Research Article

An Interesting Property of a Class of Circulant Graphs

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Suppose that \( \Pi = \text{Cay}(\mathbb{Z}_n, \Omega) \) and \( \Lambda = \text{Cay}(\mathbb{Z}_n, \Psi_m) \) are two Cayley graphs on the cyclic additive group \( \mathbb{Z}_n \), where \( n \) is an even integer, \( m = n/2 + 1 \), \( \Omega = \{ t \in \mathbb{Z}_n \mid t \text{ is odd} \} \), and \( \Psi_m = \{ n/2 \} \cup \{ n/2 \} \). In this paper, it is shown that \( \Pi \) is a distance-transitive graph, and, by this fact, we determine the adjacency matrix spectrum of \( \Lambda \) is \((n/2 + 1)^2, (1 - n/2)^2, (1)_{m=0}^{m=1}, (-1)^{n/2} \) (we write multiplicities as exponents).

1. Introduction

In this paper, graph \( \Gamma = (V, E) \) always means a simple connected graph with \( n \) vertices (without loops, multiple edges, and isolated vertices), where \( V = V(\Gamma) \) is the vertex set and \( E = E(\Gamma) \) is the edge set. Graph \( \Gamma \) is called a vertex-transitive graph, if, for any \( x, y \in V \), there is some \( \pi \in \text{Aut}(\Gamma) \), the automorphism group of \( \Gamma \), such that \( \pi(x) = y \). Let \( \Gamma \) be a graph, the complement \( \overline{\Gamma} \) of \( \Gamma \) is the graph whose vertex set is \( V(\Gamma) \) and whose edges are the pairs of nonadjacent vertices of \( \Gamma \).

It is well known that, for any graph \( \Gamma \), \( \text{Aut}(\Gamma) = \text{Aut}(\overline{\Gamma}) \) [1]. If \( \Gamma \) is a connected graph and \( \partial(u, v) \) denotes the distance in \( \Gamma \) between the vertices \( u \) and \( v \), then, for any automorphism \( \pi \) in \( \text{Aut}(\Gamma) \), we have \( \partial(u, v) = \partial(\pi(u), \pi(v)) \).

Let \( Y = \{ y_1, \ldots, y_{k+1} \} \) be a set and \( K \) be a group; then, writing \( \text{Fun}(Y, K) \) to denote the set of all functions from \( Y \) into \( K \), we can turn \( \text{Fun}(Y, K) \) into a group by defining a product:

\[
(fg)(y) = f(y)g(y) \quad \forall f, g \in \text{Fun}(Y, K) \quad \text{and} \quad y \in Y,
\]

(1)

where the product on the right is in \( K \). Since \( Y \) is finite then the group \( \text{Fun}(Y, K) \) is isomorphic to \( K^{k+1} \) (a direct product of \( k + 1 \) copies of \( K \)) via the isomorphism \( f \rightarrow (f(y_1), \ldots, f(y_{k+1})) \). Let \( H \) and \( K \) be groups and suppose that \( H \) acts on the nonempty set \( Y \). Then, the wreath product of \( K \) by \( H \) with respect to this action is defined to be the semidirect product \( \text{Fun}(Y, K) \rtimes H \) where \( H \) acts on the group \( \text{Fun}(Y, K) \) via

\[
f^x(y) = f(y^{x^{-1}}) \quad \forall f \in \text{Fun}(Y, K), \quad y \in Y \quad \text{and} \quad x \in H.
\]

(2)

We denote this group by \( K \wr_2 H \). Consider the wreath product \( G = K \wr_2 H \). If \( K \) acts on a set \( \Delta \), then we can define an action of \( G \) on \( \Delta \times Y \) by

\[
(\delta, y)^{(f, h)} = (f^{h(y)}, y^h) \quad \forall (\delta, y) \in \Delta \times Y,
\]

(3)

where \( (f, h) \in \text{Fun}(Y, K) \rtimes H = K \wr_2 H \) [2].

Let \( G \) be a group and \( H \leq G \) a subgroup of \( G \) and \( S \subseteq G \). The Schreier coset graph on \( G/H \) generated by \( S \) is the graph \( \Gamma = \Gamma(G, H, S) \) with \( V(\Gamma) = G/H = \{ gH \mid g \in G \} \) the set of left cosets of \( H \), and there is an edge \( (gH, sgH) \) for each coset \( gH \) and each \( s \in S \). If \( S \) is inverse-closed, then \( \Gamma \) is an undirected multigraph (possibly with loops). Note that if \( 1_G \) is the identity element of \( G \), then \( \Gamma(G, 1_G, S) = \Gamma(G, S) \) is the Cayley graph on \( G \) generated by \( S \). It is well known that every Cayley graph is vertex-transitive [3].

