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Research Article

Differential Calculus on ℕ-Graded Manifolds

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The differential calculus, including formalism of linear differential operators and the Chevalley–Eilenberg differential calculus, over ℕ-graded commutative rings and on ℕ-graded manifolds is developed. This is a straightforward generalization of the conventional differential calculus over commutative rings and also is the case of the differential calculus over Grassmann algebras and on $\mathbb{Z}_2$-graded manifolds. We follow the notion of an ℕ-graded manifold as a local-ringed space whose body is a smooth manifold Z. A key point is that the graded derivation module of the structure ring of graded functions on an ℕ-graded manifold is the structure ring of global sections of a certain smooth vector bundle over its body Z. Accordingly, the Chevalley–Eilenberg differential calculus on an ℕ-graded manifold provides it with the de Rham complex of graded differential forms. This fact enables us to extend the differential calculus on ℕ-graded manifolds to formalism of nonlinear differential operators, by analogy with that on smooth manifolds, in terms of graded jet manifolds of ℕ-graded bundles.

1. Introduction

This work addresses the differential calculus over ℕ-graded commutative rings and on ℕ-graded manifolds defined as local-ringed spaces. This differential calculus provides formalism of differential operators and Lagrangian theory in Grassmann-graded (even and odd) variables [1, 2].

Definition 1. Let $\mathcal{K}$ be a commutative ring. A direct sum of $\mathcal{K}$-modules

$$P = P^* = \bigoplus_{i\in\mathbb{N}} P^i \tag{1}$$

is called the ℕ-graded $\mathcal{K}$-module. Its elements p are called homogeneous of degree $[p]$ if $p \in P^{[p]}$.

Definition 2. A $\mathcal{K}$-ring $\mathcal{A}$ is called ℕ-graded if it is an ℕ-graded $\mathcal{K}$-module $\mathcal{A}^*$ (1) so that a product of homogeneous elements $\alpha\alpha'$ is a homogeneous element of degree $[\alpha] + [\alpha']$. In particular, it follows that $\mathcal{A}^0$ is a $\mathcal{K}$-ring, while $\mathcal{A}^{k>0}$ and, accordingly, $\mathcal{A}^*$ are $\mathcal{A}^0$-bimodules.

Any ℕ-graded $\mathcal{K}$-module P (1) admits the associated $\mathbb{Z}_2$-graded structure

$$\begin{aligned} P &= P_0 \oplus P_1, \\ P_0 &= \bigoplus_{i\in\mathbb{N}} P^{2i}, \\ P_1 &= \bigoplus_{i\in\mathbb{N}} P^{2i+1}. \end{aligned} \tag{2}$$

Accordingly, an ℕ-graded ring $\mathcal{A}^*$ also is the $\mathbb{Z}_2$-graded one $\mathcal{A}_*$ (Definition 18). The converse is not true. For instance, Clifford algebras are $\mathbb{Z}_2$-graded, but not ℕ-graded rings. In general, an ℕ-graded ring $\mathcal{A}^*$ can admit different ℕ- and $\mathbb{Z}_2$-graded structures (Theorem 26).

Remark 3. Hereafter, we follow the notation $\mathcal{A}^*$ (resp., $\mathcal{A}_*$) of a $\mathcal{K}$-ring $\mathcal{A}$ endowed with an ℕ-graded (resp., $\mathbb{Z}_2$-graded) structure. If there is no danger of confusion, the symbol $[\cdot]$ further stands both for ℕ- and $\mathbb{Z}_2$-degree.

We further restrict our consideration to ℕ-graded commutative rings.

Definition 4. An $\mathbb{N}$-graded $\mathcal{K}$-ring $\mathcal{A}^*$ is said to be graded commutative if

$$\alpha\beta = (-1)^{[\alpha][\beta]}\beta\alpha, \quad \alpha, \beta \in \mathcal{A}^*. \tag{3}$$

In this case, $\mathcal{A}^0$ is a commutative $\mathcal{K}$-ring, and $\mathcal{A}^*$ is an $\mathcal{A}^0$-ring.

Example 5. An $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$ is commutative if $\mathcal{A}^{2i+1} = 0$. Conversely, any commutative ring $\mathcal{A}$ is an $\mathbb{N}$-graded commutative one $\mathcal{A}^*$, where $\mathcal{A}^{>0} = 0$.

An $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$ possesses an associated $\mathbb{Z}_2$-graded commutative structure

$$\begin{gathered} \mathcal{A}_0 = \bigoplus_k \mathcal{A}^{2k}, \\ \mathcal{A}_1 = \bigoplus_k \mathcal{A}^{2k+1}, \\ k \in \mathbb{N}, \\ \alpha\beta = (-1)^{[\alpha][\beta]}\beta\alpha, \quad \alpha, \beta \in \mathcal{A}_*, \end{gathered} \tag{4}$$

in accordance with Definition 19. The converse need not be true.

The differential calculus over $\mathbb{N}$-graded commutative rings (Section 4) is a straightforward generalization of the conventional differential calculus over commutative rings, including formalism of linear differential operators and the Chevalley–Eilenberg differential calculus over rings (Section 3) [3–5]. However, this is not a particular case of the differential calculus over noncommutative rings. One can generalize a construction of the Chevalley–Eilenberg differential calculus to a case of an arbitrary ring [5–7]. However, an extension of the notion of a linear differential operator to noncommutative rings meets difficulties [5]. A key point is that multiplication in a noncommutative ring is not a zero-order differential operator.

One overcomes this difficulty in a case of $\mathbb{Z}_2$-graded commutative rings by means of reformulating the notion of linear differential operators (Remark 43). As a result, the differential calculus technique has been extended to $\mathbb{Z}_2$-graded commutative rings [5, 8, 9]. Since any $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$ possesses the associated structure (4) of a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}_*$ and the commutation relations (3) of its elements depend on their $\mathbb{Z}_2$-graded degree, the differential calculus over $\mathbb{N}$-graded commutative rings is defined similarly to that over the $\mathbb{Z}_2$-graded ones (Section 4). Herewith, a linear $\mathbb{N}$-graded differential operator, being an $\mathbb{N}$-graded $\mathcal{K}$-module homomorphism, is a $\mathbb{Z}_2$-graded homomorphism which obeys conditions (72). Consequently, it is a linear $\mathbb{Z}_2$-graded differential operator, too. However, the converse need not be true. Therefore, the differential calculus over $\mathbb{N}$-graded commutative rings can possess properties which do not characterize the $\mathbb{Z}_2$-graded differential calculus. This is just the case of $\mathbb{N}$-graded manifolds in comparison with the $\mathbb{Z}_2$-graded ones (Theorem 50).

There are different notions of graded manifolds [8, 10–13]. We follow the conventional definition of manifolds as local-ringed spaces and, by analogy with smooth manifolds [14, 15] and $\mathbb{Z}_2$-graded manifolds [5, 8, 9], define an $\mathbb{N}$-graded manifold as a local-ringed space which is a sheaf in local $\mathbb{N}$-graded commutative rings on a finite-dimensional real smooth manifold Z (Definition 47).

Since $\mathbb{Z}_2$-graded manifolds conventionally are sheaves in Grassmann algebras [8], we focus our consideration on local finitely generated $\mathbb{N}$-graded commutative rings of the following type (Remark 8).

Definition 6. An $\mathbb{N}$-graded commutative $\mathcal{K}$-ring Λ^* is called the Grassmann-graded $\mathcal{K}$-ring if it is finitely generated in degree 1 (Definition 25) so that it is the exterior algebra of $\Lambda^* = \bigwedge \Lambda^1$ of a $\mathcal{K}$-module Λ^1 (Example 9).

A Grassmann-graded $\mathcal{K}$-ring Λ^* seen as a $\mathbb{Z}_2$-graded commutative ring Λ_* is a Grassmann algebra (Definition 23). A Grassmann algebra Λ_*, in turn, can admit different associated Grassmann-graded structures Λ^*. However, since it is finitely generated in degree 1, all these structures mutually are isomorphic if $\mathcal{K}$ is a field by virtue of Theorem 26. Therefore, an $\mathbb{N}$-graded manifold also is a conventional $\mathbb{Z}_2$-graded manifold. Conversely, any $\mathbb{Z}_2$-graded manifold is isomorphic to the $\mathbb{N}$-graded one in accordance with Batchelor's Theorem 45. However, let us emphasize that though an $\mathbb{N}$-graded manifold is $\mathbb{Z}_2$-graded and *vice versa*, the differential calculus on these graded manifolds is different.

The differential calculus on an $\mathbb{N}$-graded manifold is the differential calculus over its structure $\mathbb{N}$-graded commutative ring (Section 5). A key point is that derivations of the structure ring of graded functions on an $\mathbb{N}$-graded manifold, unlike the $\mathbb{Z}_2$-graded one, are represented by sections of the smooth vector bundle $\mathcal{V}_E$ (102) over its body manifold Z (Theorem 50). As a consequence, the Chevalley–Eilenberg differential calculus on an $\mathbb{N}$-graded manifold provides it with the de Rham complex (103) of graded differential forms.

Just this fact enables us to extend the differential calculus on $\mathbb{N}$-graded manifolds to nonlinear differential operators (Section 6). We follow conventional formalism of (nonlinear) differential operators on smooth fibre bundles in terms of their jet manifolds (Appendix B) [3, 4, 16]. We develop the technique of $\mathbb{N}$-graded bundles (Definition 52) and graded jet manifolds (Definition 55). Our goal is the differential calculus $S^*_\infty[F, Y]$ (123) of graded differential forms on an $\mathbb{N}$-graded infinite order jet manifold $(J^\infty Y, \mathfrak{A}_{J^\infty F})$. A key point is that this ring $S^*_\infty[F, Y]$ is split into a bigraded variational bicomplex which provides Lagrangian theory in Grassmann-graded (even and odd) variables [1, 2, 9].

2. Algebraic Preliminary

This section summarizes the relevant basics on commutative rings [17–19] and graded commutative rings [5, 7, 8].

2.1. Commutative Rings. An algebra $\mathcal{A}$ is defined to be an additive group which additionally is provided with distributive multiplication. All algebras throughout are associative, unless they are Lie algebras and Lie superalgebras. By a ring is meant a unital algebra with a unit element $\mathbf{1} \neq 0$. Nonzero elements of a ring $\mathcal{A}$ constitute a multiplicative monoid. If it is a group, $\mathcal{A}$ is called the division ring. A field is a commutative division ring. A ring $\mathcal{A}$ is said to have no divisor of zero if an equality $ab = 0$, $a, b \in \mathcal{A}$, implies either $a = 0$ or $b = 0$. For instance, this is a case of a division ring.

A subset $\mathcal{I}$ of an algebra $\mathcal{A}$ is said to be the left (resp., right) ideal if it is a subgroup of an additive group $\mathcal{A}$ and $ab \in \mathcal{I}$ (resp., $ba \in \mathcal{I}$) for all $a \in \mathcal{A}$, $b \in \mathcal{I}$. If $\mathcal{I}$ is both a left and right ideal, it is called the two-sided ideal. For instance, any ideal of a commutative algebra is two-sided. A proper ideal of a ring is said to be maximal if it does not belong to another proper ideal. Given a two-sided ideal $\mathcal{I} \subset \mathcal{A}$, an additive factor group $\mathcal{A}/\mathcal{I}$ is an algebra.

Definition 7. A ring $\mathcal{A}$ is called local if it has a unique maximal two-sided ideal. This ideal consists of all noninvertible elements of $\mathcal{A}$.

Any division ring, for example, a field, is local. Its unique maximal ideal consists of the zero element. A homomorphism of local rings is assumed to send a maximal ideal to a maximal ideal.

Remark 8. Local rings conventionally are defined in commutative algebra [18, 19]. This notion has been extended to $\mathbb{Z}_2$-graded commutative rings, too [8]. Grassmann-graded rings in Definition 6 and Grassmann algebras in Definition 23 are local.

Given an algebra $\mathcal{A}$, an additive group P is said to be the left (resp., right) $\mathcal{A}$-module if it is provided with a distributive multiplication $\mathcal{A} \times P \to P$ by elements of $\mathcal{A}$ such that $(ab)p = a(bp)$ (resp., $p(ab) = (pa)b$) for all $a, b \in \mathcal{A}$ and $p \in P$. If $\mathcal{A}$ is a ring, one additionally assumes that $\mathbf{1}p = p = p\mathbf{1}$ for all $p \in P$. If P is both a left module over an algebra $\mathcal{A}$ and a right module over an algebra $\mathcal{A}'$, it is called the $(\mathcal{A}-\mathcal{A}')$-bimodule (the $\mathcal{A}$-bimodule if $\mathcal{A} = \mathcal{A}'$). If $\mathcal{A}$ is a commutative algebra, an $\mathcal{A}$-bimodule P is said to be commutative if $ap = pa$ for all $a \in \mathcal{A}$ and $p \in P$. Any module over a commutative algebra $\mathcal{A}$ can be brought into a commutative bimodule. Therefore, unless otherwise stated (Section 3.1), any $\mathcal{A}$-module over a commutative algebra is a commutative $\mathcal{A}$-bimodule, which is called the $\mathcal{A}$-module if there is no danger of confusion. A module over a field is called the vector space. If an algebra $\mathcal{A}$ is a commutative bimodule over a commutative ring $\mathcal{K}$, it is said to be the $\mathcal{K}$-algebra. Any algebra can be regarded as a $\mathbb{Z}$-algebra.

The following are constructions of new modules over a commutative ring $\mathcal{A}$ from the old ones.

(i) A direct sum $P \oplus P'$ of $\mathcal{A}$-modules P and P' is an additive group $P \times P$ provided with an $\mathcal{A}$-module structure

$$a\left(p, p'\right) = \left(ap, ap'\right), \quad p \in P, \ p' \in P', \ a \in \mathcal{A}. \tag{5}$$

Let $\{P_i\}_{i \in I}$ be a set of $\mathcal{A}$-modules. Their direct sum $\bigoplus P_i$ consists of elements $(\dots, p_i, \dots)$ of the Cartesian product $\prod P_i$ such that $p_i \neq 0$ at most for a finite number of indices $i \in I$.

(ii) A tensor product $P \otimes_{\mathcal{A}} Q$ of $\mathcal{A}$-modules P and Q is an additive group which is generated by elements $p \otimes q$, $p \in P$, $q \in Q$, obeying relations

$$\begin{gathered}\left(p + p'\right) \otimes q = p \otimes q + p' \otimes q, \\ p \otimes \left(q + q'\right) = p \otimes q + p \otimes q', \\ pa \otimes q = p \otimes aq, \quad p \in P, \ q \in Q, \ a \in \mathcal{A}.\end{gathered} \tag{6}$$

It is endowed with an $\mathcal{A}$-module structure

$$a\left(p \otimes q\right) = \left(ap\right) \otimes q = p \otimes \left(qa\right) = \left(p \otimes q\right) a. \tag{7}$$

If a ring $\mathcal{A}$ is treated as an $\mathcal{A}$-module, a tensor product $\mathcal{A} \otimes_{\mathcal{A}} Q$ is canonically isomorphic to Q via the assignment $a \otimes q \leftrightarrow aq$, $a \in \mathcal{A}$, $q \in Q$.

Example 9. Let Q be an $\mathcal{A}$-module. Let us consider an $\mathbb{N}$-graded module

$$\bigotimes Q = \mathcal{A} \oplus Q \oplus \cdots \oplus \left(\bigotimes_{\mathcal{A}}^{k} Q\right) \oplus \cdots. \tag{8}$$

This is an $\mathbb{N}$-graded $\mathcal{A}$-ring with respect to a tensor product $\bigotimes$. It is called the tensor algebra of an $\mathcal{A}$-module Q. Its quotient $\bigwedge Q$ with respect to an ideal generated by elements $q \otimes q' + q' \otimes q$, $q, q' \in Q$, is an $\mathbb{N}$-graded commutative algebra, called the exterior algebra of an $\mathcal{A}$-module Q.

