Research Article

Common Fixed Points via λ -Sequences in G -Metric Spaces

Yaté Ulrich Gaba^{1,2}

¹*Institut de Mathématiques et de Sciences Physiques (IMSP), UAC, 01 BP 613, Porto-Novo, Benin*

²*École Normale Supérieure (ENS) de Natitingou/UNSTIM Abomey, BP 72, Natitingou, Benin*

Correspondence should be addressed to Yaté Ulrich Gaba; gabayae2@gmail.com

Received 8 June 2017; Accepted 25 October 2017; Published 19 November 2017

Academic Editor: Kaleem R. Kazmi

Copyright © 2017 Yaté Ulrich Gaba. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

We use λ -sequences in this article to derive common fixed points for a family of self-mappings defined on a complete G -metric space. We imitate some existing techniques in our proofs and show that the tools employed can be used at a larger scale. These results generalize well known results in the literature.

1. Introduction and Preliminaries

The generalization of the Banach contraction mapping principle has been a heavily investigated branch of research. In recent years, several authors have obtained *fixed point* and *common fixed point* results for various classes of mappings in the setting of many generalized metric spaces. One of them, the G -metric space, is our focus in this paper, and fixed point results, in this setting, presented by authors like Abbas et al. [1], Gaba [2, 3], Mustafa and Sims [4], Sihag et al. [5], and many more, are enlightening on the subject. Moreover, in [6], we introduced the concept of λ -sequence which extended the idea of α -series proposed by Sihag et al. in [5]. The present article exclusively presents natural extensions of some results already given by Abbas et al. [1] and Sihag et al. [5] and therefore generalizes some recent results regarding fixed point theory in G -metric spaces. We also show how the idea of λ -sequence is used in proving some of these results. The method builds on the convergence of an appropriate series of coefficients. We also make use of a special class of homogeneous functions. Recent and similar work can also be read in [2, 3, 6, 7].

We recall here some key results that will be useful in the rest of this manuscript. The basic concepts and notations attached to the idea of G -metric spaces can be read extensively in [4], but for the convenience of the reader, we discuss the most important ones.

Definition 1 (compare [4, Definition 3]). Let X be a nonempty set, and let the function $G : X \times X \times X \rightarrow [0, \infty)$ satisfy the following properties:

- (G1) $G(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$.
- (G2) $G(x, x, y) > 0$ whenever $x, y \in X$ with $x \neq y$.
- (G3) $G(x, x, y) \leq G(x, y, z)$ whenever $x, y, z \in X$ with $z \neq y$.
- (G4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \dots$ (symmetry in all three variables).
- (G5)

$$G(x, y, z) \leq [G(x, a, a) + G(a, y, z)] \quad (1)$$

for any points $x, y, z, a \in X$.

Then, (X, G) is called a G -metric space.

The property (G3) is crucial and shall play a key role in our proofs.

Proposition 2 (compare [4, Proposition 6]). Let (X, G) be a G -metric space.

Then, for a sequence $(x_n) \subseteq X$, the following are equivalent:

- (i) (x_n) is G -convergent to $x \in X$.
- (ii) $\lim_{n, m \rightarrow \infty} G(x, x_n, x_m) = 0$.

$$(iii) \lim_{n \rightarrow \infty} G(x, x_n, x_n) = 0.$$

$$(iv) \lim_{n \rightarrow \infty} G(x_n, x, x) = 0.$$

Proposition 3 (compare [4, Proposition 9]). *In a G-metric space (X, G) , the following are equivalent:*

- (i) *The sequence $(x_n) \subseteq X$ is G-Cauchy.*
- (ii) *For each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$.*

Definition 4 (compare [4, Definition 9]). A G-metric space (X, G) is said to be complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G) .

Definition 5 (compare [4, Definition 4]). A G-metric space (X, G) is said to be symmetric if

$$G(x, y, y) = G(x, x, y), \quad \forall x, y \in X. \quad (2)$$

Definition 6 (compare [3, Definition 2.1]). A sequence $(x_n)_{n \geq 1}$ in a metric space (X, d) is a λ -sequence if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^{L-1} d(x_i, x_{i+1}) \leq \lambda L \quad \text{for each } L \geq n(\lambda) + 1. \quad (3)$$

Definition 7 (compare [2, Definition 6]). A sequence $(x_n)_{n \geq 1}$ in a G-metric space (X, G) is a λ -sequence if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^{L-1} G(x_i, x_{i+1}, x_{i+1}) \leq \lambda L \quad \text{for each } L \geq n(\lambda) + 1. \quad (4)$$

Definition 8 (compare [5, Definition 2.1]). For a sequence $(a_n)_{n \geq 1}$ of nonnegative real numbers, the series $\sum_{n=1}^{\infty} a_n$ is an α -series if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^L a_i \leq \lambda L \quad \text{for each } L \geq n(\lambda). \quad (5)$$

Remark 9. For a given λ -sequence $(x_n)_{n \geq 1}$ in a G-metric space (X, d) , the sequence $(\beta_n)_{n \geq 1}$ of nonnegative real numbers, defined by

$$\beta_i = d(x_i, x_{i+1}, x_{i+1}), \quad (6)$$

is an α -series.

