Research Article

Some Properties of Serre Subcategories in the Graded Local Cohomology Modules

Feysal Hassani and Rasul Rasuli

Department of Mathematics, Payame Noor University (PNU), Tehran, Iran

Correspondence should be addressed to Feysal Hassani; feysal.hassani.pnu@gmail.com

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Let $R = \oplus_{n \geq 0} R_n$ be a standard homogeneous Noetherian ring with local base ring $(R_0, \mathfrak{m}_0)$ and let M be a finitely generated graded R -module. Let $H_{R_+}^i(M)$ be the i th local cohomology module of M with respect to $R_+ = \oplus_{n > 0} R_n$. Let $\mathcal{S}$ be a Serre subcategory of the category of R -modules and let i be a nonnegative integer. In this paper, if $\dim R_0 \leq 1$, then we investigate some conditions under which the R -modules $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M)$, $\Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M))$ and $H_{\mathfrak{m}_0 R}^1(H_{R_+}^i(M))$ are in $\mathcal{S}$ for all $i \geq 0$. Also, we prove that if $\dim R_0 \leq 2$, then the graded R -module $H_{\mathfrak{m}_0}^1(H_{R_+}^i(M))$ is in $\mathcal{S}$ for all $i \geq 0$. Finally, we prove that if n is the biggest integer such that $H_a^i(M) \notin \mathcal{S}$, then $H_{R_+}^i(M)/\mathfrak{m}_0 H_{R_+}^i(M) \in \mathcal{S}$ for all $i \geq n$.

1. Introduction

Let $R = \oplus_{n \geq 0} R_n$ be a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. So, R_0 is a Noetherian ring and there are finitely many elements $l_1, l_2, \dots, l_r \in R_1$ such that $R = R_0[l_1, l_2, \dots, l_r]$. Let $R_+ = \oplus_{n > 0} R_n$ denote the irrelevant ideal of R and let $\mathfrak{m} = \mathfrak{m}_0 \oplus R_+$ denote the graded maximal ideal of R . Finally, let $M = \oplus_{n \in \mathbb{Z}} M_n$ be a finitely generated graded R -module. Recall that a class of R -modules is a Serre subcategory of the category of R -modules when it is closed under taking submodules, quotients, and extensions. Always, $\mathcal{S}$ stands for a Serre subcategory of the category of R -modules. Hence, if $L \rightarrow M \rightarrow N$ is an exact sequence of the category of R -modules and R -homomorphisms such that both end terms belong to $\mathcal{S}$, then M also belongs to $\mathcal{S}$. Note that the following subcategories are examples of Serre subcategory of the category of R -modules: finite R -modules; coatomic R -modules [1]; minimax R -modules [2]; α -cofinite R -modules; weakly Laskerian R -modules; Matlis reflexive R -modules; and trivially the zero R -modules. Recently, some results have been proved concerning the local cohomology modules $H_a^i(M)$ of a module M in some certain Serre subcategory of the category of modules (cf. [3–7]). Aghapournahr and Melkersson (cf. [4]) gave a condition on a Serre subcategory

$\mathcal{S}$. To give more details, let $\mathfrak{a}$ be an ideal of R , let M be an R -module, and let $(0;_M \mathfrak{a}) = \{x \in M \mid \mathfrak{a}x = 0\}$. The R -module M is said to satisfy C_a condition on $\mathcal{S}$ whenever the following condition holds:

$$\text{If } M = \Gamma_a(M) \text{ and if } (0;_M \mathfrak{a}) \text{ is in } \mathcal{S} \text{ then } M \text{ is in } \mathcal{S}. \quad (1)$$

Then $\mathcal{S}$ is said to satisfy C_a condition, whenever, every R -module satisfies C_a condition on $\mathcal{S}$. For example, the class of zero modules and Artinian R -modules satisfy the condition C_a .

In this paper, according to $\dim R_0 \leq 1$, we investigate some conditions under which the R -modules $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M)$, $\Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M))$ and $H_{\mathfrak{m}_0 R}^1(H_{R_+}^i(M))$ are in $\mathcal{S}$ for all $i \geq 0$. Next, we prove that if $\dim R_0 \leq 2$, then $H_{\mathfrak{m}_0}^1(H_{R_+}^i(M))$ is in $\mathcal{S}$ for all $i \geq 0$. Also if $\mathcal{S}$ satisfies C_a condition, then under some conditions, R -modules $H_{R_+}^i(H_{\mathfrak{m}_0 R}^1(M))$ and $\Gamma_{R_+}(H_{\mathfrak{m}_0 R}^1(M))$ belong to $\mathcal{S}$. Finally, we prove that if n is the biggest integer such that $H_a^i(M) \notin \mathcal{S}$, then $H_{R_+}^i(M)/\mathfrak{m}_0 H_{R_+}^i(M) \in \mathcal{S}$ for all $i \geq n$.

In Section 2, we give some notations and definition that we used in Section 3.

2. Preliminaries

Let R be a ring. The spectrum of R is denoted by $\text{Spec } R$, that is, the set of prime ideals of R with the Zariski topology, which is the topology where the closed sets are $V(\mathfrak{a}) = \{\mathfrak{p} \in \text{Spec } R \mid \mathfrak{a} \subseteq \mathfrak{p}\}$ for ideals $\mathfrak{a} \subseteq R$.

