded ideal of R with $I \cap S = \emptyset$; then I_S is ϕ_S -prime graded ideal of R_S .*

Proof. Let $x/s, y/t$ be homogeneous elements in R_S with $(x/s)(y/t) \in I_S - \phi_S(I_S)$; then, for some $u \in S$, $xyu \in I - \phi(I)$, so $x \in I$ or $yu \in I$, and thus $x/s \in I_S$ or $y/t \in I_S$; hence I_S is ϕ_S -prime graded ideal of R_S . $\square$

Theorem 25. *Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, and let P be ϕ -prime graded ideal of R with $h(P) \cap S = \emptyset$. If $\rho^{-1}((\phi(P))_S) \subseteq P$, then $\rho^{-1}(P_S) = P$.*

Proof. It is easy to see that $P \subseteq \rho^{-1}(P_S)$.

Conversely, let x be a homogeneous element in $\rho^{-1}(P_S)$, then for some $s \in S$, $xs \in P$. If $xs \notin \phi(P)$, then $xs \in P - \phi(P)$ and $s \notin P$ so $x \in P$. Therefore, we may assume that $xs \in \phi(P)$, so x is a homogeneous element in $\rho^{-1}((\phi(P))_S)$. Thus,

$$\rho^{-1}(P_S) \subseteq P \cup \rho^{-1}((\phi(P))_S), \quad (17)$$

and since $\rho^{-1}((\phi(P))_S) \subseteq P$, we have that $\rho^{-1}(P_S) = P$. $\square$

Lemma 26. *Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, and let I be ϕ - P -primal graded ideal of R with $h(P) \cap S = \emptyset$. If $a/s \in h(I_S) - h((\phi(I))_S)$, then $a \in I$. Moreover, if $h(\rho^{-1}((\phi(I))_S)) \subseteq h(I)$, then $I = \rho^{-1}(I_S)$.*

Proof. Let $a/s \in h(I_S) - h((\phi(I))_S)$, so $a/s = b/t$ for some $b \in h(I)$ and $t \in S$. In this case $uta = usb \in h(I)$ for some $u \in S$. If $uta \in \phi(I)$, then $a/s = uta/uts \in (\phi(I))_S$ a contradiction. So we have that $uta \in I - \phi(I)$. If $a \notin h(I)$, then ut is not ϕ -prime to I ; so $ut \in h(P) \cap S$ which contradicts the hypothesis. Therefore, $a \in I$.

For the last part, it is clear that $h(I) \subseteq h(\rho^{-1}(I_S))$. Now let a be a homogeneous element in $\rho^{-1}(I_S)$. Then $as \in h(I)$ for some $s \in S$. If $as \notin \phi(I)$ and $a \notin I$, then s is not ϕ -prime to I , so $s \in P \cap S$ a contradiction. So a must be in I . If $as \in \phi(I)$, then $a/1 = (as)/s \in (\phi(I))_S$, and so $a \in \rho^{-1}((\phi(I))_S)$. Therefore, $h(\rho^{-1}(I_S)) = h(I) \cup h(\rho^{-1}((\phi(I))_S)) = h(I)$, since $h(\rho^{-1}((\phi(I))_S)) \subseteq h(I)$. Hence $I = \rho^{-1}(I_S)$. $\square$

Lemma 27. *Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, and let I be ϕ - P -primal graded ideal of R with $h(P) \cap S = \emptyset$. Then $h(\rho^{-1}(I_S)) - h(\rho^{-1}(\phi_S(I_S))) \subseteq h(I) - h(\phi(I))$.*

Proof. Let a be a homogeneous element in $\rho^{-1}(I_S)$ such that $a \notin h(\rho^{-1}(\phi_S(I_S)))$; then $a/1 \in h(I_S) - h(\phi_S(I_S)) \subseteq h(I_S) - h((\phi(I))_S)$ and, by Lemma 26, $a \in I$. If $a \in \phi(I)$, then $a/1 \in (\phi(I))_S \subseteq \phi_S(I_S)$ implies that $a \in h(\rho^{-1}(\phi_S(I_S)))$ a contradiction. Therefore, $a \in h(I) - h(\phi(I))$. $\square$

Let R be a commutative graded ring with unity and M be R -graded module. An element $a \in h(R)$ is called a zero-divisor on M if $am = 0$ for some $m \in h(M)$. We denote by $Z_R(M)$ the set all zero-divisors of $h(R)$ on M .

Corollary 28. Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, and let I be ϕ - P -primal graded ideal of R with $h(P) \cap S = \emptyset$, $S \cap Z_R(R/\phi(I)) = \emptyset$. If $\rho^{-1}((\phi(I))_S) \subseteq I$, then $h((\nu_\phi(I))_S) - h(\phi_S(I_S)) \subseteq h(\nu_{\phi_S}(I_S))$.