Let \( \Gamma \) be a graph with automorphism group \( \text{Aut}(\Gamma) \). Say that \( \Gamma \) is symmetric graph if, for all vertices \( u, v, x, y \) of \( \Gamma \) such that \( u \) and \( v \) are adjacent, also, \( x \) and \( y \) are adjacent, and there is an automorphism \( \pi \) in \( \text{Aut}(\Gamma) \) such that \( \pi(u) = x \)
and $\pi(v) = y$. We say that $\Gamma$ is distance-transitive if, for all vertices $u, v, x, y$ of $\Gamma$ such that $\partial(u, v) = \partial(x, y)$, there is an automorphism $\pi$ in Aut($\Gamma$) satisfying $\pi(u) = x$ and $\pi(v) = y$ [3]. It is clear that hierarchy of the conditions is

$$\text{distance-transitive} \implies \text{symmetric} \implies \text{vertex-transitive}.$$  \hspace{1cm} (4)

Eigenvalues of an undirected graph $\Gamma$ are the eigenvalues of an arbitrary adjacency matrix of $\Gamma$. Harary and Schwenk [4] defined $\Gamma$ to be integral, if all of its eigenvalues are integers. For a survey of integral graphs, see [5]. In [6], the number of integral graphs on $n$ vertices is estimated. Known characterizations of integral graphs are restricted to certain graph classes; see [7].

In this paper, suppose $\Pi = \text{Cay}(Z_n, \Omega)$ and $\Lambda = \text{Cay}(Z_n, \Psi_n)$ are two Cayley graphs on the cyclic additive group $Z_n$, where $n$ is an even integer, $m = n/2 + 1$, $\Omega = \{t \in Z_n \mid t \text{ is odd}\}$, and $\Psi_n = \Omega \cup \{n/2\}$ are the inverse-closed subsets of $Z_n - \{0\}$. One of our goals in this paper is to obtain all eigenvalues of the Cayley graph $\Lambda = \text{Cay}(Z_n, \Psi_n)$. First, we determine the group automorphism of $\Pi$ and we show that $\Pi$ is a distance-transitive graph; also, by this fact, we determine the adjacency matrix spectrum of $\Pi$. Finally, according to these facts, we show that if $n \geq 8$ and $n/2$ is an even integer, then the adjacency matrix spectrum of $\Lambda$ is $(n/2+1)^2, (1-(n/2))^2, (1)^{n/2+1}, (-1)^{n/2}$ (we write multiplicities as exponents).

2. Definitions and Preliminaries

Definition 1 (see [3, 8]). For any vertex $v$ of a connected graph $\Gamma$, one defines

$$\Gamma_r(v) = \{u \in V(\Gamma) \mid \partial(u, v) = r\},$$  \hspace{1cm} (5)

where $r$ is a nonnegative integer not exceeding $d$, the diameter of $\Gamma$. It is clear that $\Gamma_0(v) = \{v\}$, and $\Gamma_r(v)$ is partitioned into the disjoint subsets $\Gamma_0(v), \ldots, \Gamma_r(v)$, for each $v \in V(\Gamma)$. The graph $\Gamma$ is called distance-regular with diameter $d$ and intersection array $[b_0, b_1, \ldots, b_{d-1}; c_1, \ldots, c_d]$, if it is regular of valency $\kappa$ and, for any two vertices $u$ and $v$ in $\Gamma$ at distance $r$, one has $|\Gamma_{r+1}(v) \cap \Gamma_r(u)| = b_r$ and $|\Gamma_{r-1}(v) \cap \Gamma_1(u)| = c_r$, $(0 \leq r \leq d)$. The numbers $b_r, c_r$, and $\kappa$, where

$$a_r = k - b_r - c_r \quad (0 \leq r \leq d),$$  \hspace{1cm} (6)

is the number of neighbours of $u$ in $\Gamma_r(v)$ for $\partial(u, v) = r$, are called the intersection numbers of $\Gamma$. Clearly $b_0 = k, b_d = c_0 = 0$, and $c_1 = 1$.

Remark 2 (see [3]). It is clear that if $\Gamma$ is distance-transitive graph, then $\Gamma$ is distance-regular.

Lemma 3 (see [3]). A connected graph $\Gamma$ with diameter $d$ and automorphism group $G = \text{Aut}(\Gamma)$ is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer $G_v$ is transitive on the set $\Gamma_r(v)$, for each $r \in \{0, 1, \ldots, d\}$, and $v \in V(\Gamma)$.