Moreover, any nonincreasing λ -sequence of elements of $\mathbb{R}^+$ endowed with the max (the max metric m refers to $m(x, y) = \max\{x, y\}$) metric is also an α -series. Therefore, λ -sequences generalize α -series, but to ease computations, we shall consider, throughout the paper, α -series (however, the reader can convince himself that using λ -sequences does not add to the complexity of the problem).

2. First Generalizations Results

We begin with the following generalization of [5, Theorem 2.1], the main result of Sihag et al.

Let Φ be the class of continuous, nondecreasing, subadditive, and homogeneous functions $F : [0, \infty) \rightarrow [0, \infty)$ such that $F^{-1}(0) = \{0\}$.

Theorem 10. *Let (X, G) be a complete G-metric space and $\{T_n\}$ be a family of self-mappings on X such that*

$$\begin{aligned} F(G(T_i x, T_j y, T_k z)) &\leq F\left(\left({}_k\Theta_{i,j}\right)\left[G(x, T_i x, T_i x) \right. \right. \\ &\quad \left. \left. + \frac{1}{2}\left[G(y, T_j y, T_j y) + G(z, T_k z, T_k z)\right]\right]\right) \\ &\quad + F\left(\left({}_k\Delta_{i,j}\right)G(x, y, z)\right), \end{aligned} \quad (7)$$

for all $x, y, z \in X$ with $x \neq y$, $0 \leq {}_k\Theta_{i,j}, {}_k\Delta_{i,j} < 1$; $i, j, k = 1, 2, \dots$, and some $F \in \Phi$, homogeneous with degree s . If

$$\sum_{i=1}^{\infty} \left[\frac{\left[({}_{i+2}\Theta_{i,i+1})^s + ({}_{i+2}\Delta_{i,i+1})^s\right]}{1 - ({}_{i+2}\Theta_{i,i+1})^s} \right] \quad (8)$$

is an α -series, then $\{T_n\}$ have a unique common fixed point in X .

Proof. We will proceed in two main steps.

Claim 1. $\{T_n\}_{n \geq 1}$ have a common fixed point in X .

For any $x_0 \in X$, we construct the sequence (x_n) by setting $x_n = T_n(x_{n-1})$, $n = 1, 2, \dots$.

We assume without loss of generality that $x_m \neq x_n$ for all $n \neq m \in \mathbb{N}$. Using (7), we obtain, for the triplet (x_0, x_1, x_2) ,

$$\begin{aligned} F(G(x_1, x_2, x_3)) &= F(G(T_1 x_0, T_2 x_1, T_3 x_2)) \\ &\leq ({}_3\Theta_{1,2})^s F\left(\left[G(x_0, x_1, x_1) + \frac{1}{2}G(x_1, x_2, x_2) \right. \right. \\ &\quad \left. \left. + \frac{1}{2}G(x_2, x_3, x_3)\right]\right) + ({}_3\Delta_{1,2})^s F(G(x_0, x_1, x_2)). \end{aligned} \quad (9)$$

By property (G3) of G , one knows that

$$\begin{aligned} G(x_i, x_{i+1}, x_{i+1}) &\leq G(x_{i-1}, x_i, x_{i+1}), \\ G(x_i, x_i, x_{i+1}) &\leq G(x_i, x_{i+1}, x_{i+2}). \end{aligned} \quad (10)$$

Hence,

$$\begin{aligned} F(G(x_1, x_2, x_3)) &= F(G(T_1 x_0, T_2 x_1, T_3 x_2)) \\ &\leq ({}_3\Theta_{1,2})^s F\left(\left[G(x_0, x_1, x_2) + \frac{1}{2}G(x_1, x_2, x_3) \right. \right. \\ &\quad \left. \left. + \frac{1}{2}G(x_1, x_2, x_3)\right]\right) + ({}_3\Delta_{1,2})^s F(G(x_0, x_1, x_2)) \\ &= ({}_3\Theta_{1,2})^s F(G(x_1, x_2, x_3)) + [({}_3\Theta_{1,2})^s \\ &\quad + ({}_3\Delta_{1,2})^s] F(G(x_1, x_2, x_0)); \end{aligned} \quad (11)$$

that is,

$$\begin{aligned} F(G(x_1, x_2, x_3)) \\ \leq \frac{[(\Theta_{1,2})^s + (\Delta_{1,2})^s]}{1 - (\Theta_{1,2})^s} F(G(x_0, x_1, x_2)). \end{aligned} \quad (12)$$

Also, we get

$$\begin{aligned} F(G(x_2, x_3, x_4)) &\leq \frac{[(\Theta_{2,3})^s + (\Delta_{2,3})^s]}{1 - (\Theta_{2,3})^s} \\ &\cdot F(G(x_1, x_2, x_3)) \leq \left[\frac{[(\Theta_{2,3})^s + (\Delta_{2,3})^s]}{1 - (\Theta_{2,3})^s} \right] \\ &\cdot \left[\frac{[(\Theta_{1,2})^s + (\Delta_{1,2})^s]}{1 - (\Theta_{1,2})^s} \right] F(G(x_0, x_1, x_2)). \end{aligned} \quad (13)$$