(i) Given a submodule Q of an $\mathcal{A}$-module P, the quotient P/Q of an additive group P by its subgroup Q also is provided with an $\mathcal{A}$-module structure. It is called the factor module.

(ii) A set $\mathrm{Hom}_{\mathcal{A}}(P, Q)$ of $\mathcal{A}$-linear morphisms of an $\mathcal{A}$-module P to an $\mathcal{A}$-module Q naturally is an $\mathcal{A}$-module. An $\mathcal{A}$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P, \mathcal{A})$ is called the dual of an $\mathcal{A}$-module P. There is a natural monomorphism $P \to P^{**}$.

A module P over a commutative ring $\mathcal{A}$ is called free if it admits a basis, that is, a linearly independent subset $I \subset P$ such that each element of P has a unique expression as a linear combination of elements of I with a finite number of nonzero coefficients from a ring $\mathcal{A}$. Any module over a field is free. Every module is isomorphic to the quotient of a free module. A module is said to be of finite rank if it is the quotient of a free module with a finite basis. One says that a module P is projective if there exists a module Q such that $P \oplus Q$ is a free module.

Theorem 10. *If P is a projective module of finite rank, then its dual P^* is so, and $P^{**} = P$.*

The forthcoming constructions of direct and inverse limits of modules over commutative rings also are extended to a case of modules over graded commutative rings.

By a directed set I is meant a set with an order relation $<$ which satisfies the following conditions: (i) $i < i$, for all $i \in I$; (ii) if $i < j$ and $j < k$, then $i < k$; (iii) for any $i, j \in I$, there exists $k \in I$ such that $i < k$ and $j < k$. It may happen that $i \neq j$, but $i < j$ and $j < i$.

A family of $\mathcal{A}$-modules $\{P_i\}_{i \in I}$, indexed by a directed set I, is called the direct system if, for any pair $i < j$, there is a morphism $r^i_j : P_i \to P_j$ such that (i) $r^i_i = \mathrm{Id}\, P_i$; (ii) $r^i_j \circ r^j_k = r^i_k$, $i < j < k$. A direct system of modules admits a direct limit.

Definition 11. This is an $\mathcal{A}$-module P_∞ together with morphisms $r^i_\infty : P_i \to P_\infty$ such that $r^i_\infty = r^j_\infty \circ r^i_j$ for all $i < j$. A module P_∞ consists of elements of a direct sum $\bigoplus P_i$ modulo the identification of elements of P_i with their images in P_j for all $i < j$.

Theorem 12. *Direct limits commute with direct sums and tensor products of modules. Namely, let $\{P_i\}$ and $\{Q_i\}$ be two direct systems of $\mathcal{A}$-modules which are indexed by the same directed set I, and let P_∞ and Q_∞ be their direct limits. Then direct limits of direct systems $\{P_i \oplus Q_i\}$ and $\{P_i \otimes Q_i\}$ are $P_\infty Q_\infty$ and $P_\infty \otimes Q_\infty$, respectively.*

Theorem 13. *A morphism of a direct system $\{P_i, r^i_j\}_I$ to a direct system $\{Q_{i'}, \rho^{i'}_{j'}\}_{I'}$ consists of an order preserving map $f : I \to I'$ and $\mathcal{A}$-module morphisms $\Phi_i : P_i \to Q_{f(i)}$ which obey compatibility conditions $\rho^{f(i)}_{f(j)} \circ \Phi_i = \Phi_j \circ r^i_j$. If P_∞ and Q_∞ are direct limits of these direct systems, there exists a unique $\mathcal{A}$-module morphism $\Phi_\infty : P_\infty \to Q_\infty$ such that $\rho^{f(i)}_\infty \circ \Phi_i = \Phi_\infty \circ r^i_\infty$.*

Theorem 14. *A construction of a direct limit morphism preserves monomorphisms and epimorphisms. If all $\Phi_i : P_i \to Q_{f(i)}$ are monomorphisms (resp., epimorphisms), so is $\Phi_\infty : P_\infty \to Q_\infty$.*

In a case of inverse systems of modules, we restrict our consideration to inverse sequences

$$P^0 \longleftarrow P^1 \longleftarrow \cdots P^i \xleftarrow{\pi^{i+1}_i} \cdots . \tag{9}$$

Its inverse limit is a module P^∞ together with morphisms $\pi^\infty_i : P^\infty \to P^i$ so that $\pi^\infty_i = \pi^j_i \circ \pi^\infty_j$ for all $i < j$. It consists of elements $(\ldots, p^i, \ldots)$, $p^i \in P^i$, of the Cartesian product $\prod P^i$ such that $p^i = \pi^j_i(p^j)$ for all $i < j$. A morphism of an inverse system $\{P_i, \pi^i_j\}$ to an inverse system $\{Q_i, \varrho^i_j\}$ consists of $\mathcal{A}$-module morphisms $\Phi_i : P_i \to Q_i$ which obey compatibility conditions

$$\Phi_j \circ \pi^i_j = \varrho^i_j \circ \Phi_i. \tag{10}$$

If P_∞ and Q_∞ are inverse limits of these inverse systems, there exists a unique $\mathcal{A}$-module morphism $\Phi_\infty : P_\infty \to Q_\infty$ such that $\Phi_j \circ \pi^\infty_j = \varrho^\infty_j \circ \Phi_\infty$. A construction of an inverse limits morphism preserves monomorphisms, but not epimorphisms.

Example 15. In particular, let $\{P_i, \pi^i_j\}$ be an inverse system of $\mathcal{A}$-modules and Q an $\mathcal{A}$-module together with $\mathcal{A}$-module morphisms $\Phi_i : Q \to P_i$ which obey compatibility conditions $\Phi_j = \pi^i_j \circ \Phi_i$. Then there exists a unique morphism $\Phi_\infty : Q \to P_\infty$ such that $\Phi_j = \pi^\infty_j \circ \Phi_\infty$.

Example 16. Let $\{P_i, \pi^i_j\}$ be an inverse system of $\mathcal{A}$-modules and Q an $\mathcal{A}$-module. Given a term P_r, let $\Phi_r : P_r \to Q$ be an $\mathcal{A}$-module morphism. It yields the pull-back morphisms

$$\pi^{r+k^*}_r \Phi_r = \Phi_r \circ \pi^{r+k}_r : P_{r+k} \longrightarrow Q \tag{11}$$

which obviously obey the compatibility conditions (10). Then there exists a unique morphism $\Phi_\infty : P_\infty \to Q$ such that $\Phi_\infty = \Phi_r \circ \pi^\infty_r$.

Example 17. Let $\{P_i\}$ be an inverse sequence of $\mathcal{A}$-modules. Given an $\mathcal{A}$-module Q, modules $\mathrm{Hom}_{\mathcal{A}}(P_i, Q)$ constitute a direct sequence whose direct limit is isomorphic to $\mathrm{Hom}_{\mathcal{A}}(P_\infty, Q)$.

2.2. $\mathbb{Z}_2$-Graded Commutative Rings. A $\mathcal{K}$-module Q is called $\mathbb{Z}_2$-graded if it is decomposed into a direct sum $Q = Q_* = Q_0 \oplus Q_1$ of modules Q_0 and Q_1, called the even and odd parts of Q_*, respectively. A $\mathbb{Z}_2$-graded $\mathcal{K}$-module is said to be free if it has a basis composed by graded-homogeneous elements.

A morphism $\Phi : P_* \to Q_*$ of $\mathbb{Z}_2$-graded $\mathcal{K}$-modules is said to be an even (resp., odd) morphism if Φ preserves (resp., changes) the $\mathbb{Z}_2$-parity of all homogeneous elements. A morphism $\Phi : P_* \to Q_*$ of $\mathbb{Z}_2$-graded $\mathcal{K}$-modules is called graded if it is represented by a sum of even and odd morphisms. A set $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ of these graded morphisms is a $\mathbb{Z}_2$-graded $\mathcal{K}$-module.

Definition 18. A $\mathcal{K}$-ring $\mathcal{A}$ is called $\mathbb{Z}_2$-graded if it is a $\mathbb{Z}_2$-graded $\mathcal{K}$-module $\mathcal{A}_*$, and a product of its homogeneous elements $\alpha\alpha'$ is a homogeneous element of degree $([a] + [a'])\mathrm{mod}\, 2$. In particular, $[\mathbf{1}] = 0$. Its even part $\mathcal{A}_0$ is a $\mathcal{K}$-ring, and the odd one $\mathcal{A}_1$ is an $\mathcal{A}_0$-bimodule.

Definition 19. A $\mathbb{Z}_2$-graded ring $\mathcal{A}_*$ is called graded commutative if

$$aa' = (-1)^{[a][a']} a'a, \quad a, a' \in \mathcal{A}_*. \tag{12}$$

Its even part $\mathcal{A}_0$ belongs to the center $\mathcal{Z}_{\mathcal{A}}$ of a ring $\mathcal{A}$.

Every $\mathbb{N}$-graded commutative $\mathcal{K}$-ring $\mathcal{A}^*$ (Definition 4) possesses the associated $\mathbb{Z}_2$-graded commutative structure $\mathcal{A}_*$ (4). For instance, the exterior algebra $\bigwedge Q$ of a $\mathcal{K}$-module Q in Example 9 is a $\mathbb{Z}_2$-graded commutative ring.

Definition 20. A $\mathbb{Z}_2$-graded commutative ring is called local if it contains a unique maximal $\mathbb{Z}_2$-graded ideal.

If $\mathcal{K}$ is a field, an exterior $\mathcal{K}$-algebra exemplifies a local $\mathbb{Z}_2$-graded commutative ring. An ideal of its nilpotents is a unique maximal ideal of its noninvertible elements which also is $\mathbb{Z}_2$-graded.

A $\mathbb{Z}_2$-graded commutative ring can admit different $\mathbb{Z}_2$-graded commutative structures $\mathcal{A}_*$ in general (Example 21).

By automorphisms of a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}_*$ are meant automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which are graded $\mathcal{K}$-module morphisms of $\mathcal{A}_*$. Obviously, they are even, and they preserve a $\mathbb{Z}_2$-graded structure of $\mathcal{A}$. However, there exist automorphisms ϕ of a $\mathcal{K}$-ring $\mathcal{A}$ which do not possess this property in general. Then $\mathcal{A}_*$ and $\phi(A_*)$ are isomorphic, but different $\mathbb{Z}_2$-graded commutative structures of a ring $\mathcal{A}$. Moreover, it may happen that a $\mathcal{K}$-ring $\mathcal{A}$ admits nonisomorphic $\mathbb{Z}_2$-graded commutative structures.

Example 21. Given a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}_*$ and its odd element κ, an automorphism

$$\begin{aligned} \phi : \mathcal{A}_0 \ni a &\longrightarrow a, \\ \mathcal{A}_1 \ni a &\longrightarrow a(\mathbf{1}+\kappa), \end{aligned} \tag{13}$$

of a $\mathcal{K}$-ring $\mathcal{A}$ does not preserve its original $\mathbb{Z}_2$-graded structure $\mathcal{A}_*$.

Given a $\mathbb{Z}_2$-graded commutative ring $\mathcal{A}_*$, a $\mathbb{Z}_2$-graded $\mathcal{A}_*$-module Q_* is defined as an $(\mathcal{A}-\mathcal{A})$-bimodule which is a $\mathbb{Z}_2$-graded $\mathcal{K}$-module such that

$$\begin{aligned} [aq] &= ([a]+[q]) \bmod 2, \\ qa &= (-1)^{[a][q]} aq, \quad a \in \mathcal{A}_*, \ q \in Q_*. \end{aligned} \tag{14}$$

The following are constructions of new $\mathbb{Z}_2$-graded $\mathcal{A}_*$-modules from the old ones.

(i) A direct sum of $\mathbb{Z}_2$-graded modules and a $\mathbb{Z}_2$-graded factor module are defined just as those of modules over a commutative ring.

(ii) A tensor product $P_* \otimes Q_*$ of $\mathbb{Z}_2$-graded $\mathcal{A}_*$-modules P_* and Q_* is their tensor product as $\mathcal{A}$-modules such that

$$\begin{aligned} &[p \otimes q] = ([p]+[q]) \bmod 2, \quad p \in P_*, \ q \in Q_*, \\ &ap \otimes q = (-1)^{[p][a]} pa \otimes q = (-1)^{[p][a]} p \otimes aq, \\ &\qquad a \in \mathcal{A}_*. \end{aligned} \tag{15}$$

In particular, the tensor algebra $\bigotimes P_*$ of a $\mathbb{Z}_2$-graded $\mathcal{A}_*$-module P_* is defined just as that (8) of a module over a commutative ring. Its quotient $\bigwedge P_*$ with respect to the ideal generated by elements

$$p \otimes p' + (-1)^{[p][p']} p' \otimes p, \quad p, p' \in P_* \tag{16}$$

is the exterior algebra of a $\mathbb{Z}_2$-graded module P_* with respect to the graded exterior product

$$p \wedge p' = -(-1)^{[p][p']} p' \wedge p. \tag{17}$$

(iii) A graded morphism $\Phi : P_* \to Q_*$ of $\mathbb{Z}_2$-graded $\mathcal{A}_*$-modules is their graded morphism as $\mathbb{Z}_2$-graded $\mathcal{K}$-modules which obeys the relations

$$\Phi(ap) = (-1)^{[\Phi][a]} a\Phi(p), \quad p \in P_*, \ a \in \mathcal{A}_*. \tag{18}$$

These morphisms form a $\mathbb{Z}_2$-graded $\mathcal{A}_*$-module $\mathrm{Hom}_{\mathcal{A}}(P_*, Q_*)$. A $\mathbb{Z}_2$-graded $\mathcal{A}_*$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P_*, \mathcal{A}_*)$ is called the dual of a $\mathbb{Z}_2$-graded $\mathcal{A}_*$-module P_*.

In the sequel, we are concerned with $\mathbb{Z}_2$-graded manifolds (Section 5). They are sheaves in Grassmann algebras which are defined as follows.

Definition 22. A $\mathbb{Z}_2$-graded $\mathcal{K}$-ring Λ_* is said to be finitely generated in degree 1 if it is a free $\mathcal{K}$-module of finite rank so that $\Lambda_0 = \mathcal{K} \oplus \Lambda_1^2$.

It follows that a $\mathcal{K}$-module Λ has a decomposition

$$\begin{aligned} \Lambda &= \mathcal{K} \oplus R, \\ R &= \Lambda_1 \oplus (\Lambda_1)^2, \end{aligned} \tag{19}$$

where R is the ideal of nilpotents of a ring Λ. A surjection $\sigma : \Lambda \to \mathcal{K}$ is called the body map.

Definition 23. A $\mathbb{Z}_2$-graded commutative $\mathcal{K}$-ring Λ_* is said to be the Grassmann algebra if it is finitely generated in degree 1 and is isomorphic to the exterior algebra $\bigwedge(R/R^2)$ (Example 9) of a $\mathcal{K}$-module R/R^2, where R is the ideal of nilpotents (19) of Λ_*.

An exterior algebra $\bigwedge Q$ of a free $\mathcal{K}$-module Q of finite rank is a Grassmann algebra. Conversely, a Grassmann algebra admits a structure of an exterior algebra $\bigwedge Q$ by a choice of its minimal generating $\mathcal{K}$-module $Q \subset \Lambda_1$, and all these structures are mutually isomorphic if $\mathcal{K}$ is a field (Theorem 26). Automorphisms of a Grassmann algebra preserve its ideal R of nilpotents and the splitting (19), but need not the odd sector Λ_1 (Example 21).