Theorem 4 (see [8]). Let $\Gamma$ be a distance-regular graph which the valency of each vertex as $k$, with diameter $d$, adjacency matrix $A$, and intersection array, is

$$\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}. \hspace{1cm} (7)$$

Then, the tridiagonal $(d + 1) \times (d + 1)$ matrix

$$j(\Gamma) = \begin{bmatrix}
    a_0 & b_0 & 0 & 0 & \cdots \\
    c_1 & a_1 & b_1 & 0 & \cdots \\
    0 & c_2 & a_2 & b_2 & \cdots \\
    \vdots & \vdots & \vdots & \vdots & \ddots \\
    c_{d-2} & a_{d-2} & b_{d-2} & 0 & \cdots \\
    0 & c_{d-1} & a_{d-1} & b_{d-1} & \cdots \\
    0 & 0 & c_d & a_d & \end{bmatrix} \hspace{1cm} (8)$$

determines all the eigenvalues of $\Gamma$.

Theorem 5 (see [9]). Let $F$ be a field and let $R$ be a commutative subring of $F^{\Omega}$, the set of all $n \times n$ matrices over $F$. Let $M \in R^{m \times m}$, then $\det(\Gamma) = \det_2(\det_3(M))$.

Theorem 6 (see [10]). Let $\Gamma$ be a graph such that contains $k + 1$ components $\Gamma_1, \ldots, \Gamma_{k+1}$. If, for any $i \in I = \{1, \ldots, k + 1\}$, $\Gamma_i \cong \Gamma_1$, then $\text{Aut}(\Gamma) \cong \text{Aut}(\Gamma_1) \wr \text{Sym}(k + 1)$.

2.1. Main Results

Proposition 7. Let $\Pi = \text{Cay}(Z_n, \Omega)$ be the Cayley graph on the cyclic group $Z_n (n \geq 4)$, where $\Omega = \{t \in Z_n \mid t \text{ is odd}\}$ is the inverse-closed subset of $Z_n - \{0\}$. Then $\text{Aut}(\Pi) \cong \text{Sym}(n/2) \wr I/2 \text{Sym}(2)$, where $I = \{1, 2\}$.

Proof. Let $V(\Pi) = \{1, \ldots, n\}$ be the vertex set of $\Pi$. By assumption, the size of the every independent sets of vertices in $\Pi$ is $n/2$, because $\Pi$ is a vertex-transitive graph and the size of every clique in graph $\Pi$ is 2. Therefore, for any $x \in V(\Pi)$, there is exactly $n/2$, $y \in V(\Pi)$ such that $x^{-1}y \notin \Omega$. Hence, if $x^{-1}y \notin \Omega$, then two vertices $x$ and $y$ are adjacent in the complement $\Pi$ of $\Pi$, so $\Pi$ contains 2 components $\Pi_1, \Pi_2$ such that $\Pi_1 \cong \Pi_2 \cong K_{n/2}$, where $K_{n/2}$ is the complete graph of $n/2$ vertices. Therefore, $\Pi = 2K_{n/2}$. Hence, by Theorem 6, $\text{Aut}(\Pi) \cong \text{Aut}(K_{n/2}) \wr I/2 \text{Sym}(2) = \text{Sym}(n/2) \wr I/2 \text{Sym}(2)$.  \hfill $\square$

Proposition 8. Let $\Pi = \text{Cay}(Z_n, \Omega)$ be the Cayley graph on the cyclic group $Z_n (n \geq 4)$, where $n$ is an even integer and $\Omega = \{t \in Z_n \mid t \text{ is odd}\}$ is the inverse-closed subset of $Z_n - \{0\}$; then $\Pi$ is a distance-transitive graph.

Proof. Suppose that $u, v, x, y$ are vertices of $\Pi$ such that $\partial(u, v) = \partial(x, y) = r$, where $r$ is a nonnegative integer not exceeding $d$, the diameter of $\Pi$. So $\partial(u, v) = \partial(x, y) = 1$ or $2$, since $d = 2$.

(a) If $\partial(u, v) = \partial(x, y) = 2$, then $u^{-1}v \notin \Omega$ and $x^{-1}y \notin \Omega$. Therefore, two vertices $u$ and $v$ are adjacent in the complement $\Pi$ of $\Pi$, also two vertices $x$ and $y$ are adjacent in the complement $\Pi$ of $\Pi$. So $\Pi$ contains 2 components $\Pi_1, \Pi_2$.
such that $\Pi_1 \equiv \Pi_2 \equiv K_{n/2}$. Therefore $\Pi \equiv 2K_{n/2}$; hence we may assume $\pi = (ux)(vy) \in Aut(\Pi) = Aut(\Pi)$, so $\pi(u) = x$ and $\pi(v) = y$.