Proof. By Lemma 26, if $\rho^{-1}((\phi(I))_S) \subseteq I$, then $\rho^{-1}(I_S) = I$. Let x/s be a homogeneous element in $(\nu_\phi(I))_S - \phi_S(I_S)$; then $x/s = y/t$, where $y \in \nu_\phi(I)$. If $y \in I$, then $y/t = x/s \in I_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$. Therefore we may assume that $y \notin I$. If $y/1 \in I_S$, then $y/t = x/s \in I_S - \phi_S(I_S) \subseteq \nu_{\phi_S}(I_S)$. Therefore we may assume that $y/1 \notin I_S$. So, $y/1$ is a homogeneous element in $(\nu_\phi(I))_S - I_S$. So $uy \in \nu_\phi(I)$ for some $u \in S$ and $uy \notin I$. So there exists $r \in h(R) - h(I)$ such that $ruy \in I - \phi(I)$. If $ry \notin I$, then $u \in \nu_\phi(I) \subseteq P$ a contradiction. Therefore, $ry \in I - \phi(I)$. So $(r/1)(y/1) \in I_S$. If $(r/1)(y/1) \in \phi_S(I_S)$, then there exists $t \in S$ with $try \in \phi(\rho^{-1}(I_S)) = \phi(I)$, so $t \in S \cap Z_R(R/\phi(I))$ a contradiction. Thus $(r/1)(y/1) \in I_S - \phi_S(I_S)$ and $r/1 \notin I_S$. So, $y/1 \in h(\nu_{\phi_S}(I_S))$. Hence $x/s = y/t \in h(\nu_{\phi_S}(I_S))$. $\square$

We recall that if I is a graded ideal in R , then $I \subseteq \rho^{-1}(I_S)$; therefore, we may assume that $(\phi(I))_S \subseteq \phi_S(I_S)$.

Under the condition that $\rho^{-1}((\phi(I))_S) \subseteq I$ for all proper graded ideals I of R , we have the following propositions.

Proposition 29. Let S be a multiplicatively closed subset of $h(R)$ with $1 \in S$, let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, and let I be ϕ - P -primal graded ideal of R with $h(P) \cap S = \emptyset$, $S \cap Z_R(R/\phi(I)) = \emptyset$. Then I_S is ϕ_S - P_S -primal graded ideal of R_S .

Proof. By Lemma 24, P_S is ϕ_S -prime graded ideal of R_S .

To show that I_S is ϕ_S - P_S -primal graded ideal of R_S , we must prove that

$$P_S = \begin{cases} \nu_{\phi_S}(I_S) + \phi_S(I_S) & \text{if } \phi_S \neq \phi_\emptyset \\ \nu_{\phi_S}(I_S) & \text{if } \phi_S = \phi_\emptyset. \end{cases} \quad (18)$$

Clearly, $\phi_S(I_S) \subseteq P_S$; let a/s be a homogenous element in $\nu_{\phi_S}(I_S)$. Then there exists $r/u \in h(R_S) - h(I_S)$ with $(r/u) \cdot (a/s) \in I_S - \phi_S(I_S) \subseteq I_S - (\phi(I))_S$, so $ra \notin \phi(I)$ and, by Lemma 26, $ra \in I$. So, $ra \in I - \phi(I)$, and $r \notin h(I)$. Thus $a \in \nu_\phi(I) \subseteq P$ and hence $a/s \in P_S$.

Conversely, let $a/s \in h(P_S)$ such that $a/s \notin \phi_S(I_S)$. Then $a \in \rho^{-1}(P_S) = P$. If $a/s \in I_S$, then $(1/1)(a/s) \in I_S - \phi_S(I_S)$, $(1/1) \notin I_S$, so a/s is not ϕ_S -prime to I_S ; thus $a/s \in \nu_{\phi_S}(I_S)$. Therefore, we may assume that $a/s \notin I_S$; that is, $ta \notin I$ for every $t \in S$. So, $a \notin I$. Therefore, $a \in h(P) - h(I) \subseteq h(\nu_\phi(I))$. Thus, $a/s \in h((\nu_\phi(I))_S) - h(\phi_S(I_S))$. Since, by Corollary 28, $h((\nu_\phi(I))_S) - h(\phi_S(I_S)) \subseteq h(\nu_{\phi_S}(I_S))$ we have that $a/s \in \nu_{\phi_S}(I_S)$. $\square$

Proposition 30. Let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function, and let J be ϕ_S - Q -primal graded ideal of R_S . Then $\rho^{-1}(Q)$ is ϕ -prime graded ideal of R and $\rho^{-1}(J)$ is ϕ -primal graded ideal of

R with the adjoint graded ideal $\rho^{-1}(Q)$ with $\rho^{-1}(Q) \cap S = \emptyset$, $S \cap Z_R(R/\phi(\rho^{-1}(J))) = \emptyset$. Moreover, $J = (\rho^{-1}(J))_S$.