A Grassmann algebra is local in accordance with Definition 20. Its ideal of nilpotents R is a unique maximal ideal which is graded in accordance with the decomposition (19).

Remark 24. Let $\mathcal{A}_*$ be a $\mathbb{Z}_2$-graded commutative ring. A $\mathbb{Z}_2$-graded $\mathcal{A}_*$-algebra $\mathcal{G}_*$ is called the Lie $\mathcal{A}_*$-superalgebra if its product $[\cdot,\cdot]$, called the Lie superbracket, obeys the rules

$$\begin{aligned} &[\varepsilon, \varepsilon'] = -(-1)^{[\varepsilon][\varepsilon']} [\varepsilon', \varepsilon], \\ &(-1)^{[\varepsilon][\varepsilon'']} [\varepsilon, [\varepsilon', \varepsilon'']] + (-1)^{[\varepsilon'][\varepsilon]} [\varepsilon', [\varepsilon'', \varepsilon]] \\ &\quad + (-1)^{[\varepsilon''][\varepsilon']} [\varepsilon'', [\varepsilon, \varepsilon']] = 0. \end{aligned} \tag{20}$$

Clearly, an even part $\mathcal{G}_0$ of a Lie superalgebra $\mathcal{G}_*$ is a Lie $\mathcal{A}_0$-algebra. Given an $\mathcal{A}_*$-superalgebra, a $\mathbb{Z}_2$-graded $\mathcal{A}_*$-module P_* is called a $\mathcal{G}_*$-module if it is provided with an $\mathcal{A}_*$-bilinear map

$$\begin{aligned} &\mathcal{G}_* \times P_* \ni (\varepsilon, p) \longrightarrow \varepsilon p \in P_*, \\ &[\varepsilon p] = ([\varepsilon] + [p]) \bmod 2, \\ &[\varepsilon, \varepsilon'] p = \left(\varepsilon \circ \varepsilon' - (-1)^{[\varepsilon][\varepsilon']} \varepsilon' \circ \varepsilon\right) p. \end{aligned} \tag{21}$$

2.3. $\mathbb{N}$-Graded Commutative Rings. Let $\mathcal{A} = \mathcal{A}^*$ be an $\mathbb{N}$-graded $\mathcal{K}$-ring (Definition 2). Seen as a $\mathcal{K}$-ring, it can admit different $\mathbb{N}$-graded structures. All these structures are isomorphic in the following case [20].

Definition 25. An $\mathbb{N}$-graded $\mathcal{K}$-ring $\mathcal{A}^*$ is called finitely generated in degree 1 if the following hold: (i) $\mathcal{A}^0 = \mathcal{K}$; (ii)

$\mathcal{A}^1$ is a free $\mathcal{K}$-module of finite rank; (iii) $\mathcal{A}^*$ is generated by $\mathcal{A}^1$; namely, if R is an ideal generated by $\mathcal{A}^1$, then there is $\mathcal{K}$-module isomorphism $\mathcal{A}/R = \mathcal{K}$, $R/R^2 = \mathcal{A}^1$.

Theorem 26. *Let $\mathcal{K}$ be a field, and let $\mathcal{A}^*$ and Λ^* be $\mathbb{N}$-graded $\mathcal{K}$-rings finitely generated in degree 1. If they are isomorphic as $\mathcal{K}$-rings, there exists their graded isomorphism $\Phi : \mathcal{A}^* \to \Lambda^*$ so that $\Phi(\mathcal{A}^i) = \Lambda^i$ for all $i \in \mathbb{N}$.*

As was mentioned above, we restrict our consideration to N-graded commutative rings (Definition 4), unless they are the differential graded ones (Section 2.4). They also possess the associated $\mathbb{Z}_2$-graded commutative structure (4).

Definition 27. An $\mathbb{N}$-graded commutative ring is called local if it contains a unique maximal $\mathbb{N}$-graded ideal.

Certainly, if an $\mathbb{N}$-graded ring $\mathcal{A}^*$ is local, the associated $\mathbb{Z}_2$-graded ring $\mathcal{A}_*$ is well. A Grassmann-graded ring over a field $\mathcal{K}$ is local. The ideal of its nilpotents is a unique maximal ideal of its noninvertible elements which also is $\mathbb{N}$-graded.

Given an $\mathbb{N}$-graded commutative ring $\mathcal{A}^*$, an $\mathbb{N}$-graded $\mathcal{A}^*$-module Q^* is defined as a graded $(\mathcal{A}^* - \mathcal{A}^*)$-bimodule which is an $\mathbb{N}$-graded $\mathcal{K}$-module such that

$$\begin{gathered} qa = (-1)^{[a][q]} aq, \\ [aq] = [a] + [q], \\ a \in \mathcal{A}^*, \ q \in Q^*, \end{gathered} \tag{22}$$

and it also is a $\mathbb{Z}_2$-graded module. A direct sum, a tensor product of $\mathbb{N}$-graded modules, and the exterior algebra $\bigwedge Q^*$ of an $\mathbb{N}$-graded module Q^* are defined similarly to those of $\mathbb{Z}_2$-graded modules (Section 2.2), and they also are a direct sum, a tensor product, and an exterior algebra of associated $\mathbb{Z}_2$-graded modules, respectively.

A morphism $\Phi : P^* \to Q^*$ of $\mathbb{N}$-graded $\mathcal{A}^*$-modules seen as $\mathcal{K}$-modules is said to be homogeneous of degree $[\Phi]$ if $[\Phi(p)] = [p] + [\Phi]$ for all homogeneous elements $p \in P^*$ and the relations (18) hold. A morphism $\Phi : P^* \to Q^*$ of $\mathbb{N}$-graded $\mathcal{A}^*$-modules as the $\mathcal{K}$-ones is called the $\mathbb{N}$-graded $\mathcal{A}^*$-module morphism if it is represented by a sum of homogeneous morphisms. Therefore, a set $\mathrm{Hom}_{\mathcal{A}}(P^*, Q^*)$ of graded morphisms $P^* \to Q^*$ is an $\mathbb{N}$-graded $\mathcal{A}^*$-module. An $\mathbb{N}$-graded $\mathcal{A}^*$-module $P^* = \mathrm{Hom}_{\mathcal{A}}(P^*, \mathcal{A}^*)$ is called the dual of an $\mathbb{N}$-graded $\mathcal{A}^*$-module P^*. Certainly, an $\mathbb{N}$-graded $\mathcal{A}^*$-module morphism of $\mathbb{N}$-graded $\mathcal{A}^*$-modules is their $\mathbb{Z}_2$-graded $\mathcal{A}_*$-module morphism as associated $\mathbb{Z}_2$-graded modules, but the converse is true.

By automorphisms of an $\mathbb{N}$-graded ring $\mathcal{A}^*$ are meant automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which preserve its $\mathbb{N}$-gradation $\mathcal{A}^*$. They also keep the associated $\mathbb{Z}_2$-structure $\mathcal{A}_*$ of $\mathcal{A}$. However, there exist automorphisms of a $\mathcal{K}$-ring $\mathcal{A}$ which do not possess these properties in general.

Let Λ^* be a Grassmann-graded $\mathcal{K}$-ring (Definition 6). Its associated $\mathbb{Z}_2$-graded commutative ring is a Grassmann algebra Λ (Definition 23). Conversely, any Grassmann algebra Λ admits the associated structure of a Grassmann-graded ring Λ^* by a choice of its minimal generating $\mathcal{K}$-module $\Lambda^1 \subset \Lambda_1$. Given a generating basis $\{c^i\}$ for a $\mathcal{K}$-module Λ^1, elements of a Grassmann-graded ring Λ^* take a form

$$a = \sum_{k=0,1,\ldots} \sum a_{i_1\cdots i_k} c^{i_1} \cdots c^{i_k}. \tag{23}$$

We agree to call $\{c^i\}$ the generating basis for the associated Grassmann algebra Λ_* which brings it into a Grassmann-graded ring Λ^*.

Given a generating basis $\{c^i\}$ for a Grassmann-graded ring Λ^*, one can show that any $\mathcal{K}$-ring automorphism is a composition of automorphisms

$$c^i \longrightarrow c'^i = \rho^i_j c^j + b^i, \tag{24}$$

where ρ is an automorphism of a $\mathcal{K}$-module Λ^1 and b^i are odd elements of $\Lambda^{>2}$ and of morphisms

$$c^i \longrightarrow c'^i = c^i (\mathbf{1} + \kappa), \quad \kappa \in \Lambda_1. \tag{25}$$

Automorphisms (24), where $b^i = 0$, are automorphisms $c'^i = \rho^i_j c^j$ of a Grassmann-graded ring Λ^*. If $b^i \neq 0$, the automorphism (24) preserves the associated $\mathbb{Z}_2$-graded structure Λ_* of Λ but does not keep its $\mathbb{N}$-graded structure Λ^*. It yields a different $\mathbb{N}$-graded structure Λ'^*, where $\{c'^i\}$ (24) is a basis for Λ'^1 and the generating basis for Λ'^*. Automorphisms (25) preserve an even sector Λ_0 of Λ^*, but not the odd one Λ_1 (Example 21). However, it follows from Theorem 26 that different $\mathbb{N}$- and $\mathbb{Z}_2$-graded structures of a Grassmann-graded ring are mutually isomorphic if $\mathcal{K}$ is a field. As a consequence, we come to the following.

Theorem 28. *Given a Grassmann-graded ring Λ^* over a field $\mathcal{K}$, there exists a finite-dimensional vector space W over $\mathcal{K}$ so that Λ^* is isomorphic to the exterior algebra $\bigwedge W$ of W (Example 9) seen as a Grassmann-graded ring generated by W.*

2.4. *Differential $\mathbb{N}$-Graded Rings.*

If an $\mathbb{N}$-graded ring also is a cochain complex, we come to the following notion [5, 17].

Definition 29. An $\mathbb{N}$-graded $\mathcal{K}$-ring Ω^* is called the differential graded ring (henceforth, DGR) if it is a cochain complex of $\mathcal{K}$-modules

$$0 \longrightarrow \mathcal{K} \longrightarrow \Omega^0 \xrightarrow{\delta} \Omega^1 \xrightarrow{\delta} \cdots \Omega^k \xrightarrow{\delta} \cdots \tag{26}$$

with respect to a coboundary operator δ which obeys the graded Leibniz rule

$$\delta(\alpha\beta) = \delta\alpha\beta + (-1)^{|\alpha|} \alpha\delta\beta. \tag{27}$$

The cochain complex (26) is called the de Rham complex of a DGR (Ω^*, δ). It also is said to be the differential graded calculus over a $\mathcal{K}$-ring Ω^0.

Given a DGR (Ω^*, δ), one considers its minimal differential graded subring $(\overline{\Omega}^*, \delta)$ which contains Ω^0. Seen as a $(\Omega^0 - \Omega^0)$-ring, it is generated by elements δa, $a \in \mathcal{A}$, and consists of monomials $\alpha = a_0\delta a_1 \cdots \delta a_k$, $a_i \in \Omega^0$, whose product obeys the juxtaposition rule

$$(a_0\delta a_1)(b_0\delta b_1) = a_0\delta(a_1b_0)\delta b_1 - a_0a_1\delta b_0\delta b_1 \tag{28}$$

in accordance with equality (27). A complex $(\overline{\Omega}^*, \delta)$ is called the minimal differential graded calculus over Ω^0. Its cohomology is said to be the de Rham cohomology of (Ω^*, δ).

One can associate a DGR to any Lie $\mathcal{K}$-algebra $\mathcal{G}$ as follows [5, 21]. Let a $\mathcal{K}$-ring Q be a $\mathcal{G}$-module so that $\mathcal{G}$ acts on Q on the left by endomorphisms

$$\begin{gathered}\mathcal{G} \times Q \ni (\varepsilon, q) \longrightarrow \varepsilon q \in Q, \\ [\varepsilon, \varepsilon'] q = (\varepsilon \circ \varepsilon' - \varepsilon' \circ \varepsilon) q, \\ \varepsilon, \varepsilon' \in \mathcal{G},\end{gathered} \tag{29}$$

(cf. Remark 24). For instance, $Q = \mathcal{K}$ and $\mathcal{G} : \mathcal{K} \to 0$. A $\mathcal{K}$-multilinear skew-symmetric map

$$c^k : \overset{k}{\times} \mathcal{G} \longrightarrow Q \tag{30}$$

is called the Q-valued k-cochain on a Lie algebra $\mathcal{G}$. These cochains form a $\mathcal{G}$-module $C^k[\mathcal{G}; Q]$. Let us put $C^0[\mathcal{G}; Q] = Q$. We obtain the cochain complex

$$\begin{gathered}0 \longrightarrow \mathcal{K} \longrightarrow Q \xrightarrow{\delta} C^1[\mathcal{G}; Q] \xrightarrow{\delta} \cdots C^k[\mathcal{G}; Q] \\ \xrightarrow{\delta} \cdots\end{gathered} \tag{31}$$

with respect to the Chevalley–Eilenberg coboundary operators

$$\begin{aligned}\delta(c^k)(\varepsilon_0, \ldots, \varepsilon_k) &= \sum_{i=0}^{k} (-1)^i \varepsilon_i c^k (\varepsilon_0, \ldots, \widehat{\varepsilon}_i, \ldots, \varepsilon_k) \\ &+ \sum_{1 \le i < j \le k} (-1)^{i+j} \\ &\cdot c^k ([\varepsilon_i, \varepsilon_j], \varepsilon_0, \ldots, \widehat{\varepsilon}_i, \ldots, \widehat{\varepsilon}_j, \ldots, \varepsilon_k),\end{aligned} \tag{32}$$

where the caret $\widehat{}$ denotes omission. Complex (31) is called the Chevalley–Eilenberg complex of a Lie algebra $\mathcal{G}$ with coefficients in a ring Q. It is a DGR with respect to the exterior product of skew-symmetric maps (30).

A construction of the Chevalley–Eilenberg complex is extended to Lie superalgebras [5, 21].

3. Differential Calculus over Commutative Rings

Conventional technique of the differential calculus over commutative rings includes formalism of linear differential operators and the Chevalley–Eilenberg differential calculus [3–5].

3.1. Differential Operators on Modules over Commutative Rings. As was mentioned above, $\mathcal{K}$ throughout is a commutative ring without a divisor of zero. Let $\mathcal{A}$ be a commutative $\mathcal{K}$-ring, and let P and Q be $\mathcal{A}$-modules. A $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ of $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can be endowed with two different $\mathcal{A}$-module structures

$$\begin{gathered}(a\Phi)(p) = a\Phi(p), \\ (\Phi \bullet a)(p) = \Phi(ap), \\ a \in \mathcal{A}, \ p \in P.\end{gathered} \tag{33}$$

We refer to the second one as an $\mathcal{A}^\bullet$-module structure. Let us put $\delta_a\Phi = a\Phi - \Phi \bullet a$, $a \in \mathcal{A}$.

Definition 30. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is called the linear s-order Q-valued differential operator on P if $(\delta_{a_0} \circ \cdots \circ \delta_{a_s})\Delta = 0$ for any tuple of $s + 1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. A set $\mathrm{Diff}_s(P, Q)$ of these operators inherits the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures (33).