The height of a prime ideal $\mathfrak{p}$, denoted by $\text{height } \mathfrak{p}$, is defined by the supremum of integers t such that there exists a chain of prime ideals $\mathfrak{p} = \mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \mathfrak{p}_2 \supsetneq \cdots \supsetneq \mathfrak{p}_t$, where $\mathfrak{p}_i \in \text{Spec } R$. The Krull dimension of R is defined as $\dim R = \sup\{\text{height } \mathfrak{p} \mid \mathfrak{p} \in \text{Spec } R\}$.

Consider the complex $K^\bullet$

$$\cdots \longrightarrow K^{i-1} \xrightarrow{\delta_K^{i-1}} K^i \xrightarrow{\delta_K^i} K^{i+1} \longrightarrow \cdots \quad (2)$$

The differential on $K^\bullet$ is denoted by δ_K , or sometimes simply δ . We say that $K^\bullet$ is bounded below if $K^i = 0$ for $i \ll 0$ and bounded above if $K^i = 0$ for $i \gg 0$; if both conditions hold, then $K^\bullet$ is bounded. On occasion, we consider a complex as a graded module with a graded endomorphism of degree 1.

Let M be an R -module. An injective resolution of M is a complex $E^\bullet$ of injective R -modules $0 \rightarrow E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots$, equipped with a quasi-isomorphism of complexes $l: M \rightarrow E^\bullet$; here one views M as a complex concentrated in degree 0. Said otherwise, a complex $E^\bullet$ of injective modules is a resolution of M if there is an exact sequence $0 \rightarrow M \xrightarrow{l} E^0 \rightarrow E^1 \rightarrow E^2 \rightarrow \cdots$.

Let M and N be R -modules. Then $\{\text{Ext}_R^i(M, -)\}_{i \geq 0}$ are the right derived functors of $\text{Hom}_R(M, -)$, and $\{\text{Tor}_i^R(-, N)\}_{i \geq 0}$ are the left derived functors of $-\otimes_R N$.

For an exact sequence of R -modules $0 \rightarrow J \rightarrow M \rightarrow L \rightarrow 0$ and each R -module N , one has exact sequences:

$$\begin{aligned} \cdots &\longrightarrow \text{Ext}_R^i(N, M) \longrightarrow \text{Ext}_R^i(N, L) \longrightarrow \text{Ext}_R^{i+1}(N, J) \\ &\longrightarrow \text{Ext}_R^{i+1}(N, M) \longrightarrow \cdots, \\ \cdots &\longrightarrow \text{Ext}_R^i(M, N) \longrightarrow \text{Ext}_R^i(J, N) \longrightarrow \text{Ext}_R^{i+1}(L, N) \\ &\longrightarrow \text{Ext}_R^{i+1}(M, N) \longrightarrow \cdots, \\ \cdots &\longrightarrow \text{Tor}_{i+1}^R(M, N) \longrightarrow \text{Tor}_{i+1}^R(L, N) \\ &\longrightarrow \text{Tor}_i^R(J, N) \longrightarrow \text{Tor}_i^R(M, N) \longrightarrow \cdots. \end{aligned} \quad (3)$$

For an R -module M , the $\mathfrak{a}$ -torsion of M is defined by $\Gamma_{\mathfrak{a}}(M) := \bigcup_{n \in \mathbb{N}} (0 :_M \mathfrak{a}^n)$. Observe that $\Gamma_{\mathfrak{a}}(M)$ is a submodule of M and $\Gamma_{\mathfrak{a}}(-)$ is a left exact $\mathfrak{a}$ -torsion functor of R -modules and R -homomorphisms. Let $n \in \mathbb{N}_0$ and let M be an R -module. Choose an injective resolution of M , so that we have an exact sequence

$$0 \longrightarrow M \xrightarrow{l} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow \cdots \quad (4)$$

Then, apply the functor $\Gamma_{\mathfrak{a}}(-)$ to the resolving cocomplex

$$\cdots \longrightarrow 0 \xrightarrow{d^{-1}} E^0 \xrightarrow{d^0} E^1 \xrightarrow{d^1} E^2 \xrightarrow{d^2} E^3 \longrightarrow \cdots \quad (5)$$

in order to obtain the cocomplex

$$\begin{aligned} \cdots &\longrightarrow 0 \xrightarrow{\Gamma_{\mathfrak{a}}(d^{-1})} \Gamma_{\mathfrak{a}}(E^0) \xrightarrow{\Gamma_{\mathfrak{a}}(d^0)} \Gamma_{\mathfrak{a}}(E^1) \xrightarrow{\Gamma_{\mathfrak{a}}(d^1)} \Gamma_{\mathfrak{a}}(E^2) \\ &\xrightarrow{\Gamma_{\mathfrak{a}}(d^2)} \cdots. \end{aligned} \quad (6)$$

Then, the n th local cohomology module of M with respect to $\mathfrak{a}$ is defined as the R -module $H_{\mathfrak{a}}^n(M) = \ker(\Gamma_{\mathfrak{a}}(d^n))/\text{Im}(\Gamma_{\mathfrak{a}}(d^{n-1}))$. If $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ is an exact sequence of R -modules and R -homomorphisms, then we have the following long exact sequence:

$$\begin{aligned} 0 &\longrightarrow H_{\mathfrak{a}}^0(L) \xrightarrow{H_{\mathfrak{a}}^0(f)} H_{\mathfrak{a}}^0(M) \xrightarrow{H_{\mathfrak{a}}^0(g)} H_{\mathfrak{a}}^0(N) \longrightarrow H_{\mathfrak{a}}^1(L) \\ &\xrightarrow{H_{\mathfrak{a}}^1(f)} H_{\mathfrak{a}}^1(M) \xrightarrow{H_{\mathfrak{a}}^1(g)} H_{\mathfrak{a}}^1(N) \longrightarrow \cdots. \end{aligned} \quad (7)$$

As a general reference to homological and commutative algebra, we refer to [8].