Repeating the above reasoning, we obtain

$$\begin{aligned} F(G(x_n, x_{n+1}, x_{n+2})) \\ \leq \prod_{i=1}^n \left[\frac{[(\Theta_{i,i+1})^s + (\Delta_{i,i+1})^s]}{1 - (\Theta_{i,i+1})^s} \right] \\ \cdot F(G(x_0, x_1, x_2)). \end{aligned} \quad (14)$$

If we set

$$r_i = \left[\frac{[(\Theta_{i,i+1})^s + (\Delta_{i,i+1})^s]}{1 - (\Theta_{i,i+1})^s} \right], \quad (15)$$

we have that

$$F(G(x_n, x_{n+1}, x_{n+2})) \leq \left[\prod_{i=1}^n r_i \right] F(G(x_0, x_1, x_2)). \quad (16)$$

Therefore, for all $l > m > n > 2$,

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n+1}, x_{n+1}) \\ &\quad + G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots \\ &\quad + G(x_{l-1}, x_{l-1}, x_l) \\ &\leq G(x_n, x_{n+1}, x_{n+2}) \\ &\quad + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots \\ &\quad + G(x_{l-2}, x_{l-1}, x_l). \end{aligned} \quad (17)$$

Using the fact that F is subadditive, we write

$$\begin{aligned} F(G(x_n, x_m, x_l)) \\ \leq \left(\left[\prod_{i=1}^n r_i \right] + \left[\prod_{i=1}^{n+1} r_i \right] + \cdots + \left[\prod_{i=1}^{l-2} r_i \right] \right) \\ \cdot F(G(x_0, x_1, x_2)) \\ = \sum_{k=0}^{l-n-2} \left[\prod_{i=1}^{n+k} r_i \right] F(G(x_0, x_1, x_2)) \\ = \sum_{k=n}^{l-2} \left[\prod_{i=1}^k r_i \right] F(G(x_0, x_1, x_2)). \end{aligned} \quad (18)$$

Now, let λ and $n(\lambda)$ be as in Definition 8; then, for $n \geq n(\lambda)$ and using the fact that the geometric mean of nonnegative real numbers is at most their arithmetic mean, it follows that

$$\begin{aligned} F(G(x_n, x_m, x_l)) \\ \leq \sum_{k=n}^{l-2} \left[\frac{1}{k} \left(\sum_{i=1}^k r_i \right) \right]^k F(G(x_0, x_1, x_2)) \\ = \left(\sum_{k=n}^{l-2} \alpha^k \right) F(G(x_0, x_1, x_2)) \\ \leq \frac{\alpha^n}{1 - \alpha} F(G(x_0, x_1, x_2)). \end{aligned} \quad (19)$$

As $n \rightarrow \infty$, we deduce that $G(x_n, x_m, x_l) \rightarrow 0$. Thus, (x_n) is a G -Cauchy sequence.

And since X is complete, there exists $u \in X$ such that (x_n) G -converges to u .

Moreover, for any positive integers k, l , we have

$$\begin{aligned} F(G(x_n, T_k u, T_l u)) &= F(G(T_n x_{n-1}, T_k u, T_l u)) \\ &\leq F\left((\Theta_{n,k}) \left[G(x_{n-1}, T_n x_{n-1}, T_n x_{n-1}) \right. \right. \\ &\quad \left. \left. + \frac{1}{2} [G(u, T_k u, T_k u) + G(u, T_l u, T_l u)] \right] \right) \\ &\quad + F((\Delta_{n,k}) G(x_{n-1}, u, u)). \end{aligned} \quad (20)$$

Letting $n \rightarrow \infty$ and using property (G3), we obtain

$$F(G(u, T_k u, T_l u)) \leq (\Theta_{n,k})^s F(G(u, T_k u, T_l u)), \quad (21)$$

and this is a contradiction, unless $u = T_k u = T_l u$, since $\Theta_{n,k} < 1$. Then, u is a common fixed point of $\{T_n\}$.

Claim 2. u is the unique common fixed point of $\{T_m\}$.

Finally, we prove the uniqueness of the common fixed point u . To this aim, let us suppose that v is another common fixed point of $\{T_m\}$; that is, $T_m(v) = v$, $\forall m \geq 1$. Then, using (7) again, we have

$$\begin{aligned} F(G(u, v, v)) &= F(G(T_m u, T_m v, T_m v)) \\ &\leq (\Delta_{m,m})^s F(G(u, v, v)), \end{aligned} \quad (22)$$

which yields $u = v$, since ${}_m\Delta_{m,m} < 1$. So, u is the unique common fixed point of $\{T_m\}$. $\square$

Theorem 11. Let (X, G) be a complete G -metric space and $\{T_n\}$ be a family of self-mappings on X such that

$$\begin{aligned} F(G(T_i^p x, T_j^p y, T_k^p z)) &\leq F({}_k\Theta_{i,j}) \\ &\cdot \left[G(x, T_i^p x, T_i^p x) \right. \\ &+ \frac{1}{2} [G(y, T_j^p y, T_j^p y) + G(z, T_k^p z, T_k^p z)] \Big] \\ &+ F({}_k\Delta_{i,j}) G(x, y, z), \end{aligned} \quad (23)$$

for all $x, y, z \in X$ with $x \neq y$, $0 \leq {}_k\Theta_{i,j}, {}_k\Delta_{i,j} < 1$; $i, j, k = 1, 2, \dots$, some positive integer p , and some $F \in \Phi$, homogeneous with degree s . If

$$\sum_{i=1}^{\infty} \left[\frac{[(i+2)\Theta_{i,i+1}]^s + (i+2)\Delta_{i,i+1}}{1 - (i+2)\Theta_{i,i+1}} \right] \quad (24)$$

is an α -series, then $\{T_n\}$ have a unique common fixed point in X .