In particular, linear zero-order differential operators obey a condition

$$\delta_a\Delta(p) = a\Delta(p) - \Delta(ap) = 0, \quad a \in \mathcal{A}, \ p \in P, \tag{34}$$

and, consequently, they coincide with $\mathcal{A}$-module morphisms $P \to Q$. A linear first-order differential operator Δ satisfies a relation

$$\begin{aligned}(\delta_b \circ \delta_a)\Delta(p) &= ba\Delta(p) - b\Delta(ap) - a\Delta(bp) \\ &+ \Delta(abp) = 0, \quad a, b \in \mathcal{A}.\end{aligned} \tag{35}$$

Of course, an s-order differential operator is of $(s+1)$-order. Therefore, there is a direct sequence

$$\begin{gathered}\mathrm{Diff}_0(P, Q) \longrightarrow \mathrm{Diff}_1(P, Q) \cdots \longrightarrow \mathrm{Diff}_r(P, Q) \\ \longrightarrow \cdots\end{gathered} \tag{36}$$

of linear Q-valued differential operators on an $\mathcal{A}$-module P. Its direct limit is an $\mathcal{A} - \mathcal{A}^\bullet$-module $\mathrm{Diff}_\infty(P, Q)$ of all linear Q-valued differential operators on P.

In particular, let $P = \mathcal{A}$. Any linear zero-order Q-valued differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is an $\mathcal{A}$-module isomorphism $\mathrm{Diff}_0(\mathcal{A}, Q) = Q$ via the association $q \to \Delta_q$, where Δ_q is given by an equality $\Delta_q(\mathbf{1}) = q$. A linear first-order differential operator Δ on $\mathcal{A}$ fulfils a condition

$$\Delta(ab) = b\Delta(a) + a\Delta(b) - ba\Delta(\mathbf{1}), \quad a, b \in \mathcal{A}. \tag{37}$$

Definition 31. It is called a Q-valued derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$; that is, it obeys the Leibniz rule

$$\Delta(ab) = \Delta(a)b + a\Delta(b), \quad a, b \in \mathcal{A}. \tag{38}$$

If ∂ is a derivation of $\mathcal{A}$, then $a\partial$ is well for any $a \in \mathcal{A}$. Hence, derivations of $\mathcal{A}$ constitute an $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the derivation module of $\mathcal{A}$.

If $Q = \mathcal{A}$, the derivation module $\mathfrak{d}\mathcal{A} = \mathfrak{d}(\mathcal{A}, \mathcal{A})$ of $\mathcal{A}$ also is a Lie algebra over a ring $\mathcal{K}$ with respect to a Lie bracket

$$[u, u'] = u \circ u' - u' \circ u, \quad u, u' \in \mathfrak{d}\mathcal{A}. \tag{39}$$

3.2. Jets of Modules. A linear s-order differential operator on an $\mathcal{A}$-module P is represented by a zero-order differential operator on a module of s-order jets of P (Theorem 33).

Given an $\mathcal{A}$-module P, let $\mathcal{A} \otimes_{\mathcal{K}} P$ be a tensor product of $\mathcal{K}$-modules $\mathcal{A}$ and P. We put

$$\delta^b (a \otimes p) = (ba) \otimes p - a \otimes (bp), \quad p \in P, \ a, b \in \mathcal{A}. \tag{40}$$

Let us denote by μ^{k+1} a submodule of $\mathcal{A} \otimes_{\mathcal{K}} P$ generated by elements

$$\delta^{b_0} \circ \cdots \circ \delta^{b_k} (a \otimes p) = a \delta^{b_0} \circ \cdots \circ \delta^{b_k} (\mathbf{1} \otimes p). \tag{41}$$

Definition 32. A k-order jet module $\mathcal{J}^k(P)$ of a module P is defined as the quotient of a $\mathcal{K}$-module $\mathcal{A} \otimes_{\mathcal{K}} P$ by μ^{k+1}. We denote its elements $c \otimes_k p$.

In particular, a first-order jet module $\mathcal{J}^1(P)$ consists of elements $c \otimes_1 p$ modulo the relations

$$\begin{aligned} \delta^a \circ \delta^b (\mathbf{1} \otimes_1 p) = ab \otimes_1 p - b \otimes_1 (ap) - a \otimes_1 (bp) \\ + \mathbf{1} \otimes_1 (abp) = 0. \end{aligned} \tag{42}$$

A $\mathcal{K}$-module $\mathcal{J}^k(\mathrm{P})$ is endowed with the $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures

$$\begin{aligned} b (a \otimes_k p) &= ba \otimes_k p, \\ b \bullet (a \otimes_k p) &= a \otimes_k (bp). \end{aligned} \tag{43}$$

There exists a module morphism

$$J^k : P \ni p \longrightarrow \mathbf{1} \otimes_k p \in \mathcal{J}^k (P) \tag{44}$$

of an $\mathcal{A}$-module P to an $\mathcal{A}^\bullet$-module $\mathcal{J}^k(P)$ such that $\mathcal{J}^k(P)$, seen as an $\mathcal{A}$-module, is generated by elements $J^k p$, $p \in P$. One can show the following [3, 4].

Theorem 33. *Any linear k-order Q-valued differential operator Δ on an $\mathcal{A}$-module P uniquely factorizes as*

$$\Delta : P \xrightarrow{J^k} \mathcal{J}^k (P) \xrightarrow{\mathfrak{f}^\Delta} Q \tag{45}$$

through the morphism J^k (44) and some $\mathcal{A}$-module homomorphism $\mathfrak{f}^\Delta : \mathcal{J}^k(P) \to Q$. The correspondence $\Delta \to \mathfrak{f}^\Delta$ defines an $\mathcal{A}$-module isomorphism

$$\mathrm{Diff}_k (P, Q) = \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{J}^k (P), Q\right). \tag{46}$$

Due to monomorphisms $\mu^r \to \mu^s$, $r > s$, there exist $\mathcal{A}$-module epimorphisms of jet modules

$$\begin{aligned} \pi^{i+1}_i : \mathcal{J}^{i+1} (P) &\longrightarrow \mathcal{J}^i (P), \\ \pi^1_0 : \mathcal{J}^1 (P) \ni a \otimes_1 p &\longrightarrow ap \in P. \end{aligned} \tag{47}$$

Thus, there is an inverse sequence

$$P \xleftarrow{\pi^1_0} \mathcal{J}^1 (P) \cdots \xleftarrow{\pi^r_{r-1}} \mathcal{J}^r (P) \longleftarrow \cdots \tag{48}$$

of jet modules. Its inverse limit $\mathcal{J}^\infty(P)$ is an $\mathcal{A}$-module together with $\mathcal{A}$-module morphisms

$$\begin{aligned} \pi^\infty_r : \mathcal{J}^\infty (P) &\to \mathcal{J}^r (P), \\ \pi^\infty_{r<s} &= \pi^s_r \circ \pi^\infty_s. \end{aligned} \tag{49}$$

In particular, let us consider a module P together with the morphisms J^r (44) which obey compatibility conditions $J^r(p) = \pi^{r+k}_r \circ J^{r+k}(p)$, $p \in P$. Then it follows from Example 15 that there exists an $\mathcal{A}$-module morphism

$$J^\infty : P \ni p \longrightarrow \left(p, J^1 p, \ldots, J^r p, \ldots\right) \in \mathcal{J}^\infty (P) \tag{50}$$

so that $J^r(p) = \pi^\infty_r \circ J^\infty(p)$. The inverse sequence (48) yields a direct sequence

$$\begin{aligned} \mathrm{Hom}_{\mathcal{A}} (P, Q) \xrightarrow{\pi^{1\,*}_0} \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{J}^1 (P), Q\right) \cdots \\ \xrightarrow{\pi^{r\;*}_{r-1}} \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{J}^r (P), Q\right) \cdots, \end{aligned} \tag{51}$$

where

$$\begin{aligned} \pi^{r\;*}_{r-1} : \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{J}^{r-1} (P), Q\right) \ni \Delta \longrightarrow \Delta \circ \pi^r_{r-1} \\ \in \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{J}^r (P), Q\right) \end{aligned} \tag{52}$$

is the pull-back $\mathcal{A}$-module morphism (11). Its direct limit is an $\mathcal{A}$-module $\mathrm{Hom}_{\mathcal{A}}(\mathcal{J}^\infty(P), Q)$ (Example 17).

Theorem 34. *One has the isomorphisms (46) of the direct systems (36) and (51) which leads to an $\mathcal{A}$-module isomorphism*

$$\mathrm{Diff}_\infty (P, Q) = \mathrm{Hom}_{\mathcal{A}} \left(\mathcal{J}^\infty (P), Q\right) \tag{53}$$

of their direct limits in accordance with Theorem 14.

Proof. Any element $\Delta_\infty = \Delta \in \mathrm{Diff}_\infty(P, Q)$ factorizes as

$$\Delta_\infty : P \xrightarrow{J^\infty} \mathcal{J}^\infty (P) \xrightarrow{\mathfrak{f}^\Delta_\infty} Q \tag{54}$$

through the morphism J^∞ (50) and an $\mathcal{A}$-module homomorphism $\mathfrak{f}^\Delta_\infty = \mathfrak{f}^\Delta \circ \pi^\infty_k$ (Example 16) in accordance with the commutative diagram

$$\begin{tikzcd} & \mathcal{J}^\infty(P) \arrow[d] \arrow[dr, "\mathfrak{f}^\Delta_\infty"] & \\ P \arrow[ur, "J^\infty"] \arrow[r, "J^k"'] & \mathcal{J}^k(P) \arrow[r, "\mathfrak{f}^\Delta"'] & Q \end{tikzcd} \tag{55}$$

□

3.3. Chevalley–Eilenberg Differential Calculus over Commutative Rings. Since the derivation module $\mathfrak{d}\mathcal{A}$ of a commutative $\mathcal{K}$-ring $\mathcal{A}$ is a Lie $\mathcal{K}$-algebra, one can associate to $\mathcal{A}$ the following DGR (59), called the Chevalley–Eilenberg differential calculus over $\mathcal{A}$.

Given a Lie $\mathcal{K}$-algebra $\mathfrak{d}\mathcal{A}$, let us consider the Chevalley–Eilenberg complex $C^*[\mathfrak{d}\mathcal{A}; \mathcal{A}]$ (31) of $\mathfrak{d}\mathcal{A}$ with coefficients in a ring $\mathcal{A}$ regarded as a $\mathfrak{d}\mathcal{A}$-module [5, 7]. This complex contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-multilinear skew-symmetric maps

$$\mathcal{O}^k[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}\left(\overset{k}{\times}\mathfrak{d}\mathcal{A},\mathcal{A}\right) \ni \phi : \overset{k}{\times}\mathfrak{d}\mathcal{A} \longrightarrow \mathcal{A} \tag{56}$$

with respect to the Chevalley–Eilenberg coboundary operator (32):

$$\begin{aligned} d\phi(u_0,\ldots,u_k) &= \sum_{i=0}^{k}(-1)^i u_i\left(\phi\left(u_0,\ldots,\widehat{u}_i,\ldots,u_k\right)\right) \\ &+ \sum_{i<j}(-1)^{i+j}\phi\left(\left[u_i,u_j\right],u_0,\ldots,\widehat{u}_i,\ldots,\widehat{u}_j,\ldots,u_k\right). \end{aligned} \tag{57}$$

Indeed, it is readily justified that if ϕ (56) is an $\mathcal{A}$-multilinear map, $d\phi$ (57) is well. In particular,

$$(da)(u) = u(a), \quad a \in \mathcal{O}^0[\mathfrak{d}\mathcal{A}] = \mathcal{A}. \tag{58}$$

It follows that $d(\mathbf{1}) = 0$; that is, d is an $\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$-valued derivation of $\mathcal{A}$.

Let us define an $\mathbb{N}$-graded $\mathcal{A}$-module

$$\mathcal{O}^*[\mathfrak{d}\mathcal{A}] = \bigoplus_{i\in\mathbb{N}}\mathcal{O}^i[\mathfrak{d}\mathcal{A}]. \tag{59}$$

It is provided with the structure of an $\mathbb{N}$-graded $\mathcal{A}$-ring with respect to a product

$$\begin{aligned} &\phi\wedge\phi'(u_1,\ldots,u_{r+s}) \\ &\quad= \sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\mathrm{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi\left(u_{i_1},\ldots,u_{i_r}\right) \\ &\quad\cdot\phi'\left(u_{j_1},\ldots,u_{j_s}\right), \\ &\qquad \phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}],\ \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}],\ u_k\in\mathfrak{d}\mathcal{A}, \end{aligned} \tag{60}$$

where $\mathrm{sgn}^{\cdots}_{\cdots}$ denotes the sign of a permutation. This product obeys relations

$$\begin{aligned} &\phi\wedge\phi' = (-1)^{|\phi||\phi'|}\phi'\wedge\phi, \\ &d\left(\phi\wedge\phi'\right) = d(\phi)\wedge\phi' + (-1)^{|\phi|}\phi\wedge d\left(\phi'\right), \\ &\qquad \phi,\phi'\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}]. \end{aligned} \tag{61}$$

By the first one, $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is an $\mathbb{N}$-graded commutative ring. Relation (61) shows that $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is a DGR (Definition 29), called the Chevalley–Eilenberg differential calculus over a $\mathcal{K}$-ring $\mathcal{A}$.

Since $\mathcal{O}^1[\mathfrak{d}\mathcal{A}] = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A}) = \mathfrak{d}\mathcal{A}^*$ and, consequently, $\mathfrak{d}\mathcal{A} \subset \mathfrak{d}\mathcal{A}^{**} = \mathcal{O}^1[\mathfrak{d}\mathcal{A}]^*$, we have the interior product $u\rfloor\phi = \phi(u)$, $u\in\mathfrak{d}\mathcal{A}$, $\phi\in\mathcal{O}^1[\mathfrak{d}\mathcal{A}]$. It is extended as

$$\begin{aligned} &(u\rfloor\phi)(u_1,\ldots,u_{k-1}) = k\phi(u,u_1,\ldots,u_{k-1}), \\ &\qquad u\in\mathfrak{d}\mathcal{A},\ \phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}], \end{aligned} \tag{62}$$

to a DGR $(\mathcal{O}^*[\mathfrak{d}\mathcal{A}], d)$, and obeys a relation

$$u\rfloor(\phi\wedge\sigma) = u\rfloor\phi\wedge\sigma + (-1)^{|\phi|}\phi\wedge u\rfloor\sigma. \tag{63}$$

With the interior product (62), one defines a derivation

$$\begin{aligned} &\mathbf{L}_u(\phi) = d(u\rfloor\phi) + u\rfloor d\phi, \quad \phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}], \\ &\mathbf{L}_u(\phi\wedge\sigma) = \mathbf{L}_u(\phi)\wedge\sigma + \phi\wedge\mathbf{L}_u\sigma, \end{aligned} \tag{64}$$

of an $\mathbb{N}$-graded ring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ for any $u\in\mathfrak{d}\mathcal{A}$. Then one can think of elements of $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ as being differential forms over $\mathcal{A}$.

The minimal Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}$ over a ring $\mathcal{A}$ consists of the monomials $a_0da_1\wedge\cdots\wedge da_k$, $a_i\in\mathcal{A}$. Its de Rham complex

$$0\longrightarrow\mathcal{K}\longrightarrow\mathcal{A}\xrightarrow{d}\mathcal{O}^1\mathcal{A}\xrightarrow{d}\cdots\mathcal{O}^k\mathcal{A}\xrightarrow{d}\cdots \tag{65}$$

is called the de Rham complex of a $\mathcal{K}$-ring $\mathcal{A}$.

3.4. Differential Calculus over $C^\infty(X)$. Let X be a smooth manifold (Remark 35) and $C^\infty(X)$ an $\mathbb{R}$-ring of real smooth functions on X. The differential calculus on a smooth manifold X is defined as that over a ring $C^\infty(X)$.

Remark 35. Throughout the work, smooth manifolds are finite-dimensional real manifolds. We follow the notion of a manifold without boundary. A smooth manifold customarily is assumed to be Hausdorff and second-countable topological space. Consequently, it is a locally compact countable at infinity space and a paracompact space, which admits the partition of unity by smooth real functions. Unless otherwise stated, manifolds are assumed to be connected.