3. Main Results

In this paper, $R = \bigoplus_{n \geq 0} R_n$ is a Noetherian homogeneous ring with local base ring $(R_0, \mathfrak{m}_0)$. We start off this section with some technical lemmas to rich our main aims in this work.

Lemma 1. *Let $H \in \mathcal{S}$ be an R -module and let M_0 be a finitely generated R_0 -module. Then for any $i \geq 0$ the R -module $\text{Tor}_i^{R_0}(M_0, H) \in \mathcal{S}$.*

Proof. Let

$$\begin{aligned} \cdots &\longrightarrow F_{i+1} \longrightarrow F_i \longrightarrow F_{i-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \\ &\longrightarrow M \longrightarrow 0 \end{aligned} \quad (8)$$

be a minimal free resolution of M_0 . If $F_i = R^{n_i}$ for some integer n_i , then $\text{Tor}_i^{R_0}(M_0, H)$ is an R_0 -subquotient of $F_i \otimes_{R_0} H \cong H^{n_i}$. Thus $\text{Tor}_i^{R_0}(M_0, H) \in \mathcal{S}$. $\square$

Lemma 2. *If $M, R/\mathfrak{a} \in \mathcal{S}$, then $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$ for all $i \geq 0$.*

Proof. Let $F_\bullet : \cdots \rightarrow F_1 \rightarrow F_0 \rightarrow 0$ be a finite free resolution of $R/\mathfrak{a}^n$. If $F_i = R^{n_i}$ for some integer n_i , then $\text{Ext}_R^i(R/\mathfrak{a}^n, M) = H^i(\text{Hom}_R(F_\bullet, M))$ is a subquotient of M^{n_i} . Since $\mathcal{S}$ is a Serre class, $\text{Ext}_R^i(R/\mathfrak{a}^n, M) \in \mathcal{S}$. From definition, we have $H_{\mathfrak{a}}^i(M) = \lim_{n \in \mathbb{N}} \text{Ext}_R^i(R/\mathfrak{a}^n, M)$ and so $H_{\mathfrak{a}}^i(M) \in \mathcal{S}$ for all $i \geq 0$. $\square$

An immediate result of the above lemmas is the following result.

Corollary 3. *If $\dim(R_0) = 0$ and $M \in \mathcal{S}$, then $R_0 = R/R_+$ is in $\mathcal{S}$ and $H_{R_+}^i(M) \in \mathcal{S}$ for all $i \geq 0$.*

Lemma 4. *If $M \in \mathcal{S}$, then $H_{R_+}^i(\Gamma_{\mathfrak{m}_0 R}(M)) \in \mathcal{S}$ as an R -module for all $i \geq 0$.*

Proof. Since $\Gamma_{\mathfrak{m}_0 R}(M)$ is a finitely generated $\mathfrak{m}_0$ -torsion module, there exists an $n \geq 0$ such that $\mathfrak{m}_0^n \Gamma_{\mathfrak{m}_0 R}(M) = 0$. Then we obtain the $R/\mathfrak{m}_0^n$ -module $\Gamma_{\mathfrak{m}_0 R}(M)$. If $i \geq 0$, then from the Graded Independence Theorem (cf. ([8], 13.1.6)) there is an isomorphism of graded R -modules $H_{R_+}^i(\Gamma_{\mathfrak{m}_0 R}(M)) \cong H_{(R/\mathfrak{m}_0^n)_+}^i(\Gamma_{\mathfrak{m}_0 R}(M))$. Since $\dim(R/\mathfrak{m}_0^n)_0 = \dim R_0/\mathfrak{m}_0^n = 0$, Corollary 3 implies that $H_{R_+}^i(\Gamma_{\mathfrak{m}_0 R}(M)) \in \mathcal{S}$. $\square$

Lemma 5. Let $i \geq 0$ and $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M/\Gamma_{\mathfrak{m}_0 R}(M)) \in \mathcal{S}$; then $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M) \in \mathcal{S}$.