Proof. It follows from Theorem 10 that the family $\{T_n^p\}$ have a unique common fixed point x^* . Now, for any positive integers i, j , $i \neq j$,

$$\begin{aligned} T_i(x^*) &= T_i T_i^p(x^*) = T_i^p T_i(x^*), \\ T_j(x^*) &= T_j T_j^p(x^*) = T_j^p T_j(x^*); \end{aligned} \quad (25)$$

that is, $T_i(x^*)$ and $T_j(x^*)$ are also fixed points for T_i^p and T_j^p (remember that any fixed point of T_i^p is a fixed point of T_j^p for $i \neq j$; cf. Theorem 10). Since the common fixed point of $\{T_n^p\}$ is unique, we deduce that

$$x^* = T_i(x^*) = T_j(x^*) \quad \forall i. \quad (26)$$

$\square$

The next result, corollary of Theorem 10, corresponds to the result presented by Sihag et al. [5, Theorem 2.1].

Corollary 12 (compare [5, Theorem 2.1]). Let (X, G) be a complete G -metric space and $\{T_n\}$ be a family of self-mappings on X such that

$$\begin{aligned} G(T_i x, T_j y, T_k z) &\leq ({}_k\Theta_{i,j}) [G(x, T_i x, T_i x) + G(y, T_j y, T_j z)] \\ &+ ({}_k\Delta_{i,j}) G(x, y, z), \end{aligned} \quad (27)$$

for all $x, y, z \in X$ with $x \neq y$, $0 \leq {}_k\Theta_{i,j}, {}_k\Delta_{i,j} < 1$, $i, j, k = 1, 2, \dots$. If

$$\sum_{i=1}^{\infty} \left[\frac{[(i+2)\Theta_{i,i+1}] + (i+2)\Delta_{i,i+1}}{1 - (i+2)\Theta_{i,i+1}} \right] \quad (28)$$

is an α -series, then $\{T_n\}$ have a unique common fixed point in X .

Proof. In Theorem 10, take $F = Id_{[0,\infty)}$ (the identity map on $[0, \infty)$), $j = k$ and $y = z$. $\square$

Example 13. Let $X = [0, 1]$ and $G(x, y, z) = \max\{x, y, z\}$ whenever $x, y \in [0, 1]$ and $G(x, x, x) = 0$ for $x \in [0, 1]$. Clearly, (X, G) is a complete G -metric space.

Following the notation in the definition, we set the following.

${}_k\Theta_{i,j} = (1/(1+2^n))^2$, where $\eta = \min\{i, j, k\}$ and ${}_k\Delta_{i,j} = 0$. We also define $T_i(x) = x/16^i$ for all $x \in X$ and $i = 1, 2, \dots$ and $F : [0, \infty) \rightarrow [0, \infty)$, $x \mapsto \sqrt{x}$. Then, F is continuous, non-decreasing, subadditive, and homogeneous of degree $s = 1/2$ and $F^{-1}(0) = \{0\}$. Assume $i < j \leq k$ and $x > y \geq z$. Hence, we have

$$\begin{aligned} F(D(T_i(x), T_j(y), T_j(z))) &= \sqrt{\frac{x}{16^i}}, \\ F(\delta_{i,j} [D(x, T_i(x), T_i(x)) + D(y, T_j(y), T_j(z))]) &= \sqrt{\left(\frac{1}{1+2^i}\right)^2 (x+y)}. \end{aligned} \quad (29)$$

Therefore, condition (27) is satisfied for all $x, y \in X$ with $x \neq y$. Moreover, since F is homogeneous of degree $s = 1/2$, the sequence

$$s_i = \frac{\delta_{i,i+1}^s}{1 - \delta_{i,i+1}^s} = \frac{1}{2^i} \quad (30)$$

satisfies the condition on (8); that is, it is an α -series.

Then, by Corollary 12, $\{T_n\}$ has a common fixed point, which is this case $x^* = 0$.

3. Second Generalizations Results

The next generalization is that of [1, Theorem 2.1], the main result of Abbas et al. Instead of considering three maps, we consider a family of maps like in the previous case. Moreover, to show the reader that λ -sequences do not add to the complexity of the problem, we shall use them in the next statement.