Similarly to a sheaf C^0_X of continuous functions (Example A.3), a sheaf C^∞_X of smooth real functions on X is defined. Its stalk C^∞_x at $x \in X$ has a unique maximal ideal of germs of functions vanishing at x. Therefore, C^∞_X is a local-ringed space (Definition A.2). Though a sheaf C^∞_X exists on a topological space X, it fixes a unique smooth manifold structure on X as follows.

Theorem 36. *Let X be a paracompact topological space and $(X,\mathfrak{R})$ a local-ringed space. Let X admit an open cover $\{U_i\}$ such that a sheaf $\mathfrak{R}$ restricted to each U_i is isomorphic to a local-ringed space $(\mathbb{R}^n, C^\infty_{R^n})$. Then X is an n-dimensional smooth manifold together with a natural isomorphism of local-ringed spaces $(X,\mathfrak{R})$ and (X, C^∞_X).*

One can think of this result as being an equivalent definition of smooth real manifolds in terms of local-ringed spaces. A smooth manifold X also is algebraically reproduced as a certain subspace of the spectrum of a real ring $C^\infty(X)$ of smooth real functions on X [7, 14].

Moreover, the well-known Serre–Swan theorem (Theorem 37) states the categorial equivalence between the vector bundles over a smooth manifold X and projective modules of finite rank over the ring $C^\infty(X)$ of smooth real functions on X. This theorem originally has been proved in the case of a compact manifold X, but it is generalized to an arbitrary smooth manifold [7, 22].

Theorem 37. *Let X be a smooth manifold. A $C^\infty(X)$-module P is a projective module of finite rank iff it is isomorphic to the structure module $Y(X)$ of global sections of some smooth vector bundle $Y \to X$ over X.*

In particular, the derivation module of a real ring $C^\infty(X)$ coincides with a $C^\infty(X)$-module $\mathcal{T}_1(X)$ of vector fields on X, that is, the structure module of sections of the tangent bundle TX of X. Hence, it is a projective $C^\infty(X)$-module of finite rank. Its $C^\infty(X)$-dual $\mathcal{O}^1(X) = \mathcal{T}_1(X)^*$ is the structure module $\mathcal{O}^1(X)$ of the cotangent bundle T^*X of X which is a module of one-form on X and, conversely, $\mathcal{T}_1(X) = \mathcal{O}^1(X)^*$ (Theorem 10). It follows that the Chevalley–Eilenberg differential calculus over a real ring $C^\infty(X)$ is exactly the DGR $(\mathcal{O}^*(X), d)$ of exterior forms on X, where the Chevalley–Eilenberg coboundary operator d (57) coincides with the exterior differential. Accordingly, the de Rham complex (65) of a real ring $C^\infty(X)$ is the de Rham complex of a DGR $\mathcal{O}^*(X)$ of exterior forms on X. The cohomology of $\mathcal{O}^*(X)$ is called the de Rham cohomology $H^*_{\mathrm{DR}}(X)$ of a manifold X.

Let $Y \to X$ be a vector bundle and $Y(X)$ its structure module. An r-order jet manifold J^rY of $Y \to X$ (Appendix B) also is a smooth vector bundle, and its structure module $J^rY(X)$ is exactly the r-order jet module $\mathcal{J}^r(Y(X))$ of a $C^\infty(X)$-module $Y(X)$ (Definition 32) [3, 4].

In view of this fact and by virtue of Theorem 33, a liner k-order differential operator (Definition 30) on a projective $C^\infty(X)$-module P of finite rank with values in a projective C^∞(X)-module Q of finite rank is represented by a linear bundle morphism $J^kY \to E$ over X of a jet bundle $J^kY \to X$ to a vector bundle $E \to X$, where $Y \to X$ and $E \to X$ are smooth vector bundles whose structure modules $Y(X)$ and $E(X)$ are isomorphic to P and Q, respectively, in accordance with Theorem 37.

This construction is generalized to a case of nonlinear differential operators [3, 4, 16].

Definition 38. Let $Y \to X$ and $E \to X$ be smooth fibre bundles. A bundle morphism $\Delta : J^kY \to E$ over X is called the E-valued k-order differential operator on Y. This differential operator sends each section s of $Y \to X$ to the section $\Delta \circ J^ks$ of $E \to X$.

Jet manifolds J^kY of a fibre bundle $Y \to X$ constitute the inverse sequence (B.7) whose inverse limit is an infinite order jet manifold $J^\infty Y$ (Definition B.1). Then any k-order E-valued differential operator Δ on a fibre bundle Y (Definition 38) is defined by a continuous bundle map

$$\Delta \circ \pi^\infty_r : J^\infty Y \underset{X}{\longrightarrow} E. \tag{66}$$

For instance, differential operators in Lagrangian theory on fibre bundles, for example, Euler–Lagrange operators, are represented by certain exterior forms on finite order jet manifolds [4, 16].

The inverse sequence (B.7) of jet manifolds yields the direct sequence of DGRs $\mathcal{O}^*_r = \mathcal{O}^*(J^rY)$ of exterior forms on finite order jet manifolds

$$\begin{aligned} &\mathcal{O}^*(X) \xrightarrow{\pi^*} \mathcal{O}^*(Y) \xrightarrow{\pi^{1*}_0} \mathcal{O}^*_1 \longrightarrow \cdots \mathcal{O}^*_{r-1} \xrightarrow{\pi^{r}_{r-1}{}^*} \mathcal{O}^*_r \\ &\longrightarrow \cdots, \end{aligned} \tag{67}$$

where $\pi^r_{r-1}{}^*$ are the pull-back monomorphisms. Its direct limit

$$\mathcal{O}^*_\infty Y = \overrightarrow{\lim}\, \mathcal{O}^*_r \tag{68}$$

(Definition 11) consists of all exterior forms on finite order jet manifolds modulo the pull-back identification. In accordance with Theorem 13, $\mathcal{O}^*_\infty Y$ is a DGR which inherits operations of the exterior differential d and the exterior product $\wedge$ of DGRs $\mathcal{O}^*_r$.

Theorem 39. *The cohomology $H^*(\mathcal{O}^*_\infty Y)$ of the de Rham complex*

$$0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{O}^0_\infty Y \xrightarrow{d} \mathcal{O}^1_\infty Y \xrightarrow{d} \cdots \tag{69}$$

of a DGR $\mathcal{O}^*_\infty Y$ equals the de Rham cohomology $H^*_{DR}(Y)$ of a fibre bundle Y.

Proof. The result follows from the fact that Y is a strong deformation retract of any jet manifold J^rY and, consequently, the de Rham cohomology of J^rY equals that of Y [23]. □

One can think of elements of $\mathcal{O}^*_\infty Y$ as being differential forms on an infinite order jet manifold $J^\infty Y$. A DGR $\mathcal{O}^*_\infty Y$ is split into a variational bicomplex. Its cohomology provides the global first variational formula for Lagrangians and Euler–Lagrange operators of a Lagrangian formalism on a fibre bundle Y [1, 16, 24, 25].

4. Differential Calculus over $\mathbb{N}$-Graded Commutative Rings

The differential calculus over $\mathbb{N}$-graded commutative rings is defined similarly to that over commutative rings, but it differs from the differential calculus over noncommutative rings (Remark 43). It also provides the differential calculus over a $\mathbb{Z}_2$-graded commutative ring endowed with a fixed $\mathbb{N}$-graded structure (Remark 44).

Let $\mathcal{K}$ be a commutative ring without a divisor of zero and $\mathcal{A}$ an $\mathbb{N}$-graded commutative $\mathcal{K}$-ring. Let P and Q be $\mathbb{N}$-graded $\mathcal{A}$-modules. An $\mathbb{N}$-graded $\mathcal{K}$-module $\mathrm{Hom}_{\mathcal{K}}(P, Q)$ of $\mathbb{N}$-graded $\mathcal{K}$-module homomorphisms $\Phi : P \to Q$ can admit the two $\mathbb{N}$-graded $\mathcal{A}$-module structures

$$\begin{aligned} (a\Phi)(p) &= a\Phi(p), \\ (\Phi \bullet a)(p) &= \Phi(ap), \\ &\quad a \in \mathcal{A},\ p \in P, \end{aligned} \tag{70}$$

called $\mathcal{A}$- and $\mathcal{A}^\bullet$-module structures, respectively. Let us put

$$\delta_a \Phi = a\Phi - (-1)^{[a][\Phi]}\, \Phi \bullet a, \quad a \in \mathcal{A}. \tag{71}$$

Definition 40. An element $\Delta \in \mathrm{Hom}_{\mathcal{K}}(P, Q)$ is said to be the linear Q-valued $\mathbb{N}$-graded differential operator of order s on P if

$$\delta_{a_0} \circ \cdots \circ \delta_{a_s} \Delta = 0 \tag{72}$$

for any tuple of $s+1$ elements $a_0, \ldots, a_s$ of $\mathcal{A}$. A set $\mathrm{Diff}_s(P, Q)$ of these operators inherits the $\mathbb{N}$-graded $\mathcal{A}$-module structures (70).

For instance, zero-order $\mathbb{N}$-graded differential operators obey a condition

$$\delta_a \Delta(p) = a\Delta(p) - (-1)^{[a][\Delta]} \Delta(ap) = 0, \quad a \in \mathcal{A}, \; p \in P; \tag{73}$$

that is, they coincide with graded $\mathcal{A}$-module morphisms $P \to Q$. A first-order $\mathbb{N}$-graded differential operator Δ satisfies a condition

$$\begin{aligned} \delta_a \circ \delta_b \Delta(p) &= ab\Delta(p) - (-1)^{([b]+[\Delta])[a]} b\Delta(ap) \\ &\quad - (-1)^{[b][\Delta]} a\Delta(bp) \\ &\quad + (-1)^{[b][\Delta]+([\Delta]+[b])[a]} \Delta(bap) = 0, \\ &\qquad a, b \in \mathcal{A}, \; p \in P. \end{aligned} \tag{74}$$

Graded differential operators on an $\mathbb{N}$-graded $\mathcal{A}$-module P form a direct system of $\mathbb{N}$-graded $(\mathcal{A} - \mathcal{A}^\bullet)$-modules.

$$\mathrm{Diff}_0(P, Q) \xrightarrow{\mathrm{in}} \mathrm{Diff}_1(P, Q) \cdots \xrightarrow{\mathrm{in}} \mathrm{Diff}_r(P, Q) \longrightarrow \cdots . \tag{75}$$

Its limit $\mathrm{Diff}_\infty(P, Q)$ is a $\mathbb{Z}_2$-graded module of all Q-valued graded differential operators on P.

In particular, let $P = \mathcal{A}$. Any zero-order Q-valued $\mathbb{N}$-graded differential operator Δ on $\mathcal{A}$ is defined by its value $\Delta(\mathbf{1})$. Then there is a graded $\mathcal{A}$-module isomorphism $\mathrm{Diff}_0(\mathcal{A}, Q) = Q$ via the association $q \to \Delta_q$, where Δ_q is given by the equality $\Delta_q(\mathbf{1}) = q$. A first-order Q-valued $\mathbb{N}$-graded differential operator Δ on $\mathcal{A}$ fulfils a condition

$$\begin{aligned} \Delta(ab) &= \Delta(a) b + (-1)^{[a][\Delta]} a\Delta(b) \\ &\quad - (-1)^{([b]+[a])[\Delta]} ab\Delta(\mathbf{1}), \quad a, b \in \mathcal{A}. \end{aligned} \tag{76}$$

Definition 41. It is called the Q-valued $\mathbb{N}$-graded derivation of $\mathcal{A}$ if $\Delta(\mathbf{1}) = 0$, that is, if it obeys the graded Leibniz rule

$$\Delta(ab) = \Delta(a) b + (-1)^{[a][\Delta]} a\Delta(b), \quad a, b \in \mathcal{A}. \tag{77}$$

Any first-order $\mathbb{N}$-graded differential operator on $\mathcal{A}$ falls into a sum

$$\Delta(a) = \Delta(\mathbf{1}) a + [\Delta(a) - \Delta(\mathbf{1}) a] \tag{78}$$

of a zero-order graded differential operator $\Delta(\mathbf{1})a$ and an $\mathbb{N}$-graded derivation $\Delta(a) - \Delta(\mathbf{1})a$. If ∂ is an $\mathbb{N}$-graded derivation of $\mathcal{A}$, then $a\partial$ is so for any $a \in \mathcal{A}$. Hence, $\mathbb{N}$-graded derivations of $\mathcal{A}$ constitute an $\mathbb{N}$-graded $\mathcal{A}$-module $\mathfrak{d}(\mathcal{A}, Q)$, called the graded derivation module.

If $Q = \mathcal{A}$, the $\mathbb{N}$-graded derivation module $\mathfrak{d}\mathcal{A} = \mathfrak{d}(\mathcal{A}, \mathcal{A})$ also is a Lie $\mathcal{K}$-superalgebra (Remark 24) with respect to the superbracket

$$[u, u'] = u \circ u' - (-1)^{[u][u']} u' \circ u, \quad u, u' \in \mathcal{A}. \tag{79}$$

Example 42. Let Λ be a Grassmann-graded ring provided with an odd generating basis $\{c^i\}$. Its $\mathbb{N}$-graded derivations are defined in full by their action on the generating elements c^i. Let us consider odd derivations

$$\begin{aligned} \partial_i(c^j) &= \delta_i^j, \\ \partial_i \circ \partial_j &= -\partial_j \circ \partial_i. \end{aligned} \tag{80}$$

Then any $\mathbb{N}$-graded derivation of Λ takes a form

$$u = u^i \partial_i, \quad u_i \in \mathcal{A}. \tag{81}$$

Graded derivations (81) constitute the free $\mathbb{N}$-graded Λ-module $\mathfrak{d}\Lambda$ of finite rank. It also is a finite-dimensional Lie superalgebra over $\mathcal{K}$ with respect to the superbracket (79). Any $\mathbb{N}$-graded differential operator on a Grassmann-graded ring is a composition of graded derivations.

Remark 43. It should be emphasized that though an $\mathbb{N}$-graded commutative ring is a particular noncommutative ring, $\mathbb{N}$-graded differential operators in accordance with Definition 40 are not differential operators over a noncommutative ring [5–7]. For instance, $\mathbb{N}$-graded derivations of $\mathcal{A}$ obey the graded Leibniz rule (77) which differs from the Leibniz rule

$$\partial(ab) = \partial(a) b + a\partial(b) \tag{82}$$

(cf. (38)) in the noncommutative differential calculus.