Proof. Let $L := M/\Gamma_{\mathfrak{m}_0 R}(M)$. Then we have the exact rows:

$$H_{R_+}^i(\Gamma_{\mathfrak{m}_0 R}(M)) \xrightarrow{f} H_{R_+}^i(M) \xrightarrow{g} H_{R_+}^i(L) \quad (9)$$

$$\begin{aligned} &\longrightarrow H_{R_+}^{i+1}(\Gamma_{\mathfrak{m}_0 R}(M)), \\ 0 &\longrightarrow \frac{H_{R_+}^i(M)}{\text{Im } f} \longrightarrow H_{R_+}^i(L) \\ &\longrightarrow \frac{H_{R_+}^i(L)}{\text{Im } g} \longrightarrow 0, \end{aligned} \quad (10)$$

such that $h : H_{R_+}^i(M) \rightarrow H_{R_+}^i(M)/\text{Im } f$ is an epimorphism and $k : H_{R_+}^i(L)/\text{Im } g \rightarrow H_{R_+}^{i+1}(\Gamma_{\mathfrak{m}_0 R}(M))$ is a monomorphism. Since $T := H_{R_+}^i(L)/\text{Im } g$ is a submodule of $H_{R_+}^{i+1}(\Gamma_{\mathfrak{m}_0 R}(M))$, Lemma 4 leads to $T \in \mathcal{S}$. Also, from Lemma 1, we get $\text{Tor}_1^{R_0}(R_0/\mathfrak{m}_0, T) \in \mathcal{S}$. The short exact sequence (10) deduces the following exact sequence:

$$\begin{aligned} \text{Tor}_1^{R_0}\left(\frac{R_0}{\mathfrak{m}_0}, T\right) &\longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \frac{H_{R_+}^i(M)}{\text{Im } f} \\ &\longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^i(L). \end{aligned} \quad (11)$$

According to our assumption, $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(L) \in \mathcal{S}$ and then $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M)/\text{Im } f \in \mathcal{S}$. From the exact sequence (9), we have the exact sequence $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M/\Gamma_{\mathfrak{m}_0 R}(M)) \rightarrow R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M) \rightarrow R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M)/\text{Im } f$, and hence $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M) \in \mathcal{S}$. $\square$

Theorem 6. Let $\dim R_0 \leq 1$ and $i \geq 0$; then the following assertions hold:

- (1) If $M \in \mathcal{S}$, then $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M) \in \mathcal{S}$.
- (2) If $x \in \mathfrak{m}_0$, such that $\sqrt{xR_0} = \mathfrak{m}_0$, and $H_{(R_+, x)}^i(M) \in \mathcal{S}$, then the R -modules $\Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M))$ and $H_{\mathfrak{m}_0 R}^1(H_{R_+}^i(M))$ are in $\mathcal{S}$.

Proof. (1) If $\dim R_0 = 0$, then from Corollary 3 we have $H_{R_+}^i(M) \in \mathcal{S}$ and then from Lemma 1 we get $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M) \in \mathcal{S}$. Let $\dim R_0 = 1$ and $\Gamma_{\mathfrak{m}_0 R}(M) = 0$.

Then there exists an M -regular element $x \in \mathfrak{m}_0$ which avoids all minimal primes of R_0 . Now, the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow \frac{M}{xM} \longrightarrow 0 \quad (12)$$

induces an exact sequence

$$\begin{aligned} H_{R_+}^{i-1}\left(\frac{M}{xM}\right) &\longrightarrow H_{R_+}^i(M) \xrightarrow{x} H_{R_+}^i(M) \\ &\xrightarrow{f} H_{R_+}^i\left(\frac{M}{xM}\right). \end{aligned} \quad (13)$$

Since x avoids all minimal primes of R_0 , $\dim R_0/xR_0 = 0$. Then by the Independence Theorem, $H_{R_+}^i(M/xM) \in \mathcal{S}$ and then $\text{Im } f \in \mathcal{S}$. The exact sequence (13) gives the exact sequence

$$\begin{aligned} \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^i(M) &\xrightarrow{x} \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^i(M) \\ &\longrightarrow \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \text{Im } f \longrightarrow 0. \end{aligned} \quad (14)$$

As $x \in \mathfrak{m}_0$, the multiplication map x is zero. Then there is an isomorphism

$$\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^i(M) \cong \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} \text{Im } f. \quad (15)$$

Therefore, $R_0/\mathfrak{m}_0 \otimes_{R_0} H_{R_+}^i(M) \in \mathcal{S}$.

(2) Let $x \in \mathfrak{m}_0$ such that $\sqrt{xR_0} = \mathfrak{m}_0$. Therefore, for all $i \geq 0$, there is an exact sequence of graded R -modules

$$\begin{aligned} H_{R_+}^{i-1}(M) &\xrightarrow{f_x^{i-1}} H_{R_+}^{i-1}(M)_x \longrightarrow H_{(R_+, x)}^i(M) \\ &\longrightarrow H_{R_+}^i(M) \xrightarrow{f_x^i} H_{R_+}^i(M)_x \end{aligned} \quad (16)$$

such that f_x^{i-1} and f_x^i are the natural homomorphisms (cf. [8], 13.1.12). Now from

$$\begin{aligned} \text{coker}(f_x^{i-1}) &\cong H_{xR}^1(H_{R_+}^i(M)) \cong H_{\mathfrak{m}_0 R}^1(H_{R_+}^i(M)), \\ \ker(f_x^i) &\cong \Gamma_{xR}(H_{R_+}^i(M)) = \Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M)), \end{aligned} \quad (17)$$

we obtain an exact sequence

$$\begin{aligned} 0 &\longrightarrow H_{\mathfrak{m}_0 R}^1(H_{R_+}^{i-1}(M)) \longrightarrow H_{(R_+, x)}^i(M) \\ &\longrightarrow \Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M)) \longrightarrow 0. \end{aligned} \quad (18)$$