Theorem 14. Let X be a complete G -metric space (X, G) and $\{T_n\}$ be a sequence of self-mappings on X . Assume that there exist three sequences (a_n) , (b_n) , and (c_n) of elements of X such that

$$\begin{aligned} G(T_i x, T_j y, T_k z) &\leq ({}_k\Delta_{i,j}) G(x, y, z) + ({}_k\Theta_{i,j}) \\ &\cdot [G(T_i x, x, x) + G(y, T_j y, y) + G(z, z, T_k z)] \\ &+ ({}_k\Delta_{i,j}) \\ &\cdot [G(T_i x, y, z) + G(x, T_j y, z) + G(x, y, T_k z)], \end{aligned} \quad (31)$$

for all $x, y, z \in X$ with $0 \leq {}_k\Delta_{i,j} + 3({}_k\Theta_{i,j}) + 4({}_k\Lambda_{i,j}) < 1/2$, $i, j, k = 1, 2, \dots$, where ${}_k\Delta_{i,j} = G(a_i, a_j, a_k)$, ${}_k\Theta_{i,j} = G(b_i, b_j, b_k)$, and ${}_k\Lambda_{i,j} = G(c_i, c_j, c_k)$. If the sequence (r_i) , where

$$r_i = \left[\frac{[(i+2)\Delta_{i,i+1}] + 2[(i+2)\Theta_{i,i+1}] + 3[(i+2)\Lambda_{i,i+1}]}{1 - (i+2)\Theta_{i,i+1} - (i+2)\Lambda_{i,i+1}} \right], \quad (32)$$

is a nonincreasing λ -sequence of $\mathbb{R}^+$ endowed with the max (the max metric m refers to $m(x, y) = \max\{x, y\}$) metric, then $\{T_n\}$ have a unique common fixed point in X . Moreover, any fixed point of T_i is a fixed point of T_j for $i \neq j$.

Proof. We will proceed in two main steps.

Claim 1. Any fixed point of T_i is also a fixed point of T_j and T_k for $i \neq j \neq k \neq i$.

Assume that x^* is a fixed point of T_i and suppose that $T_j x^* \neq x^*$ and $T_k x^* \neq x^*$. Then,

$$\begin{aligned} G(x^*, T_j x^*, T_k x^*) &= G(T_i x^*, T_j x^*, T_k x^*) \leq ({}_k\Lambda_{i,j}) \\ &\cdot G(x^*, x^*, x^*) + ({}_k\Theta_{i,j}) [G(T_i x^*, x^*, x^*) \\ &+ G(x^*, T_j x^*, x^*) + G(x^*, x^*, T_k x^*)] + ({}_k\Lambda_{i,j}) \\ &\cdot [G(T_i x^*, x^*, x^*) + G(x^*, T_j x^*, x^*) \\ &+ G(x^*, x^*, T_k x^*)] \leq [({}_k\Theta_{i,j}) + ({}_k\Lambda_{i,j})] \\ &\cdot [G(x^*, T_j x^*, T_k x^*) + G(x^*, T_j x^*, T_k x^*)] \\ &\leq [(2{}_k\Theta_{i,j}) + (2{}_k\Lambda_{i,j})] [G(x^*, T_j x^*, T_k x^*)], \end{aligned} \quad (33)$$

which is a contradiction unless $T_i x^* = x^* = T_j x^* = T_k x^*$.

Claim 2. For any $x_0 \in X$, we construct the sequence (x_n) by setting $x_n = T_n(x_{n-1})$, $n = 1, 2, \dots$. We assume without loss of generality that $x_n \neq x_m$ for all $n \neq m$. Using (31), we obtain

$$\begin{aligned} G(x_1, x_2, x_3) &= G(T_1 x_0, T_2 x_1, T_3 x_2) \leq ({}_3\Delta_{1,2}) \\ &\cdot G(x_0, x_1, x_2) + ({}_3\Theta_{1,2}) \\ &\cdot [G(x_1, x_0, x_0) + G(x_1, x_2, x_1) + G(x_2, x_2, x_3)] \\ &+ ({}_3\Lambda_{1,2}) \\ &\cdot [G(x_1, x_1, x_2) + G(x_0, x_2, x_2) + G(x_0, x_1, x_3)]. \end{aligned} \quad (34)$$

By property (G3), one can write

$$\begin{aligned} G(x_1, x_2, x_3) &= G(T_1 x_0, T_2 x_1, T_3 x_2) \leq ({}_3\Delta_{1,2}) \\ &\cdot G(x_0, x_1, x_2) + ({}_3\Theta_{1,2}) \\ &\cdot [G(x_1, x_0, x_2) + G(x_1, x_2, x_0) + G(x_1, x_2, x_3)] \\ &+ ({}_3\Lambda_{1,2}) \\ &\cdot [G(x_1, x_0, x_2) + G(x_0, x_1, x_2) + G(x_0, x_1, x_3)]. \end{aligned} \quad (35)$$