Since the graded derivation module $\mathfrak{d}\mathcal{A}$ of an $\mathbb{N}$-graded commutative ring $\mathcal{A}$ is a Lie $\mathcal{K}$-superalgebra, one can consider the Chevalley–Eilenberg complex

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A} \xrightarrow{d} C^1[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \cdots C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] \xrightarrow{d} \cdots , \tag{83}$$

where a $\mathcal{K}$-ring $\mathcal{A}$ is seen as a $\mathfrak{d}\mathcal{A}$-module [5, 21]. Its cochains are $\mathcal{A}$-modules

$$C^k[\mathfrak{d}\mathcal{A}; \mathcal{A}] = \mathrm{Hom}_{\mathcal{K}}\left(\bigwedge^k \mathfrak{d}\mathcal{A}, \mathcal{A}\right) \tag{84}$$

of $\mathcal{K}$-linear $\mathbb{N}$-graded morphisms of graded exterior products $\bigwedge^k \mathfrak{d}\mathcal{A}$ of an $\mathbb{N}$-graded $\mathcal{K}$-module $\mathfrak{d}\mathcal{A}$ to $\mathcal{A}$. Let us bring homogeneous elements of $\bigwedge^k \mathfrak{d}\mathcal{A}$ into a form

$$\varepsilon_1 \wedge \cdots \varepsilon_r \wedge \epsilon_{r+1} \wedge \cdots \wedge \epsilon_k, \quad \varepsilon_i \in \mathfrak{d}\mathcal{A}_0, \; \epsilon_j \in \mathfrak{d}\mathcal{A}_1. \tag{85}$$

Then a Chevalley–Eilenberg coboundary operator d of the complex (83) reads

$$
\begin{aligned}
dc(\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s) &= \sum_{i=1}^{r}(-1)^{i-1}\varepsilon_i c(\varepsilon_1 \\
&\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\epsilon_s)+\sum_{j=1}^{s}(-1)^{r}\epsilon_j c\left(\varepsilon_1\wedge\cdots\right. \\
&\left.\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s\right)+\sum_{1\le i<j\le r}(-1)^{i+j}c\left(\left[\varepsilon_i,\varepsilon_j\right]\right. \\
&\left.\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\widehat{\varepsilon}_j\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\wedge\epsilon_s\right) \\
&+\sum_{1\le i<j\le s}c\left(\left[\epsilon_i,\epsilon_j\right]\wedge\varepsilon_1\wedge\cdots\wedge\varepsilon_r\wedge\epsilon_1\right. \\
&\left.\wedge\cdots\widehat{\epsilon}_i\cdots\widehat{\epsilon}_j\cdots\wedge\epsilon_s\right)+\sum_{1\le i<r,1\le j\le s}(-1)^{i+r+1} \\
&\cdot c\left(\left[\varepsilon_i,\epsilon_j\right]\wedge\varepsilon_1\wedge\cdots\widehat{\varepsilon}_i\cdots\wedge\varepsilon_r\wedge\epsilon_1\wedge\cdots\widehat{\epsilon}_j\cdots\right. \\
&\left.\wedge\epsilon_s\right),
\end{aligned}\tag{86}
$$

where the caret $\widehat{}$ denotes omission.

It is easily justified that complex (83) contains a subcomplex $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ of $\mathcal{A}$-linear $\mathbb{N}$-graded morphisms. The $\mathbb{N}$-graded module $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ is provided with the graded exterior product

$$
\begin{aligned}
&\phi\wedge\phi'(u_1,\ldots,u_{r+s}) \\
&\quad=\sum_{i_1<\cdots<i_r;j_1<\cdots<j_s}\operatorname{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}\phi\left(u_{i_1},\ldots,u_{i_r}\right) \\
&\quad\cdot\phi'\left(u_{j_1},\ldots,u_{j_s}\right), \\
&\qquad\phi\in\mathcal{O}^r[\mathfrak{d}\mathcal{A}],\ \phi'\in\mathcal{O}^s[\mathfrak{d}\mathcal{A}],\ u_k\in\mathfrak{d}\mathcal{A},
\end{aligned}\tag{87}
$$

where $u_1,\ldots,u_{r+s}$ are graded-homogeneous elements of $\mathfrak{d}\mathcal{A}$ and

$$
\begin{aligned}
&u_1\wedge\cdots\wedge u_{r+s} \\
&\quad=\operatorname{sgn}^{i_1\cdots i_r j_1\cdots j_s}_{1\cdots r+s}u_{i_1}\wedge\cdots\wedge u_{i_r}\wedge u_{j_1}\wedge\cdots\wedge u_{j_s}.
\end{aligned}\tag{88}
$$

The graded Chevalley–Eilenberg differential d (86) and the graded exterior product $\bigwedge$ (87) bring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ into a differential bigraded ring (henceforth, DBGR) whose elements obey relations

$$
\begin{aligned}
\phi\wedge\phi' &= (-1)^{|\phi||\phi'|+[\phi][\phi']}\phi'\wedge\phi, \\
d\left(\phi\wedge\phi'\right) &= d\phi\wedge\phi'+(-1)^{|\phi|}\phi\wedge d\phi'.
\end{aligned}\tag{89}
$$

It is called the graded Chevalley–Eilenberg differential calculus over an $\mathbb{N}$-graded commutative $\mathcal{K}$-ring $\mathcal{A}$. In particular, we have

$$
\mathcal{O}^1[\mathfrak{d}\mathcal{A}]=\operatorname{Hom}_{\mathcal{A}}(\mathfrak{d}\mathcal{A},\mathcal{A})=\mathfrak{d}\mathcal{A}^*.\tag{90}
$$

One can extend this duality relation to the graded interior product of $u\in\mathfrak{d}\mathcal{A}$ with any element $\phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ by the rules

$$
\begin{aligned}
&u\rfloor(bda)=(-1)^{[u][b]}bu(a),\quad a,b\in\mathcal{A}, \\
&u\rfloor\left(\phi\wedge\phi'\right)=(u\rfloor\phi)\wedge\phi'+(-1)^{|\phi|+[\phi][u]}\phi \\
&\qquad\wedge\left(u\rfloor\phi'\right).
\end{aligned}\tag{91}
$$

As a consequence, any graded derivation $u\in\mathfrak{d}\mathcal{A}$ of $\mathcal{A}$ yields a graded derivation

$$
\begin{aligned}
&\mathbf{L}_u\phi=u\rfloor d\phi+d(u\rfloor\phi), \\
&\qquad\phi\in\mathcal{O}^*[\mathfrak{d}\mathcal{A}],\ u\in\mathfrak{d}\mathcal{A}, \\
&\mathbf{L}_u\left(\phi\wedge\phi'\right)=\mathbf{L}_u(\phi)\wedge\phi'+(-1)^{[u][\phi]}\phi\wedge\mathbf{L}_u\left(\phi'\right),
\end{aligned}\tag{92}
$$

termed the graded Lie derivative of a differential bigraded ring $\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$.

The minimal graded Chevalley–Eilenberg differential calculus $\mathcal{O}^*\mathcal{A}\subset\mathcal{O}^*[\mathfrak{d}\mathcal{A}]$ over a $\mathbb{N}$-graded commutative ring $\mathcal{A}$ consists of monomials $a_0da_1\wedge\cdots\wedge da_k$, $a_i\in\mathcal{A}$. The corresponding complex

$$
0\longrightarrow\mathcal{K}\longrightarrow\mathcal{A}\xrightarrow{d}\mathcal{O}^1\mathcal{A}\xrightarrow{d}\cdots\mathcal{O}^k\mathcal{A}\xrightarrow{d}\cdots\tag{93}
$$

is called the bigraded de Rham complex of an $\mathbb{N}$-graded commutative $\mathcal{K}$-ring $\mathcal{A}$.

Remark 44. Let us note that if $\mathcal{A}=\mathcal{A}^0$ is a commutative ring, $\mathbb{N}$-graded differential operators and the graded Chevalley–Eilenberg differential calculus are reduced to the familiar commutative differential calculus. On the other hand, $\mathbb{N}$-graded modules possess the associated structure of $\mathbb{Z}_2$-graded modules, and their $\mathbb{N}$-graded homomorphisms are $\mathbb{Z}_2$-graded homomorphisms of this $\mathbb{Z}_2$-graded structure. Moreover, $\mathbb{N}$-graded commutativity conditions in fact are the $\mathbb{Z}_2$-graded ones. Therefore, the differential calculus over $\mathbb{N}$-graded rings is the $\mathbb{Z}_2$-graded differential calculus when the associated $\mathbb{N}$-structure hold fixed. In particular, any $\mathbb{N}$-graded differential operator also is a $\mathbb{Z}_2$-graded differential operator, but the converse need not be true. For instance, the differential calculus over a Grassmann-graded $\mathcal{K}$-ring $\mathcal{A}^*$ (Example 42) is the differential calculus over an associated Grassmann algebra $\mathcal{A}_*$. Namely, the derivations (81) of $\mathcal{A}^*$ also are derivations of a Grassmann algebra $\mathcal{A}^*$, and any $\mathbb{Z}_2$-graded differential operator on $\mathcal{A}_*$ is a composition of these derivations.

5. $\mathbb{N}$-Graded Manifolds

As was mentioned above, we define an $\mathbb{N}$-graded manifold by analogy with smooth and $\mathbb{Z}_2$-graded manifolds as a local-ringed space which is a sheaf in local $\mathbb{N}$-graded commutative rings on a finite-dimensional real smooth manifold X.

We start with the notion of a $\mathbb{Z}_2$-graded manifold [5, 8, 9]. It is defined as a local-ringed space $(Z,\mathfrak{A})$ (Definition A.2), where Z is an n-dimensional smooth manifold, and $\mathfrak{A} = \mathfrak{A}_0 \oplus \mathfrak{A}_1$ is a sheaf of real Grassmann algebras such that

(i) there is the exact sequence of sheaves

$$\begin{gathered} 0 \longrightarrow \mathscr{R} \longrightarrow \mathfrak{A} \xrightarrow{\sigma} C_Z^\infty \longrightarrow 0, \\ \mathscr{R} = \mathfrak{A}_1 + (\mathfrak{A}_1)^2, \end{gathered} \tag{94}$$

where C_Z^∞ is the sheaf of smooth real functions on Z;

(ii) $\mathscr{R}/\mathscr{R}^2$ is a locally free sheaf of C_Z^∞-modules of finite rank (with respect to pointwise operations), and the sheaf $\mathfrak{A}$ is locally isomorphic to the exterior product $\bigwedge_{C_Z^\infty}(\mathscr{R}/\mathscr{R}^2)$.

A sheaf $\mathfrak{A}$ is called the structure sheaf of a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$, and a manifold Z is said to be the body of $(Z, \mathfrak{A})$. Sections of the sheaf $\mathfrak{A}$ are termed the graded functions on a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$. They make up a $\mathbb{Z}_2$-graded commutative $C^\infty(Z)$-ring $\mathfrak{A}(Z)$ called the structure ring of $(Z, \mathfrak{A})$.

Given a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$, by the sheaf $\mathfrak{d}\mathfrak{A}$ of graded derivations of $\mathfrak{A}$ is meant a subsheaf of endomorphisms of the structure sheaf $\mathfrak{A}$ such that any section $u \in \mathfrak{d}\mathfrak{A}(U)$ of $\mathfrak{d}\mathfrak{A}$ over an open subset $U \subset Z$ is a graded derivation of the real $\mathbb{Z}_2$-graded commutative ring $\mathfrak{A}(U)$, that is, $u \in \mathfrak{d}(\mathfrak{A}(U))$. Conversely, one can show that, given open sets $U' \subset U$, there is a surjection of the graded derivation derivation modules $\mathfrak{d}(\mathfrak{A}(U)) \to \mathfrak{d}(\mathfrak{A}(U'))$ [8]. It follows that any graded derivation of a local $\mathbb{Z}_2$-graded commutative ring $\mathfrak{A}(U)$ also is a local section over U of a sheaf $\mathfrak{d}\mathfrak{A}$. Global sections of $\mathfrak{d}\mathfrak{A}$ are called graded vector fields on a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$. They make up a graded derivation module $\mathfrak{d}\mathfrak{A}(Z)$ of a real $\mathbb{Z}_2$-graded commutative ring $\mathfrak{A}(Z)$. This module is a real Lie superalgebra (Remark 24) with respect to the superbracket (79).

Turn now to $\mathbb{N}$-graded manifolds. By virtue of the well-known Batchelor theorem [5, 8], $\mathbb{Z}_2$-graded manifolds possess the following structure.

Theorem 45. *Let $(Z, \mathfrak{A})$ be a $\mathbb{Z}_2$-graded manifold. There exists a vector bundle $E \to Z$ with an m-dimensional typical fibre V such that the structure sheaf $\mathfrak{A}$ of $(Z, \mathfrak{A})$ as a sheaf in real rings is isomorphic to the structure sheaf $\mathfrak{A}_E = \bigwedge E_Z^*$ of germs of sections of the exterior bundle*

$$\bigwedge E^* = (Z \times \mathbb{R}) \underset{Z}{\oplus} E^* \underset{Z}{\oplus} \bigwedge E^* \underset{Z}{\oplus} \overset{2}{\bigwedge} E^* \cdots, \tag{95}$$

whose typical fibre is the Grassmann algebra $\Lambda = \bigwedge V^*$ in Theorem 28.

It should be emphasized that Batchelor's isomorphism in Theorem 45 fails to be canonical.

Combining Batchelor Theorem 45 and classical Serre–Swan Theorem 37, we come to the following Serre–Swan theorem for $\mathbb{Z}_2$-graded manifolds [7].

Theorem 46. *Let Z be a smooth manifold. A $\mathbb{Z}_2$-graded commutative $C^\infty(Z)$-ring $\mathscr{A}$ is isomorphic to the structure ring of a $\mathbb{Z}_2$-graded manifold with a body Z iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank.*

In fact, the structure sheaf $\mathfrak{A}_E$ of a $\mathbb{Z}_2$-graded manifold $(Z, \mathfrak{A})$ in Theorem 45 is a sheaf in Grassmann-graded rings Λ^*, whose $\mathbb{N}$-graded structure is fixed. Therefore, we come to the following notion of $\mathbb{N}$-graded manifolds.

Definition 47. An $\mathbb{N}$-graded manifold is defined to be a $\mathbb{Z}_2$-graded manifold whose Batchelor isomorphism $(Z, \mathfrak{A}_E)$ is fixed.

Thus, Theorem 45 states that a $\mathbb{Z}_2$-graded manifold is isomorphic to the $\mathbb{N}$-graded one.

An $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$ is said to be modelled over a vector bundle $E \to Z$, and E is called its characteristic vector bundle. Its structure ring $\mathscr{A}_E$ is the structure module

$$\mathscr{A}_E = \mathfrak{A}_E(Z) = \bigwedge E^*(Z) \tag{96}$$

of sections of the exterior bundle $\bigwedge E^*$ (95). Automorphisms of an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$ are restricted to those induced by automorphisms of its characteristic vector bundles $E \to Z$.

Accordingly, Serre–Swan Theorem 46 can be formulated for the $\mathbb{N}$-graded ones.

Theorem 48. *Let Z be a smooth manifold. An $\mathbb{N}$-graded commutative $C^\infty(Z)$-ring $\mathscr{A}$ is isomorphic to the structure ring of an $\mathbb{N}$-graded manifold with a body Z iff it is the exterior algebra of some projective $C^\infty(Z)$-module of finite rank.*

Remark 49. One can treat a local-ringed space $(Z, \mathfrak{A}_0 = C_Z^\infty)$ as a trivial $\mathbb{N}$-graded manifold whose characteristic vector bundle is $E = Z \times \{0\}$. Its structure module is a ring $C^\infty(Z)$ of smooth real functions on Z.

Given an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$, every trivialization chart $(U; z^A, y^a)$ of its characteristic vector bundle $E \to Z$ yields a splitting domain $(U; z^A, c^a)$ of $(Z, \mathfrak{A}_E)$. Graded functions on such a chart are Λ-valued functions

$$f = \sum_{k=0}^{m} \frac{1}{k!} f_{a_1 \cdots a_k}(z) c^{a_1} \cdots c^{a_k}, \tag{97}$$

where $f_{a_1 \cdots a_k}(z)$ are smooth functions on U and $\{c^a\}$ is the fibre basis for E^*. One calls $\{z^A, c^a\}$ the local basis for an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$. Transition functions $y'^a = \rho_b^a(z^A) y^b$ of bundle coordinates on $E \to Z$ induce the corresponding transformation $c'^a = \rho_b^a(z^A) c^b$ of the associated local basis for an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$ and the according coordinate transformation law of graded functions (97).

The following is an essential peculiarity of an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$ in comparison with the $\mathbb{Z}_2$-graded ones.