Therefore, $\Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M))$ and $H_{\mathfrak{m}_0 R}^1(H_{R_+}^i(M))$ are in $\mathcal{S}$. $\square$

Corollary 7. Let $\dim R_0 \leq 1$ and $i \geq 0$. Moreover, let $\mathfrak{q}_0 \subseteq R_0$ be an $\mathfrak{m}_0$ -primary ideal. Then

- (1) the R -module $R_0/\mathfrak{m}_0/\mathfrak{q}_0 H_{R_+}^i(M)$ is in $\mathcal{S}$;
- (2) the R -module $(0 :_{H_{R_+}^i(M)} \mathfrak{q}_0)$ is in $\mathcal{S}$.

Proof. (1) Since $\mathfrak{q}_0$ is an $\mathfrak{m}_0$ -primary ideal, there is some $n \geq 0$ such that $\mathfrak{m}_0^n \subseteq \mathfrak{q}_0$. Then it suffices to prove that $R_0/\mathfrak{m}_0^n \otimes_{R_0} H_{R_+}^i(M) \in \mathcal{S}$. We proceed the assertion by induction on n . For $n = 1$, use part (1) of Theorem 6. Now suppose, inductively, that $n > 1$ and the result has been proved for all values smaller than n . We have the exact sequence

$$\begin{aligned} \frac{\mathfrak{m}_0^{n-1}}{\mathfrak{m}_0^n} \otimes_{R_0} H_{R_+}^i(M) &\longrightarrow \frac{R_0}{\mathfrak{m}_0^n} \otimes_{R_0} H_{R_+}^i(M) \\ &\longrightarrow \frac{R_0}{\mathfrak{m}_0^{n-1}} \otimes_{R_0} H_{R_+}^i(M), \end{aligned} \quad (19)$$

and the natural isomorphism

$$\begin{aligned} \frac{\mathfrak{m}_0^{n-1}}{\mathfrak{m}_0^n} \otimes_{R_0} H_{R_+}^i(M) &\cong \mathfrak{m}_0^{n-1} \\ &\otimes_{R_0} \left(\frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^i(M) \right). \end{aligned} \quad (20)$$

Now, use part (1) of Theorem 6 and Lemma 1.

(2) Since $(0 :_{H_{R_+}^i(M)} \mathfrak{q}_0) \subseteq \Gamma_{\mathfrak{m}_0 R}(H_{R_+}^i(M))$, part (2) of Theorem 6 implies that $(0 :_{H_{R_+}^i(M)} \mathfrak{q}_0) \in \mathcal{S}$. $\square$

The following result is an extension of Theorem 6 to the case $\dim R_0 = 2$.

Theorem 8. *If $\dim R_0 \leq 2$, then $H_{\mathfrak{m}_0}^1(H_{R_+}^i(M)) \in \mathcal{S}$ for all $i \geq 0$.*

Proof. If $\dim R_0 \leq 1$, this statement is the same as Theorem 6. Let $\dim R_0 = 2$, then there is a system of parameters (x, y) of R_0 such that $H_{\mathfrak{m}_0 R}^1(H_{R_+}^i(M)) = H_{(x,y)R}^1(H_{R_+}^i(M))$. From the exact sequence

$$\begin{aligned} 0 \longrightarrow H_{yR}^1(H_{R_+}^{i-1}(M)) &\longrightarrow H_{(y,R_+)}^i(M) \\ &\longrightarrow \Gamma_{yR}(H_{R_+}^i(M)) \longrightarrow 0, \end{aligned} \quad (21)$$

which obtains an epimorphism $H_{xR}^1(H_{(y,R_+)}^i(M)) \rightarrow H_{xR}^1(\Gamma_{yR}(H_{R_+}^i(M)))$, also, there is a monomorphism

$$H_{xR}^1(H_{(y,R_+)}^i(M)) \longrightarrow H_{(x,y,R_+)}^{i+1}(M) = H_{\mathfrak{m}}^{i+1}(M). \quad (22)$$

Since $H_{\mathfrak{m}}^{i+1}(M) \in \mathcal{S}$, $H_{xR}^1(\Gamma_{yR}(H_{R_+}^i(M))) \in \mathcal{S}$. Application of the functor $\Gamma_{xR}(-)$ to the monomorphism $H_{yR}^1(H_{R_+}^i(M)) \rightarrow H_{(y,R_+)}^{i+1}(M)$ yields a monomorphism $\Gamma_{xR}(H_{yR}^1(H_{R_+}^i(M))) \rightarrow \Gamma_{xR}(H_{(y,R_+)}^{i+1}(M))$. Also, there is an epimorphism

$$H_{\mathfrak{m}}^{i+1}(M) = H_{(x,y,R_+)}^{i+1}(M) \longrightarrow \Gamma_{xR}(H_{(y,R_+)}^{i+1}(M)). \quad (23)$$