Again, since

$$\begin{aligned} G(x_0, x_1, x_3) &\leq G(x_0, x_2, x_2) + G(x_2, x_1, x_3) \\ &\leq G(x_0, x_1, x_2) + G(x_2, x_1, x_3), \end{aligned} \quad (36)$$

we obtain

$$\begin{aligned} G(x_1, x_2, x_3) &= G(T_1 x_0, T_2 x_1, T_3 x_2) \leq ({}_3\Delta_{1,2}) \\ &\cdot G(x_0, x_1, x_2) + ({}_3\Theta_{1,2}) [G(x_1, x_0, x_2) \\ &+ G(x_1, x_2, x_0) + G(x_1, x_2, x_3)] + ({}_3\Lambda_{1,2}) \\ &\cdot [G(x_1, x_0, x_2) + G(x_0, x_1, x_2) + G(x_0, x_1, x_2) \\ &+ G(x_2, x_1, x_3)]; \end{aligned} \quad (37)$$

that is,

$$\begin{aligned} [1 - ({}_3\Theta_{1,2}) - ({}_3\Lambda_{1,2})] G(x_1, x_2, x_3) \\ \leq [({}_3\Delta_{1,2}) + 2({}_3\Theta_{1,2}) + 3({}_3\Lambda_{1,2})] G(x_0, x_1, x_2). \end{aligned} \quad (38)$$

Hence,

$$\begin{aligned} G(x_1, x_2, x_3) \\ \leq \frac{[({}_3\Delta_{1,2}) + 2({}_3\Theta_{1,2}) + 3({}_3\Lambda_{1,2})]}{1 - ({}_3\Theta_{1,2}) - ({}_3\Lambda_{1,2})} G(x_0, x_1, x_2). \end{aligned} \quad (39)$$

Also, we get

$$\begin{aligned} G(x_2, x_3, x_4) &\leq \frac{[({}_4\Delta_{2,3}) + 2({}_4\Theta_{2,3}) + 3({}_4\Lambda_{2,3})]}{1 - ({}_4\Theta_{2,3}) - ({}_4\Lambda_{2,3})} \\ &\cdot G(x_1, x_2, x_3) \\ &\leq \left[\frac{[({}_4\Delta_{2,3}) + 2({}_4\Theta_{2,3}) + 3({}_4\Lambda_{2,3})]}{1 - ({}_4\Theta_{2,3}) - ({}_4\Lambda_{2,3})} \right] \\ &\cdot \left[\frac{[({}_3\Delta_{1,2}) + 2({}_3\Theta_{1,2}) + 3({}_3\Lambda_{1,2})]}{1 - ({}_3\Theta_{1,2}) - ({}_3\Lambda_{1,2})} \right] \\ &\cdot G(x_0, x_1, x_2). \end{aligned} \quad (40)$$

Repeating the above reasoning, we obtain

$$\begin{aligned} G(x_n, x_{n+1}, x_{n+2}) \\ \leq \prod_{i=1}^n \left[\frac{[(i+2)\Delta_{i,i+1}] + 2[(i+2)\Theta_{i,i+1}] + 3[(i+2)\Lambda_{i,i+1}]}{1 - (i+2)\Theta_{i,i+1} - (i+2)\Lambda_{i,i+1}} \right] \\ \cdot G(x_0, x_1, x_2). \end{aligned} \quad (41)$$

If we set

$$r_i = \left[\frac{[(i+2)\Delta_{i,i+1}] + 2[(i+2)\Theta_{i,i+1}] + 3[(i+2)\Lambda_{i,i+1}]}{1 - (i+2)\Theta_{i,i+1} - (i+2)\Lambda_{i,i+1}} \right], \quad (42)$$

we have that

$$G(x_n, x_{n+1}, x_{n+2}) \leq \left[\prod_{i=1}^n r_i \right] G(x_0, x_1, x_2). \quad (43)$$

Therefore, for all $l > m > n > 2$,

$$\begin{aligned} G(x_n, x_m, x_l) &\leq G(x_n, x_{n+1}, x_{n+1}) \\ &+ G(x_{n+1}, x_{n+2}, x_{n+2}) + \cdots + G(x_{l-1}, x_{l-1}, x_l) \\ &\leq G(x_n, x_{n+1}, x_{n+2}) + G(x_{n+1}, x_{n+2}, x_{n+3}) + \cdots \\ &+ G(x_{l-2}, x_{l-1}, x_l), \end{aligned} \quad (44)$$

$$\begin{aligned} G(x_n, x_m, x_l) &\leq \left(\left[\prod_{i=1}^n r_i \right] + \left[\prod_{i=1}^{n+1} r_i \right] + \cdots + \left[\prod_{i=1}^{l-2} r_i \right] \right) \\ &\cdot G(x_0, x_1, x_2) = \sum_{k=0}^{l-n-2} \left[\prod_{i=1}^{n+k} r_i \right] G(x_0, x_1, x_2) \\ &= \sum_{k=n}^{l-2} \left[\prod_{i=1}^k r_i \right] G(x_0, x_1, x_2). \end{aligned} \quad (45)$$

Now, let λ and $n(\lambda)$ be as in Definition 8; then, for $n \geq n(\lambda)$ and using the fact that the geometric mean of nonnegative real numbers is at most their arithmetic mean, it follows that

$$\begin{aligned} G(x_n, x_m, x_l) &\leq \sum_{k=n}^{l-2} \left[\frac{1}{k} \left(\sum_{i=1}^k r_i \right) \right]^k G(x_0, x_1, x_2) \\ &= \left(\sum_{k=n}^{l-2} \lambda^k \right) G(x_0, x_1, x_2) \\ &\leq \frac{\lambda^n}{1-\lambda} G(x_0, x_1, x_2). \end{aligned} \quad (46)$$

As $n \rightarrow \infty$, we deduce that $G(x_n, x_m, x_l) \rightarrow 0$. Thus, (x_n) is a G -Cauchy sequence.