(b) If $\partial(u, v) = \partial(x, y) = 1$, then, by Lemma 3, it is sufficient to show that vertex-stabilizer $G_v$ is transitive on set $\Pi_v$ for every $r \in \{0, 1, 2\}$ and every $v \in V(\Pi)$, because $\Pi$ is a vertex-transitive graph. In this case, let $V(\Pi) = \{1, 2, \ldots, n\}$ be the vertex set of $\Pi$ and $G = Aut(\Pi)$. Consider the vertex $v = 1$ in $V(\Pi)$; then $\Pi_0(v) = \{1\}$, $\Pi_0(v) = \{t \in \mathbb{Z}_n \mid t \text{ is even}\}$, and $\Pi_2(v) = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$. Let $H$ be the group that is generated by all elements of sets $\Pi_1(v)$ and $\Pi_2(v)$, say $H = \{(2, 4, \ldots, n), (3, 5, \ldots, n - 1)\}$. It is clear that $H$ is a subgroup of $Aut(\Pi)$, so the group $H$ is transitive on the set $\Pi_v$ for each $r \in \{0, 1, 2\}$. Note that if $1 \neq v \in V(\Pi)$, then, we can show that vertex-stabilizer $G_v$ is transitive on the set $\Pi_v$ for each $r \in \{0, 1, 2\}$, because $\Pi$ is a vertex-transitive graph.

**Proposition 9.** Let $\Pi = Cay(\mathbb{Z}_n, \Omega)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ ($n \geq 4$), where $n$ is an even integer and $\Omega = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$ is the inverse-closed subset of $\mathbb{Z}_n - \{0\}$; then $\Pi$ is an integral graph.

**Proof.** By Remark 2, it is clear that $\Pi$ is distance-regular, because $\Pi$ is a distance-transitive graph. Let $V(\Pi) = \{1, 2, \ldots, n\}$ be the vertex set of $\Pi$. Consider the vertex $v = 1$ in $V(\Pi)$; then $\Pi_0(v) = \{1\}$, $\Pi_1(v) = \{t \in \mathbb{Z}_n \mid t \text{ is even}\}$, and $\Pi_2(v) = \{t \in \mathbb{Z}_n \mid t \text{ is odd}\}$. Let be $u$ in $V(\Pi)$ such that $\partial(u, v) = 0$; then $u = 1$ and $\Pi_1(v) \cap \Pi_1(u) = n/2$; hence $b_0 = n/2$ and, by Definition 1, $a_0 = n/2 - b_0 = 0$. Also, if $u$ in $V(\Pi)$ and $\partial(u, v) = 1$, then two vertices $u, v$ are adjacent in $\Pi$, so $\Pi_0(v) \cap \Pi_1(u) = 1$ and $\Pi_1(v) \cap \Pi_1(u) = n/2 - 1$; hence $c_1 = 1$, $b_1 = n/2 - 1$, and $a_1 = n/2 - b_1 - c_1$. Finally, if $u \in V(\Pi)$ and $\partial(u, v) = 2$, then two vertices $u, v$ are not adjacent in $\Pi$, so $\Pi_1(v) \cap \Pi_1(u) = n/2$; hence $c_2 = n/2$ and $a_2 = n/2 - c_2 = 0$. So the intersection array of $\Pi$ is $(n/2, n/2 - 1; 1, n/2)$. Therefore, by Theorem 4, the tridiagonal $(3 \times 3)$ matrix

$$
\begin{bmatrix}
0 & n/2 & 0 \\
1 & 0 & n/2 - 1 \\
0 & n/2 & 0
\end{bmatrix}
$$

(9)
determines all the eigenvalues of $\Pi$. It is clear that all the eigenvalues of $\Pi$ are $n/2, -n/2, 0$, and their multiplicities are $1, 1, n - 2$, respectively. So $\Pi$ is an integral graph. $
$

**Conclusion 10.** Let $\Pi = Cay(\mathbb{Z}_n, \Omega)$ be the Cayley graph on the cyclic group $\mathbb{Z}_n$ as before with the adjacency matrix $M = [A_A A]$, and characteristic polynomial $\Phi_{\Pi}(x)$ then is

$$
\Phi_{\Pi}(x) = \det(xI_{n/2} - 2A)
$$

(10)
References