Because $H_{\mathfrak{m}}^{i+1}(M) \in \mathcal{S}$, $\Gamma_{xR}(H_{yR}^1(H_{R_+}^i(M))) \in \mathcal{S}$. The exact sequence

$$\begin{aligned} 0 \longrightarrow H_{xR}^1(\Gamma_{yR}(H_{R_+}^i(M))) &\longrightarrow H_{(x,y)R}^1(H_{R_+}^i(M)) \\ &\longrightarrow \Gamma_{xR}(H_{yR}^1(H_{R_+}^i(M))) \longrightarrow 0 \end{aligned} \quad (24)$$

implies that $H_{\mathfrak{m}_0}^1(H_{R_+}^i(M)) \in \mathcal{S}$. $\square$

Proposition 9. *Let $M \in \mathcal{S}$ and M satisfies C_a condition on $\mathcal{S}$. If R_0 is a local ring of dimension one, then $H_{R_+}^i(H_{\mathfrak{m}_0 R}^1(M)) \in \mathcal{S}$ for all $i \geq 0$.*

Proof. Since $H_{\mathfrak{m}_0 R}^1(M) \cong H_{\mathfrak{m}_0 R}^1(M/\Gamma_{\mathfrak{m}_0 R}(M))$, one can assume that $\Gamma_{\mathfrak{m}_0 R}(M) = 0$. Then, there exists an element $x_0 \in \mathfrak{m}_0$ such that it is an M -sequence. By the exact sequence

$$0 \longrightarrow M \xrightarrow{x_0} M \longrightarrow \frac{M}{x_0 M} \longrightarrow 0 \quad (25)$$

and by application of the functor $H_{R_+}^i(-)$ to this exact sequence, we have the following exact sequence

$$\begin{aligned} H_{R_+}^i\left(\Gamma_{\mathfrak{m}_0 R}\left(\frac{M}{x_0 M}\right)\right) &\longrightarrow H_{R_+}^i(H_{\mathfrak{m}_0 R}^i(M)) \\ &\xrightarrow{x_0} H_{R_+}^i(H_{\mathfrak{m}_0 R}^i(M)) \longrightarrow H_{R_+}^{i+1}\left(\Gamma_{\mathfrak{m}_0 R}\left(\frac{M}{x_0 M}\right)\right). \end{aligned} \quad (26)$$

Lemma 4 leads to $H_{R_+}^i(\Gamma_{\mathfrak{m}_0 R}(M/x_0 M)) \in \mathcal{S}$ and then $(0 :_{H_{R_+}^i(H_{\mathfrak{m}_0 R}^1(M))} x_0) \in \mathcal{S}$. Furthermore, by the basic properties of local cohomology, we have $\Gamma_a(H_{R_+}^i(H_{\mathfrak{m}_0 R}^1(M))) = H_{R_+}^i(H_{\mathfrak{m}_0 R}^1(M))$. Since $H_{R_+}^i(H_{\mathfrak{m}_0 R}^1(M))$ satisfies C_a condition on $\mathcal{S}$, $H_{R_+}^i(H_{\mathfrak{m}_0 R}^1(M)) \in \mathcal{S}$ for all $i \geq 0$. $\square$

Proposition 10. *Suppose that M satisfies C_a condition on $\mathcal{S}$ and n is the least integer such that $H_{\mathfrak{m}_0 R}^n(M) \notin \mathcal{S}$. Then $\Gamma_{R_+}(H_{\mathfrak{m}_0 R}^1(M)) \in \mathcal{S}$ for all $i \leq n$.*

Proof. If $i < n$, then $H_{\mathfrak{m}_0 R}^i(M) \in \mathcal{S}$ and therefore $\Gamma_{R_+}(H_{\mathfrak{m}_0 R}^i(M)) \in \mathcal{S}$. Let $i = n$ and from the isomorphism $H_{\mathfrak{m}_0 R}^i(M) \cong H_{\mathfrak{m}_0 R}^i(M/\Gamma_{\mathfrak{m}_0 R}(M))$ for all i , one can suppose that $\Gamma_{\mathfrak{m}_0 R}(M) = 0$. Therefore, there exists an element $x \in \mathfrak{m}_0$ such that it is an M -sequence. In view of the exact sequence

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow \frac{M}{xM} \longrightarrow 0, \quad (27)$$

we have the exact sequence

$$\begin{aligned} H_{\mathfrak{m}_0 R}^{n-1}(M) &\xrightarrow{f} H_{\mathfrak{m}_0 R}^{n-1}\left(\frac{M}{xM}\right) \longrightarrow H_{\mathfrak{m}_0 R}^n(M) \\ &\xrightarrow{x} H_{\mathfrak{m}_0 R}^n(M). \end{aligned} \quad (28)$$

By the above sequence, we obtain the exact sequence

$$0 \longrightarrow \ker f \longrightarrow H_{\mathfrak{m}_0 R}^{n-1}(M) \longrightarrow \operatorname{Im} f \longrightarrow 0, \quad (29)$$

and isomorphism $\operatorname{Im} f \cong (0 :_{H_{\mathfrak{m}_0 R}^n(M)} x)$. Since $H_{\mathfrak{m}_0 R}^{n-1}(M) \in \mathcal{S}$, $\operatorname{Im} f \in \mathcal{S}$ and $(0 :_{H_{\mathfrak{m}_0 R}^n(M)} x) \in \mathcal{S}$. Thus $\Gamma_{R_+}(0 :_{H_{\mathfrak{m}_0 R}^n(M)} x) = (0 :_{\Gamma_{R_+}(H_{\mathfrak{m}_0 R}^n(M))} x) \in \mathcal{S}$. Then $\Gamma_{R_+}(H_{\mathfrak{m}_0 R}^n(M)) \in \mathcal{S}$, because $\Gamma_{R_+}(H_{\mathfrak{m}_0 R}^n(M))$ is x -torsion. $\square$

Definition 11. Let $\mathfrak{a}$ be a graded ideal of R and $M \in \mathcal{S}$. Define $t_{\mathfrak{a}}^{\mathcal{S}}(M) = \sup\{i \mid H_{\mathfrak{a}}^i(M) \notin \mathcal{S}\}$.