Moreover, since X is complete, there exists $u \in X$ such that (x_n) G -converges to u .

If there exists n_0 such that $T_{n_0} u = u$, then, by Claim 1, the proof of existence is complete.

Otherwise, for any positive integers k, l , we have

$$\begin{aligned} G(x_n, T_k u, T_l u) &= G(T_n x_{n-1}, T_k u, T_l u) \leq ({}_l \Delta_{n,k}) \\ &\cdot G(x_{n-1}, u, u) + ({}_l \Theta_{n,k}) [G(T_n x_{n-1}, x_{n-1}, x_{n-1}) \\ &+ G(u, T_k u, u) + G(u, u, T_l u)] + ({}_l \Lambda_{n,k}) \\ &\cdot [G(T_n x_{n-1}, u, u) + G(x_{n-1}, T_k u, u) \\ &+ G(x_{n-1}, u, T_l u)]. \end{aligned} \quad (47)$$

Letting $n \rightarrow \infty$ and using property (G3), we obtain

$$\begin{aligned} G(u, T_k u, T_l u) &\leq ({}_l \Theta_{n,k}) \\ &\cdot [G(u, T_k u, u) + G(u, u, T_l u)] + ({}_l \Lambda_{n,k}) \\ &\cdot [G(u, T_k u, u) + G(u, u, T_l u)] \\ &\leq [(2_k \Theta_{i,j}) + (2_k \Lambda_{i,j})] \\ &\cdot [G(u, T_k u, T_l u) + G(u, T_k u, T_l u)], \end{aligned} \quad (48)$$

and this is a contradiction, unless $u = T_k u = T_l u$.

Finally, we prove the uniqueness of the common fixed point u . To this aim, let us suppose that v is another common fixed point of T_m ; that is, $T_m(v) = v$, $\forall m \geq 1$. Then, using (31), we have

$$\begin{aligned} G(u, v, v) &= G(T_n u, T_k v, T_l v) \\ &\leq ({}_l \Delta_{n,k}) G(u, v, v) + 3 ({}_l \Lambda_{n,k}) G(u, v, v), \end{aligned} \quad (49)$$

which yields $u = v$. So, u is the unique common fixed point of $\{T_m\}$. $\square$

Following the same lines of the proof of Theorem 11, one can prove the next theorem.

Theorem 15. Let X be a complete G -metric space (X, G) and $\{T_n\}$ be a sequence of self-mappings on X . Assume that there exist three sequences (a_n) , (b_n) , and (c_n) of elements of X such that

$$\begin{aligned} G(T_i^p x, T_j^p y, T_k^p z) &\leq ({}_k \Delta_{i,j}) G(x, y, z) + ({}_k \Theta_{i,j}) \\ &\cdot [G(T_i^p x, x, x) + G(y, T_j^p y, y) + G(z, z, T_k^p z)] \\ &+ ({}_k \Lambda_{i,j}) \\ &\cdot [G(T_i^p x, y, z) + G(x, T_j^p y, z) + G(x, y, T_k^p z)], \end{aligned} \quad (50)$$

for all $x, y, z \in X$ with $0 \leq {}_k \Delta_{i,j} + 3({}_k \Theta_{i,j}) + 4({}_k \Lambda_{i,j}) < 1/2$, $i, j, k = 1, 2, \dots$, some positive integer p , where ${}_k \Delta_{i,j} = G(a_i, a_j, a_k)$, ${}_k \Theta_{i,j} = G(b_i, b_j, b_k)$, and ${}_k \Lambda_{i,j} = G(c_i, c_j, c_k)$. If the sequence (r_i) , where

$$r_i = \left[\frac{[(i+2) \Delta_{i,i+1}] + 2[(i+2) \Theta_{i,i+1}] + 3[(i+2) \Lambda_{i,i+1}]}{1 - (i+2) \Theta_{i,i+1} - (i+2) \Lambda_{i,i+1}} \right], \quad (51)$$

is a nonincreasing λ -sequence of $\mathbb{R}^+$ endowed with the max (the max metric m refers to $m(x, y) = \max\{x, y\}$) metric, then $\{T_n\}$ have a unique common fixed point in X . Moreover, any fixed point of T_i is a fixed point of T_j for $i \neq j$.

The next result, corollary of Theorem 14, corresponds to the result presented by Abbas et al. [1, Theorem 2.1].

Corollary 16. Let X be a complete G -metric space (X, G) , f, g, h mappings on X . Assume that there exist three nonnegative real numbers a, b , and c such that

$$\begin{aligned} & G(fx, gy, hz) \\ & \leq aG(x, y, z) \\ & + b[G(fx, x, x) + G(y, gy, y) + G(z, z, hz)] \\ & + c[G(fx, y, z) + G(x, gy, z) + G(x, y, hz)], \end{aligned} \quad (52)$$

for all $x, y, z \in X$ with $0 \leq a + 3b + 4c < 1$.