Proof. To show that $\rho^{-1}(Q)$ is ϕ -prime graded ideal of R , it is enough to prove that $\rho^{-1}(J)$ is ϕ -primal graded ideal of R with the adjoint graded ideal $\rho^{-1}(Q)$. Then, by using Corollary 7, $\rho^{-1}(Q)$ will be ϕ -prime graded ideal of R .

Now, to prove that $\rho^{-1}(J)$ is ϕ -primal graded ideal of R we must show that

$$\rho^{-1}(Q) = \begin{cases} \nu_\phi(\rho^{-1}(J)) + \phi(\rho^{-1}(J)) & \text{if } \phi \neq \phi_\emptyset \\ \nu_\phi(\rho^{-1}(J)) & \text{if } \phi = \phi_\emptyset. \end{cases} \quad (19)$$

But $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(J) \subseteq \rho^{-1}(Q)$. Let a be a homogenous element in $\nu_\phi(\rho^{-1}(J))$ with $a \notin \phi(\rho^{-1}(J))$; then $a/1 \in (\nu_\phi(\rho^{-1}(J)))_S - \phi_S(J)$, since $S \cap Z_R(R/\phi(I)) = \emptyset$ and, by Corollary 28, $(\nu_\phi(\rho^{-1}(J)))_S - \phi_S(J) \subseteq \nu_{\phi_S}(J)$. Thus, $a/1 \in \nu_{\phi_S}(J) \subseteq Q$ and hence $a \in \rho^{-1}(Q)$.

Conversely, let a be a homogeneous element in $\rho^{-1}(Q)$. Then $a/1$ in Q . We may assume that $a \notin \phi(\rho^{-1}(J))$, since $S \cap Z_R(R/\phi(\rho^{-1}(J))) = \emptyset$, $a/1 \notin \phi_S(J)$. If $a/1 \in J$, then $(a/1) \in J - \phi_S(J)$ and since $\phi(\rho^{-1}(J)) \subseteq \rho^{-1}(\phi_S(J))$, $a \in \rho^{-1}(J) - \rho^{-1}(\phi_S(J)) \subseteq \rho^{-1}(J) - \phi(\rho^{-1}(J))$, but $1 \notin \rho^{-1}(J)$, so $a \in \nu_\phi(\rho^{-1}(J))$. If $a/1 \notin J$, then $a/1 \in Q - J$ and so $a/1 \in \nu_{\phi_S}(J)$. Let x/s be a homogeneous element in $R_S - J$, with $(a/1)(x/s) \in J - \phi_S(J)$; then $ax \in \rho^{-1}(J) - \rho^{-1}(\phi_S(J)) \subseteq \rho^{-1}(J) - \phi(\rho^{-1}(J))$, since $ax/1 \in J$ and $ax/1 \notin \phi_S(J)$, for if $ax/1 \in \phi_S(J)$, then $ax/s \in \phi_S(J)$, a contradiction. Thus we have that $ax \in \rho^{-1}(J) - \phi(\rho^{-1}(J))$ and $x \notin \rho^{-1}(J)$, since $x/s \notin J$. Therefore, $a \in \nu_\phi(\rho^{-1}(J))$ and so $\rho^{-1}(J)$ is ϕ -primal graded ideal of R with the adjoint graded ideal $\rho^{-1}(Q)$.

Finally, we show that $J = (\rho^{-1}(J))_S$. Clearly, $J \subseteq (\rho^{-1}(J))_S$. Conversely, let x/s be a homogeneous element in $(\rho^{-1}(J))_S$. Then $xt \in \rho^{-1}(J)$ for some $t \in S$. Thus, $xt/1 \in \rho(\rho^{-1}(J)) = J$, and hence $(xt/1)(1/st) = x/s \in J$. Therefore, $J = (\rho^{-1}(J))_S$. $\square$

Under the condition that $\rho^{-1}((\phi(I))_S) \subseteq I$ for all proper graded ideals I of R and by using Propositions 29 and 30 we have the following result.

Corollary 31. Let R be a commutative graded ring with unity. Let S be a multiplicatively closed subset of $h(R)$, and let $\phi : \mathfrak{F}(R) \rightarrow \mathfrak{F}(R) \cup \{\emptyset\}$ be any function. Then there is one-to-one correspondence between ϕ - P^I -primal graded ideals I of R and ϕ_S - P_S^I -primal graded ideals I_S of R_S , where P^I is ϕ -prime graded ideal of R with $P^I \cap S = \emptyset$, $S \cap Z_R(R/\phi(I)) = \emptyset$.

I would like to remark that “ ϕ -theory” introduced here has nothing in common with “ ϕ -theories” considered previously by Badawi and Lucas in [7].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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