Lemma 12. Let $x \in \mathfrak{m}$ be a homogeneous nonzero divisor of M . Then

$$t_{R_+}^{\mathcal{S}}\left(\frac{M}{xM}\right) \leq t_{R_+}^{\mathcal{S}}(M). \quad (30)$$

Proof. Let $\deg x = d$. Since x is a nonzero divisor of M , there is an exact sequence

$$0 \longrightarrow M(-d) \xrightarrow{x} M \longrightarrow \frac{M}{xM} \longrightarrow 0. \quad (31)$$

Application of the functor $H_{R_+}^i(-)$ to the above exact sequence induces the exact sequence

$$\begin{aligned} H_{R_+}^i(M) &\longrightarrow H_{R_+}^i\left(\frac{M}{xM}\right) \xrightarrow{x} H_{R_+}^{i+1}(M)(-d) \\ &\longrightarrow H_{R_+}^{i+1}(M). \end{aligned} \quad (32)$$

If $t_{R_+}^{\mathcal{S}}(M) = n$, then for all $i > n$ we have $H_{R_+}^i(M) \in \mathcal{S}$. This means that $H_{R_+}^i(M/xM) \in \mathcal{S}$. Thus, $t_{R_+}^{\mathcal{S}}(M/xM) \leq n$. $\square$

Remark 13. (1) Any local flat morphism of local Noetherian rings is faithfully flat. So, if $\hat{R}_0$ is flat over R_0 and $\hat{\mathfrak{m}}_0 = \mathfrak{m}_0 \hat{R}_0$, then $\hat{R}_0$ is faithfully flat over R_0 . Moreover, if $(\hat{R}_0, \hat{\mathfrak{m}}_0)$ is a faithfully flat local R_0 -algebra, then from ([9], Theorem 1) we obtain that $A \in \mathcal{S}$ of the graded R -module if and only if $\hat{A} := (\hat{R}_0 \otimes_{R_0} A) \in \mathcal{S}$ of the graded module over $\hat{R} := \hat{R}_0 \otimes_{R_0} R$.

(2) If $(\hat{R}_0, \hat{\mathfrak{m}}_0)$ is a faithfully flat local R_0 -algebra, then $t_{R_+}^{\mathcal{S}}(M) = t_{(\hat{R}_0 \otimes_{R_0} R)_+}^{\mathcal{S}}(\hat{R}_0 \otimes_{R_0} M)$.

Theorem 14. Let $n = t_{R_+}^{\mathcal{S}}(M)$; then $H_{R_+}^i(M)/\mathfrak{m}_0 H_{R_+}^i(M) \in \mathcal{S}$ for all $i \geq n$.

Proof. We proceed by induction on $d = \dim R_0$. If $i > n$, then $H_{R_+}^i(M) \in \mathcal{S}$ and then $H_{R_+}^i(M)/\mathfrak{m}_0 H_{R_+}^i(M) \in \mathcal{S}$. Let $i = n$. If $d = 1$, then the result follows Theorem 6. Assume inductively that the result has been proved for all values smaller than d , so we prove it for d . Let x be an indeterminate and let $\hat{R}_0 := R_0[x]_{\mathfrak{m}_0 R_0}$, $\hat{\mathfrak{m}} = \mathfrak{m}_0 \hat{R}_0$, $\hat{R} = \hat{R}_0 \otimes_{R_0} R$ and $\hat{M} := \hat{R}_0 \otimes_{R_0} M$. Then by the flat base change property of local cohomology, we have

$$\hat{R}_0 \otimes_{R_0} \frac{H_{R_+}^n(M)}{\mathfrak{m}_0 H_{R_+}^n(M)} \cong \frac{H_{\hat{R}_+}^n(M)}{\hat{\mathfrak{m}}_0 H_{\hat{R}_+}^n(\hat{M})}. \quad (33)$$

Since $\hat{R}_0$ is a faithfully flat local R_0 -algebra, in view of Remark 13 (1) it is enough to show that $H_{\hat{R}_+}^n(M)/\hat{\mathfrak{m}}_0 H_{\hat{R}_+}^n(\hat{M}) \in \mathcal{S}$. Then, we may replace R and M by $\hat{R}$ and $\hat{M}$, respectively. Hence we assume that $R_0/\mathfrak{m}_0$ is infinite residue field. From application of the functor $H_{R_+}^i(-)$ over the exact sequence

$$0 \longrightarrow \Gamma_{\mathfrak{m}_0}(M) \longrightarrow M \longrightarrow \frac{M}{\Gamma_{\mathfrak{m}_0}(M)} \longrightarrow 0, \quad (34)$$

we have the following exact sequence:

$$\begin{aligned} H_{R_+}^i(\Gamma_{\mathfrak{m}_0}(M)) &\xrightarrow{f} H_{R_+}^i(M) \xrightarrow{g} H_{R_+}^i\left(\frac{M}{\Gamma_{\mathfrak{m}_0}(M)}\right) \\ &\xrightarrow{h} H_{R_+}^{i+1}(\Gamma_{\mathfrak{m}_0}(M)). \end{aligned} \quad (35)$$