Theorem 50. *Derivations of the structure module $\mathscr{A}_E$ of an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$ are represented by sections of the vector bundle (102).*

Proof. Due to the canonical splitting $VE = E\times E$, the vertical tangent bundle VE of $E \to Z$ can be provided with fibre bases $\{\partial/\partial c^a\}$, which are the duals of bases $\{c^a\}$. Then graded derivations of $\mathscr{A}_E$ on a splitting domain $(U; z^A, c^a)$ of $(Z, \mathfrak{A}_E)$ read

$$u = u^A\partial_A + u^a\frac{\partial}{\partial c^a}, \qquad (98)$$

where u^λ, u^a are local graded functions on U. In particular,

$$\begin{aligned} \frac{\partial}{\partial c^a}\circ\frac{\partial}{\partial c^b} &= -\frac{\partial}{\partial c^b}\circ\frac{\partial}{\partial c^a},\\ \partial_A\circ\frac{\partial}{\partial c^a} &= \frac{\partial}{\partial c^a}\circ\partial_A. \end{aligned} \qquad (99)$$

The graded derivations (98) act on graded functions $f \in \mathfrak{A}_E(U)$ (97) by a rule

$$\begin{aligned} u\left(f_{a\cdots b}c^a\cdots c^b\right) &= u^A\partial_A\left(f_{a\cdots b}\right)c^a\cdots c^b\\ &\quad + u^k f_{a\cdots b}\frac{\partial}{\partial c^k}\rfloor\left(c^a\cdots c^b\right). \end{aligned} \qquad (100)$$

This rule implies the corresponding transformation law

$$\begin{aligned} u'^A &= u^A,\\ u'^a &= \rho^a_j u^j + u^A\partial_A\left(\rho^a_j\right)c^j \end{aligned} \qquad (101)$$

of graded derivations (98). It follows that they can be represented by sections of a vector bundle

$$\mathscr{V}_E = \bigwedge E^*\underset{E}{\otimes} TE \longrightarrow Z. \qquad (102)$$

Thus, the graded derivation module $\mathfrak{d}\mathscr{A}_E$ is isomorphic to the structure module $\mathscr{V}_E(Z)$ of global sections of the vector bundle $\mathscr{V}_E \to Z$ (102). □

Given the structure ring $\mathscr{A}_E$ of graded functions on an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$ and the real Lie superalgebra $\mathfrak{d}\mathscr{A}_E$ of its graded derivations, let us consider the graded Chevalley–Eilenberg differential calculus

$$\begin{aligned} &\mathscr{S}^*[E;Z] = \mathscr{O}^*\left[\mathfrak{d}\mathscr{A}_E\right]\\ &\qquad 0 \longrightarrow \mathbb{R} \longrightarrow \mathscr{A}_E \xrightarrow{d} \mathscr{S}^1[E;Z]\\ &\qquad \xrightarrow{d}\cdots\mathscr{S}^k[E;Z]\xrightarrow{d}\cdots, \end{aligned} \qquad (103)$$

over $\mathscr{S}^0[E;Z] = \mathscr{A}_E$. Since a module $\mathfrak{d}\mathscr{A}_E$ is isomorphic to the structure module of sections of a vector bundle $\mathscr{V}_E \to Z$, elements of $\mathscr{S}^*[E;Z]$ are represented by sections of the exterior bundle $\bigwedge\overline{\mathscr{V}}_E$ of the $\mathscr{A}_E$-dual

$$\overline{\mathscr{V}}_E = \bigwedge E^*\underset{E}{\otimes} T*E \longrightarrow Z \qquad (104)$$

of $\mathscr{V}_E$. With respect to the dual fibre bases $\{dz^A\}$ for T^*Z and $\{dc^b\}$ for E^*, sections of $\overline{\mathscr{V}}_E$ (104) take a coordinate form

$$\begin{aligned} \phi &= \phi_A dz^A + \phi_a dc^a,\\ \phi'_a &= \rho^{-1b}{}_a\phi_b,\\ \phi'_A &= \phi_A + \rho^{-1b}{}_a\partial_A\left(\rho^a_j\right)\phi_b c^j. \end{aligned} \qquad (105)$$

The duality relation $\mathscr{S}^1[E;Z] = \mathfrak{d}\mathscr{A}^*_E$ (90) is given by a graded interior product

$$u\rfloor\phi = u^A\phi_A + (-1)^{[\phi_a]}u^a\phi_a. \qquad (106)$$

Elements of $\mathscr{S}^*[E;Z]$ are called graded differential forms on an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$. Seen as an $\mathscr{A}_E$-ring, the DBGR $\mathscr{S}^*[E;Z]$ (103) on a splitting domain (z^A, c^a) is locally generated by the graded one-form dz^A, dc^i such that

$$\begin{aligned} dz^A\wedge dc^i &= -dc^i\wedge dz^A,\\ dc^i\wedge dc^j &= dc^j\wedge dc^i. \end{aligned} \qquad (107)$$

Accordingly, the graded Chevalley–Eilenberg coboundary operator d (86), termed the graded exterior differential, reads

$$d\phi = dz^A\wedge\partial_A\phi + dc^a\wedge\frac{\partial}{\partial c^a}\phi, \qquad (108)$$

where derivations $\partial_\lambda, \partial/\partial c^a$ act on coefficients of graded exterior forms by formula (100), and they are graded commutative with graded forms dz^A and dc^a. Formulas (89), (90), and (92) hold.

Theorem 51. *The DBGR $\mathscr{S}^*[E;Z]$ (103) is a minimal differential calculus over $\mathscr{A}_E$; that is, it is generated by elements df, $f \in \mathscr{A}_E$.*

Proof. Since $\mathfrak{d}\mathscr{A}_E = \mathscr{V}_E(Z)$, it is a projective $C^\infty(Z)$- and $\mathscr{A}_E$-module of finite rank, and so is its $\mathscr{A}_E$-dual $\mathscr{S}^1[E;Z]$ (Theorem 10). Hence, $\mathfrak{d}\mathscr{A}_E$ is the $\mathscr{A}_E$-dual of $\mathscr{S}^1[E;Z]$ and, consequently, $\mathscr{S}^1[E;Z]$ is generated by elements df, $f \in \mathscr{A}_E$. □

Cohomology of the DBGR $\mathscr{S}^*[E;Z]$ (103) is called the de Rham cohomology of an $\mathbb{N}$-graded manifold $(Z, \mathfrak{A}_E)$. It equals the de Rham cohomology of its body Z [9]. In particular, there exist both a cochain monomorphism $\mathscr{O}^*(Z) \to \mathscr{S}^*[E;Z]$ and a body epimorphism $\mathscr{S}^*[E;Z] \to \mathscr{O}^*(Z)$.

6. $\mathbb{N}$-Graded Bundles and Jet Manifolds

A morphism of (both $\mathbb{Z}_2$- and $\mathbb{N}$-) graded manifolds $(Z, \mathfrak{A}) \to (Z', \mathfrak{A}')$ is defined as that of local-ringed spaces $\phi: Z \to Z'$, $\widehat{\Phi}: \mathfrak{A}' \to \phi_*\mathfrak{A}$, where ϕ is a manifold morphism and $\widehat{\Phi}$ is a sheaf morphism of $\mathfrak{A}'$ to the direct image $\phi_*\mathfrak{A}$ of $\mathfrak{A}$ onto Z' (Appendix A). This morphism of graded manifolds is said to be (i) a monomorphism if ϕ is an injection and $\widehat{\Phi}$ is an epimorphism and (ii) an epimorphism if ϕ is a surjection and $\widehat{\Phi}$ is a monomorphism.

An epimorphism of graded manifolds $(Z,\mathfrak{A}) \to (Z',\mathfrak{A}')$, where $Z \to Z'$ is a fibre bundle is called the graded bundle [26, 27]. In this case, a sheaf monomorphism $\widehat{\Phi}$ induces a monomorphism of canonical presheaves $\overline{\mathfrak{A}}' \to \overline{\mathfrak{A}}$, which associates to each open subset $U \subset Z$ the ring of sections of $\mathfrak{A}'$ over $\phi(U)$. Accordingly, there is a pull-back monomorphism of the structure rings $\mathfrak{A}'(Z') \to \mathfrak{A}(Z)$ of graded functions on graded manifolds $(Z',\mathfrak{A}')$ and $(Z,\mathfrak{A})$.

In particular, let $(Y,\mathfrak{A})$ be a graded manifold whose body is a fibre bundle $Y \to X$. Let us consider a trivial graded manifold $(X,\mathfrak{A}_0 = C^\infty_X)$ (Remark 49). Then we have a graded bundle

$$(Y,\mathfrak{A}) \longrightarrow (X, C^\infty_X). \tag{109}$$

Let us denote it by $(X,Y,\mathfrak{A})$. Given a graded bundle $(X,Y,\mathfrak{A})$, the local basis for a graded manifold $(Y,\mathfrak{A})$ can take a form (x^λ, y^i, c^a), where (x^λ, y^i) are bundle coordinates of $Y \to X$.

Definition 52. One agrees to call the graded bundle (109) over a trivial graded manifold (X, C^∞_X) the graded bundle over a smooth manifold.

If $Y \to X$ is a vector bundle, the graded bundle (109) is a particular case of graded fibre bundles in [26, 28] when their base is a trivial graded manifold.

Remark 53. Let $Y \to X$ be a fibre bundle. Then a trivial graded manifold (Y, C^∞_Y) together with a real ring monomorphism $C^\infty(X) \to C^\infty(Y)$ is a graded bundle (X,Y,C^∞_Y) of trivial graded manifolds $(Y,C^\infty_Y) \to (X, C^\infty_X)$.

Remark 54. A graded manifold $(X,\mathfrak{A})$ itself can be treated as the graded bundle $(X,X,\mathfrak{A})$ (109) associated with the identity smooth bundle $X \to X$.

Let $E \to Z$ and $E' \to Z'$ be vector bundles and $\Phi : E \to E'$ their bundle morphism over a morphism $\phi : Z \to Z'$. Then a section s^* of the dual bundle $E'^* \to Z'$ yields the pull-back section $\Phi^* s^*$ of the dual bundle $E^* \to Z$ by a law

$$v_z \rfloor \Phi^* s^* (z) = \Phi(v_z) \rfloor s^*(\varphi(z)), \quad v_z \in E_z. \tag{110}$$

A bundle morphism (Φ,ϕ) induces a morphism of $\mathbb{N}$-graded manifolds

$$(Z,\mathfrak{A}_E) \longrightarrow (Z',\mathfrak{A}_{E'}). \tag{111}$$

This is a pair $(\phi, \widehat{\Phi} = \phi_* \circ \Phi^*)$ of a morphism ϕ of body manifolds and the composition $\phi_* \circ \Phi^*$ of the pull-back $\mathscr{A}_{E'} \ni f \to \Phi^* f \in \mathscr{A}_E$ of graded functions and the direct image ϕ_* of a sheaf $\mathfrak{A}_E$ onto Z'. Relative to local bases (z^A, c^a) and (z'^A, c'^a) for $(Z,\mathfrak{A}_E)$ and $(Z',\mathfrak{A}_{E'})$, the morphism (111) of $\mathbb{N}$-graded manifolds reads $z' = \phi(z)$, $\widehat{\Phi}(c'^a) = \Phi^a_b(z)c^b$.

The graded manifold morphism (111) is a monomorphism (resp., epimorphism) if Φ is a bundle injection (resp., surjection). In particular, the graded manifold morphism (111) is a graded bundle if Φ is a fibre bundle. Let $\mathscr{A}_{E'} \to \mathscr{A}_E$ be the corresponding pull-back monomorphism of the structure rings. By virtue of Theorem 51 it yields a monomorphism of the DBGRs

$$\mathcal{S}^*[E';Z'] \longrightarrow \mathcal{S}^*[E;Z]. \tag{112}$$

Let $(Y,\mathfrak{A}_F)$ be an $\mathbb{N}$-graded manifold modelled over a vector bundle $F \to Y$. This is an $\mathbb{N}$-graded bundle $(X,Y,\mathfrak{A}_F)$:

$$(Y,\mathfrak{A}_F) \longrightarrow (X, C^\infty_X), \tag{113}$$

modelled over a composite bundle

$$F \longrightarrow Y \longrightarrow X. \tag{114}$$

The structure ring of graded functions on an $\mathbb{N}$-graded manifold $(Y,\mathfrak{A}_F)$ is the graded commutative $C^\infty(X)$-ring $\mathscr{A}_F = \bigwedge F^*(Y)$ (96). Let the composite bundle (114) be provided with adapted bundle coordinates (x^λ, y^i, q^a) possessing transition functions

$$\begin{gathered} x'^\lambda(x^\mu), \\ y'^i(x^\mu, y^j), \\ q'^a = \rho^a_b(x^\mu, y^j)\, q^b. \end{gathered} \tag{115}$$

Then the corresponding basis for an $\mathbb{N}$-graded manifold $(Y,\mathfrak{A}_F)$ is (x^λ, y^i, c^a) together with transition functions $c'^a = \rho^a_b(x^\mu, j^j)c^b$. We call it the local basis for an $\mathbb{N}$-graded bundle $(X,Y,\mathfrak{A}_F)$.

As was shown above, the differential calculus on a fibre bundle $Y \to X$ is formulated in terms of jet manifolds J^*Y of Y. Being fibre bundles over X, they can be regarded as trivial graded bundles $(X, J^kY, C^\infty_{J^kY})$. We describe their partners in a case of $\mathbb{N}$-graded bundles as follows.

Let us note that, given an $\mathbb{N}$-graded manifold $(X,\mathfrak{A}_E)$ and its structure ring $\mathscr{A}_E$, one can define the jet module $J^1\mathscr{A}_E$ of a $C^\infty(X)$-ring $\mathscr{A}_E$ [5, 7]. It is a module of global sections of the jet bundle $J^1(\bigwedge E^*)$. A problem is that $J^1\mathscr{A}_E$ fails to be a structure ring of some graded manifold. By this reason, we have suggested a different construction of jets of graded manifolds (Definition 55), though it is applied only to $\mathbb{N}$-graded manifolds [2, 9].

Let $(X,\mathscr{A}_E)$ be an $\mathbb{N}$-graded manifold modelled over a vector bundle $E \to X$. Let us consider a k-order jet manifold J^kE of E. It is a vector bundle over X. Then let $(X,\mathscr{A}_{J^kE})$ be an $\mathbb{N}$-graded manifold modelled over $J^kE \to X$.

Definition 55. One calls $(X,\mathscr{A}_{J^kE})$ the graded jet manifold of an $\mathbb{N}$-graded manifold $(X,\mathscr{A}_E)$.

Given a splitting domain $(U; x^\lambda, c^a)$ of an $\mathbb{N}$-graded manifold $(Z,\mathscr{A}_E)$, the adapted splitting domain of a graded jet manifold $(X,\mathscr{A}_{J^kE})$ reads

$$\begin{gathered} \left(U; x^\lambda, c^a, c^a_\lambda, c^a_{\lambda_1\lambda_2}, \dots, c^a_{\lambda_1\cdots\lambda_k}\right), \\ c'^a_{\lambda\lambda_1\cdots\lambda_r} = \rho^a_b(x)\, c^a_{\lambda\lambda_1\cdots\lambda_r} + \partial_\lambda \rho^a_b(x)\, c^a_{\lambda_1\cdots\lambda_r}. \end{gathered} \tag{116}$$

As was mentioned above, a graded manifold is a particular graded bundle over its body (Remark 54). Then Definition 55 of graded jet manifolds is generalized to $\mathbb{N}$-graded bundles over smooth manifolds as follows. Let $(X, Y, \mathfrak{A}_F)$ be the $\mathbb{N}$-graded bundle (113) modelled over the composite bundle (114). It is readily observed that the jet manifold J^rF of $F \to X$ is a vector bundle $J^rF \to J^rY$ coordinated by $(x^\lambda, y^i_\Lambda, q^a_\Lambda)$, $0 \leq |\Lambda| \leq r$. Let $(J^rY, \mathfrak{A}_{J^rF})$ be an $\mathbb{N}$-graded manifold modelled over this vector bundle. Its local generating basis is $(x^\lambda, y^i_\Lambda, c^a_\Lambda)$, $0 \leq |\Lambda| \leq r$.