Then, f, g, h have a unique common fixed point in X . Moreover, any fixed point of f is a fixed point of g and h and vice versa.

Proof. In Theorem 14, take $T_1 = f, T_2 = g, T_3 = h$. Also, set

$$\begin{aligned} {}_3\Delta_{1,2} &= a, \\ {}_3\Theta_{1,2} &= b, \\ {}_3\Lambda_{1,2} &= c. \end{aligned} \quad (53)$$

Hence, we have

$$\begin{aligned} 0 \leq a + 3b + 4c &< \frac{1}{2} \implies \\ 0 \leq a + 3b + 4c &< 1 \iff \\ 0 \leq r_i = r &= \left[\frac{a + 2b + 3c}{1 - b - c} \right] < 1. \end{aligned} \quad (54)$$

The sequence $r_i = r$ is constant, so, in Definition 8, if we choose $\lambda = 1/2$ and $n(\lambda) = 1$, it is clear that $\sum_{i=1}^{\infty} r_i$ is an α -series. Indeed, since

$$\left[\frac{a + 2b + 3c}{1 - b - c} \right] < a + 3b + 4c < \frac{1}{2}, \quad (55)$$

therefore, for any $L \geq n(\lambda) + 1 = 1 + 1 = 2$,

$$\sum_{i=1}^{L-1} r_i = \sum_{i=1}^{L-1} r < \frac{1}{2} (L - 1) \leq \frac{1}{2} L. \quad (56)$$

□

We conclude this manuscript with the following result, whose proof is straightforward, following the steps of the proofs of the earliest results.

Theorem 17. Let X be a complete G -metric space (X, G) and $\{T_n\}$ be a sequence of self-mappings on X . Assume that there exist three sequences (a_n) , (b_n) , and (c_n) of elements of X such that

$$\begin{aligned} & F[G(T_i^p x, T_j^p y, T_k^p z)] \leq F[({}_k\Delta_{i,j})G(x, y, z) \\ & + ({}_k\Theta_{i,j})[G(T_i^p x, x, x) + G(y, T_j^p y, y) \\ & + G(z, z, T_k^p z)] + ({}_k\Lambda_{i,j})[G(T_i^p x, y, z) \\ & + G(x, T_j^p y, z) + G(x, y, T_k^p z)]], \end{aligned} \quad (57)$$

for all $x, y, z \in X$ with $0 \leq ({}_k\Delta_{i,j})^s + 3({}_k\Theta_{i,j})^s + 4({}_k\Lambda_{i,j})^s < 1/2$, $i, j, k = 1, 2, \dots$, some positive integer p , and some $F \in \Phi$, homogeneous with degree s , where ${}_k\Delta_{i,j} = G(a_i, a_j, a_k)$, ${}_k\Theta_{i,j} = G(b_i, b_j, b_k)$, and ${}_k\Lambda_{i,j} = G(c_i, c_j, c_k)$. If the sequence (r_i) where

$$r_i = \left[\frac{[(i+2)\Delta_{i,i+1}]^s + 2[(i+2)\Theta_{i,i+1}]^s + 3[(i+2)\Lambda_{i,i+1}]^s}{1 - (i+2)\Theta_{i,i+1}^s - (i+2)\Lambda_{i,i+1}^s} \right] \quad (58)$$

is a nonincreasing λ -sequence of $\mathbb{R}^+$ endowed with the max (the max metric m refers to $m(x, y) = \max\{x, y\}$) metric, then $\{T_n\}$ have a unique common fixed point in X . Moreover, any fixed point of T_i is a fixed point of T_j for $i \neq j$.

In addition to the examples provided by Abbas et al. and Sihag et al., illustrations of all the above results can be read in [2, Example 2.5] and [6, Example 2.8].

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Acknowledgments

This work was carried out with financial support from the government of Canada's International Development Research Centre (IDRC) and within the framework of the AIMS Research for Africa Project.

References

- [1] M. Abbas, T. Nazir, and P. Vetro, "Common fixed point results for three maps in G -metric spaces," *Filomat*, vol. 25, no. 4, pp. 1–17, 2011.
- [2] Y. U. Gaba, " λ -sequences and fixed point theorems in G -metric type spaces," *Journal of Nigerian Mathematical Society*, vol. 35, pp. 303–311, 2016.
- [3] Y. U. Gaba, "New contractive conditions for maps in G -metric type spaces," *Advances in Analysis*, vol. 1, no. 2, 2016.
- [4] Z. Mustafa and B. Sims, "A new approach to generalized metric spaces," *Journal of Nonlinear Convex Analysis*, vol. 7, pp. 289–297, 2006.
- [5] V. Sihag, R. K. Vats, and C. Vetro, "A fixed point theorem in G -metric spaces via α -series," *Quaestiones Mathematicae*, vol. 37, no. 3, pp. 429–434, 2014.
- [6] Y. U. Gaba, "Metric type spaces and λ -sequences," *Quaestiones Mathematicae*, vol. 40, no. 1, pp. 49–55, 2017.
- [7] L. Gajić and Z. Lozanov-Crvenković, "A fixed point result for mappings with contractive iterate at a point in G -metric spaces," *Filomat*, vol. 25, no. 2, pp. 53–58, 2011.