From Lemma 4, we have that $H_{R_+}^i(\Gamma_{\mathfrak{m}_0}(M)) \in \mathcal{S}$ for all i . Now, let $U = \text{Im } f$, $V = \text{Im } g$, and $W = \text{Im } h$. It follows from Lemma 1 that $\text{Tor}_i^{R_0}(R_0/\mathfrak{m}_0, U)$ and $\text{Tor}_i^{R_0}(R_0/\mathfrak{m}_0, W)$ are in $\mathcal{S}$ for all i . If $H_{R_+}^i(M/\Gamma_{\mathfrak{m}_0}(M))/\mathfrak{m}_0 H_{R_+}^i(M/\Gamma_{\mathfrak{m}_0}(M)) \in \mathcal{S}$, then $V/\mathfrak{m}_0 V \in \mathcal{S}$. This implies that $H_{R_+}^i(M)/\mathfrak{m}_0 H_{R_+}^i(M) \in \mathcal{S}$. Since $t^{\mathcal{S}_{R_+}}(M) = t^{\mathcal{S}_{R_+}}(M/\Gamma_{\mathfrak{m}_0}(M))$, we may assume that $\Gamma_{\mathfrak{m}_0}(M) = \Gamma_{\mathfrak{m}_0 R}(M) = 0$. If x_0 is a nonzero divisor of M and a part of a system of parameters of $\mathfrak{m}_0$, then we have the exact sequence

$$0 \longrightarrow M \xrightarrow{x_0} M \longrightarrow \frac{M}{x_0 M} \longrightarrow 0. \quad (36)$$

Application of the functor $H_{R_+}^n(-)$ to the above obtained exact sequence induces the exact sequence

$$\begin{aligned} H_{R_+}^n(M) &\xrightarrow{x_0} H_{R_+}^n(M) \longrightarrow H_{R_+}^n\left(\frac{M}{x_0 M}\right) \\ &\xrightarrow{k} H_{R_+}^{n+1}(M). \end{aligned} \quad (37)$$

Let $\ker k = X$ and $\text{Im } k = Y$. Since $H_{R_+}^{n+1}(M) \in \mathcal{S}$, the graded R -module Y is in $\mathcal{S}$ and therefore from Lemma 1 we get $H_{R_+}^n(M/x_0 M)/\mathfrak{m}_0 H_{R_+}^n(M/x_0 M) \in \mathcal{S}$ if and only if $X/\mathfrak{m}_0 X \in \mathcal{S}$. On the other hand, application of the functor $R_0/\mathfrak{m}_0 \otimes_{R_0} -$ induces the exact sequence

$$\begin{aligned} \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^n(M) &\xrightarrow{p} \frac{R_0}{\mathfrak{m}_0} \otimes_{R_0} H_{R_+}^n(M) \longrightarrow \frac{X}{\mathfrak{m}_0 X} \\ &\longrightarrow 0, \end{aligned} \quad (38)$$

such that $p := id_{R_0/\mathfrak{m}_0} \otimes_{R_0} x_0$. Since $x_0 \in \mathfrak{m}_0$, the map p is zero and $H_{R_+}^n(M)/\mathfrak{m}_0 H_{R_+}^n(M) \cong X/\mathfrak{m}_0 X$. Now, similar to the above arguments, one can conclude that $H_{R_+}^n(M)/\mathfrak{m}_0 H_{R_+}^n(M) \in \mathcal{S}$ if and only if $H_{R_+}^n(M/x_0 M)/\mathfrak{m}_0 H_{R_+}^n(M/x_0 M) \in \mathcal{S}$. Let $\overline{R_0} = R_0/x_0 R_0$ and $\overline{\mathfrak{m}_0} = \mathfrak{m}_0/x_0 \mathfrak{m}_0$. Using the Independence Theorem for graded local cohomology, we have $H_{R_+}^n(M/x_0 M) \cong H_{(\overline{R_0} \otimes_{R_0} R)_+}^n(M/x_0 M)$.

Note that the local base ring of the graded ring $\overline{R_0} \otimes_{R_0} R$ is $(\overline{R_0}, \overline{\mathfrak{m}_0})$ and $\dim \overline{R_0} = d - 1$. By the following isomorphism

$$\frac{H_{R_+}^n(M/x_0 M)}{\mathfrak{m}_0 H_{R_+}^n(M/x_0 M)} \cong \frac{H_{(\overline{R_0} \otimes_{R_0} R)_+}^n(M/x_0 M)}{\overline{\mathfrak{m}_0} H_{(\overline{R_0} \otimes_{R_0} R)_+}^n(M/x_0 M)} \quad (39)$$

together with Lemma 12 and using induction hypothesis, this module is in $\mathcal{S}$. $\square$

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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