Definition 56. One calls $(J^rY, \mathfrak{A}_{J^rF})$ the graded jet manifold of an $\mathbb{N}$-graded bundle $(X, Y, \mathfrak{A}_F)$.

In particular, let $Y \to X$ be a smooth bundle seen as a trivial graded bundle (X, Y, C^∞_Y) modelled over a composite bundle $Y \times \{0\} \to Y \to X$. Then its graded jet manifold is a trivial graded bundle $(X, J^rY, C^\infty_{J^rY})$, that is, the jet manifold J^rY of Y. Thus, Definition 56 of graded jet manifolds of $\mathbb{N}$-graded bundles is compatible with Definition 2 of jets of fibre bundles.

The affine bundles $J^{r+1}Y \to J^rY$ (B.7) and the corresponding fibre bundles $J^{r+1}F \to J^rF$ also yield the $\mathbb{N}$-graded bundles

$$\left(J^{r+1}Y, \mathfrak{A}_{J^{r+1}F}\right) \longrightarrow \left(J^rY, \mathfrak{A}_{J^rF}\right), \tag{117}$$

including the sheaf monomorphisms

$$\pi^{r+1*}_r \mathfrak{A}_{J^rF} \longrightarrow \mathfrak{A}_{J^{r+1}F}, \tag{118}$$

where $\pi^{r+1*}_r \mathfrak{A}_r$ is the pull-back onto $J^{r+1}Y$ of the continuous fibre bundle $\mathfrak{A}_{J^rF} \to J^rY$. The sheaf monomorphism (118) induces a monomorphism of canonical presheaves $\overline{\mathfrak{A}}_{J^rF} \to \overline{\mathfrak{A}}_{J^{r+1}F}$, which associates to each open subset $U \subset J^{r+1}Y$ the ring of sections of $\mathfrak{A}_{J^{r+1}F}$ over $\pi^{r+1}_r(U)$. Accordingly, there is a pull-back monomorphism of the structure rings

$$\begin{gathered} \mathcal{S}^0_r[F;Y] \longrightarrow \mathcal{S}^0_{r+1}[F;Y], \\ \mathcal{S}^0_k[F;Y] = \mathcal{S}^0\left[J^kF; J^kY\right], \end{gathered} \tag{119}$$

of graded functions on graded manifolds $(J^rY, \mathfrak{A}_{J^rF})$ and $(J^{r+1}Y, \mathfrak{A}_{J^{r+1}F})$. As a consequence, we have the inverse sequence of graded manifolds

$$\begin{gathered} \left(Y, \mathfrak{A}_F\right) \longleftarrow \left(J^1Y, \mathfrak{A}_{J^1F}\right) \longleftarrow \cdots \left(J^{r-1}Y, \mathfrak{A}_{J^{r-1}F}\right) \\ \longleftarrow \left(J^rY, \mathfrak{A}_{J^rF}\right) \longleftarrow \cdots. \end{gathered} \tag{120}$$

One can think of its inverse limit $(J^\infty Y, \mathfrak{A}_{J^\infty F})$ as being the graded infinite order jet manifold whose body is an infinite order jet manifold $J^\infty Y$ and whose structure sheaf $\mathfrak{A}_{J^\infty F}$ is a sheaf of germs of graded functions on $\mathbb{N}$-graded manifolds $(J^*Y, \mathfrak{A}_{J^*F})$. However $(J^\infty Y, \mathfrak{A}_{J^\infty F})$ fails to be an $\mathbb{N}$-graded manifold in a strict sense because the inverse limit $J^\infty Y$ of the sequence (B.7) is a Fréche manifold, but not the smooth one.

By virtue of Theorem 51, the differential calculus $\mathcal{S}^*_r[F;Y]$ of graded differential forms on $\mathbb{N}$-graded manifolds $(J^rY, \mathfrak{A}_{J^rF})$ is minimal. Therefore, the monomorphisms of structure rings (119) yield the pull-back monomorphisms (112) of DBGRs

$$\begin{gathered} \pi^{r+1*}_r : \mathcal{S}^*_r[F;Y] \longrightarrow \mathcal{S}^*_{r+1}[F;Y], \\ \mathcal{S}^*_k[F;Y] = \mathcal{S}^*\left[J^kF; J^kY\right]. \end{gathered} \tag{121}$$

As a consequence, we have a direct system of DBGRs

$$\begin{gathered} \mathcal{S}^*[F;Y] \xrightarrow{\pi^*} \mathcal{S}^*_1[F;Y] \longrightarrow \cdots \mathcal{S}^*_{r-1}[F;Y] \\ \xrightarrow{\pi^{r*}_{r-1}} \mathcal{S}^*_r[F;Y] \longrightarrow \cdots. \end{gathered} \tag{122}$$

The DBGR $\mathcal{S}^*_\infty[F;Y]$ associated with an $\mathbb{N}$-graded bundle $(X, Y, \mathfrak{A}_F)$ is defined as the direct limit

$$\mathcal{S}^*_\infty[F;Y] = \overrightarrow{\lim}\, \mathcal{S}^*_r[F;Y] \tag{123}$$

of the direct system (122). It includes all graded differential forms $\phi \in \mathcal{S}^*_r[F;Y]$ on graded manifolds $(J^rY, \mathfrak{A}_{J^rF})$ modulo the monomorphisms (121). Its elements obey relations (89).

Monomorphisms $\mathcal{O}^*(J^rY) \to \mathcal{S}^*_r[F;Y]$ yield a monomorphism of direct system (67) to direct system (122) and, consequently, a monomorphism

$$\mathcal{O}^*_\infty Y \longrightarrow \mathcal{S}^*_\infty[F;Y] \tag{124}$$

of their direct limits. Accordingly, body epimorphisms $\mathcal{S}^*_r[F;Y] \to \mathcal{O}^*_r Y$ yield an epimorphism

$$\mathcal{S}^*_\infty[F;Y] \longrightarrow \mathcal{O}^*_\infty Y. \tag{125}$$

It is readily observed that the morphisms (124)–(125) are cochain morphisms between the de Rham complex (69) of $\mathcal{O}^*_\infty Y$ and the de Rham complex

$$\begin{gathered} 0 \longrightarrow \mathbb{R} \longrightarrow \mathcal{S}^0_\infty[F;Y] \xrightarrow{d} \mathcal{S}^1_\infty[F;Y] \cdots \\ \xrightarrow{d} \mathcal{S}^k_\infty[F;Y] \to \cdots \end{gathered} \tag{126}$$

of a DBGR $\mathcal{S}^*_\infty[F;Y]$. Moreover, the corresponding homomorphisms of cohomology groups of these complexes are isomorphisms as follows [9].

Theorem 57. *There is an isomorphism $H^*(\mathcal{S}^*_\infty[F;Y]) = H^*_{DR}(Y)$ of the cohomology $H^*(\mathcal{S}^*_\infty[F;Y])$ of the de Rham complex (126) to the de Rham cohomology $H^*_{DR}(Y)$ of Y.*

As was mentioned above, the $\mathcal{S}^*_\infty[F;Y]$ (123) is split into a graded variational bicomplex which provides Lagrangian theory in Grassmann-graded (even and odd) variables [1, 2, 9].

Appendix

A. Local-Ringed Spaces

We follow the terminology of [29, 30]. Unless otherwise stated, all presheaves and sheaves are considered on the same topological space X.

A sheaf on a topological space X is a topological fibre bundle $\pi : S \to X$ in modules over a commutative ring $\mathcal{K}$, where a surjection π is a local homeomorphism and fibres S_x, $x \in X$, called the stalks, are provided with the discrete topology. Global sections of a sheaf S constitute a $\mathcal{K}$-module $S(X)$, called the structure module of S.

A presheaf $S_{\{U\}}$ on a topological space X is defined if a module S_U over a commutative ring $\mathcal{K}$ is assigned to every open subset $U \subset X$ ($S_\emptyset = 0$) and if, for any pair of open subsets $V \subset U$, there exists a restriction morphism

$$\begin{gathered} r_V^U : S_U \longrightarrow S_V, \\ r_U^U = \operatorname{Id} S_U, \\ r_W^U = r_W^V r_V^U, \\ W \subset V \subset U. \end{gathered} \tag{A.1}$$

Every presheaf $S_{\{U\}}$ on a topological space X yields a sheaf on X whose stalk S_x at a point $x \in X$ is the direct limit of modules $S_U, x \in U$, with respect to restriction morphisms r_V^U. It means that, for each open neighborhood U of a point x, every element $s \in S_U$ determines an element $s_x \in S_x$, called the germ of s at x. Two elements $s \in S_U$ and $s' \in S_V$ belong to the same germ at x iff there exists an open neighborhood $W \subset U \cap V$ of x such that $r_W^U s = r_W^V s'$.

Example A.1. Let $C^0_{\{U\}}$ be a presheaf of continuous real functions on a topological space X. Two such functions s and s' define the same germ s_x if they coincide on an open neighborhood of x. Hence, we obtain a sheaf C^0_X of continuous functions on X.

Every sheaf S defines a presheaf $S(\{U\})$ of modules $S(U)$ of its local sections. It is called the canonical presheaf of S. The sheaf generated by the canonical presheaf of a sheaf S is S.

A direct sum and a tensor product of presheaves (as families of modules) and sheaves (as fibre bundles in modules) are naturally defined. By virtue of Theorem 12, a direct sum (resp., a tensor product) of presheaves generates a direct sum (resp., a tensor product) of sheaves.

A morphism of a presheaf $S_{\{U\}}$ to a presheaf $S'_{\{U\}}$ on a topological space X is defined as a set of module morphisms $\gamma_U : S_U \to S'_U$ which commute with restriction morphisms. A morphism of presheaves yields a morphism of sheaves generated by these presheaves. This is a bundle morphism over X such that $\gamma_x : S_x \to S'_x$ is the direct limit of morphisms $\gamma_U, x \in U$. Conversely, any morphism of sheaves $S \to S'$ on a topological space X yields a morphism of canonical presheaves of local sections of these sheaves. Let $\operatorname{Hom}(S|_U, S'|_U)$ be a commutative group of sheaf morphisms $S|_U \to S'|_U$ for any open subset $U \subset X$. These groups are assembled into a presheaf and define the sheaf $\operatorname{Hom}(S, S')$ on X. There is a monomorphism

$$\operatorname{Hom}\left(S, S'\right)(U) \longrightarrow \operatorname{Hom}\left(S(U), S'(U)\right). \tag{A.2}$$

A sheaf $\mathfrak{R}$ on a topological space X is said to be the ringed space if its stalk $\mathfrak{R}_x$ at each point $x \in X$ is a commutative ring [5, 7, 31]. A ringed space often is denoted by a pair $(X, \mathfrak{R})$ of a topological space X and a sheaf $\mathfrak{R}$ of rings on X which are called the body and the structure sheaf of a ringed space, respectively.

Definition A.2. A ringed space is said to be the local-ringed space (the geometric space in the terminology of [31]) if it is a sheaf of local rings.

Example A.3. A sheaf C^0_X of germs of continuous real functions on a topological space X (Example A.1) is a local-ringed space. Its stalk C^0_x, $x \in X$, contains a unique maximal ideal of germs of functions vanishing at x.

Morphisms of local-ringed spaces are defined to be particular morphisms of sheaves on different topological spaces as follows. Let $\varphi : X \to X'$ be a continuous map. Given a sheaf S on X, its direct image $\varphi_* S$ on X' is generated by the presheaf of assignments $X' \supset U' \to S(\varphi^{-1}(U'))$ for any open subset $U' \subset X'$. Conversely, given a sheaf S' on X', its inverse image $\varphi^* S'$ on X is defined as the pull-back onto X of a continuous fibre bundle S' over X'; that is, $\varphi^* S'_x = S_{\varphi(x)}$. This sheaf is generated by the presheaf which associates to any open subset $V \subset X$ the direct limit of modules $S'(U)$ over all open subsets $U \subset X'$ such that $V \subset f^{-1}(U)$.

By a morphism of ringed spaces $(X, \mathfrak{R}) \to (X', \mathfrak{R}')$ is meant a pair (φ, Φ) of a continuous map $\varphi : X \to X'$ and a sheaf morphism $\Phi : \mathfrak{R}' \to \varphi_* \mathfrak{R}$ or, equivalently, a sheaf morphism $\varphi^* \mathfrak{R}' \to \mathfrak{R}$ [31]. Restricted to each stalk, a sheaf morphism Φ is assumed to be a ring homomorphism. A morphism of ringed spaces is said to be (i) a monomorphism if φ is an injection and Φ is an epimorphism and (ii) an epimorphism if φ is a surjection, while Φ is a monomorphism.

Given a local-ringed space $(X, \mathfrak{R})$, a sheaf P on X is called the sheaf of $\mathfrak{R}$-modules if every stalk $P_x, x \in P_x, x \in X$, is an $\mathfrak{R}_x$-module or, equivalently, if $P(U)$ is an $\mathfrak{R}(U)$-module for any open subset $U \subset X$. A sheaf of $\mathfrak{R}$-modules P is said to be locally free if there exists an open neighborhood U of every point $x \in X$ such that $P(U)$ is a free $\mathfrak{R}(U)$-module. If all these free modules are of the same finite rank, one says that P is of constant rank. The structure module of a locally free sheaf is called the locally free module.

B. Jet Manifolds

Let $Y \to X$ be a smooth fibre bundle provided with bundle coordinates (x^λ, y^i). An r-order jet manifold $J^r Y$ of sections of a fibre bundle $Y \to X$ is defined as the disjoint union of equivalence classes $j^r_x s$ of sections s of $Y \to X$ which are identified by $r + 1$ terms of their Taylor series at points of X [4, 16, 32]. The set $J^r Y$ is endowed with an atlas of the adapted coordinates

$$\begin{gathered} \left(x^\lambda, y^i_\Lambda\right), \\ y^i_\Lambda \circ s = \partial_\Lambda s^i(x), \\ 0 \le |\Lambda| \le r, \end{gathered} \tag{B.1}$$

$$y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial' x^\lambda} d_\mu y'^i_\Lambda, \tag{B.2}$$

where the symbol d_λ stands for the higher order total derivative

$$\begin{aligned} d_\lambda &= \partial_\lambda + \sum_{0\leq|\Lambda|\leq r-1} y^i_{\Lambda+\lambda}\partial_i^\Lambda, \\ d'_\lambda &= \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu. \end{aligned} \tag{B.3}$$

These derivatives act on exterior forms on J^rY and obey the relations

$$\begin{aligned} \left[d_\lambda, d_\mu\right] &= 0, \\ d_\lambda(\phi\wedge\sigma) &= d_\lambda(\phi)\wedge\sigma + \phi\wedge d_\lambda(\sigma), \\ d_\lambda(dx^\mu) &= 0, \\ d_\lambda\left(dy^i_\Lambda\right) &= dy^i_{\lambda+\Lambda}. \end{aligned} \tag{B.4}$$

We use the compact notation $d_\Lambda = d_{\lambda_r}\circ\cdots\circ d_{\lambda_1}$, $\Lambda = (\lambda_r\cdots\lambda_1)$.

The coordinates (B.1) bring a set J^rY into a smooth manifold. They are compatible with the natural surjections

$$\pi^r_{r-1} : J^rY \longrightarrow J^{r-1}Y. \tag{B.5}$$

A glance at the transition functions (B.2) shows that they are affine bundles. It follows that Y is a strong deformation retract of any finite order jet manifold J^rY.

Given fibre bundles Y and Y' over X, every bundle morphism $\Phi : Y \to Y'$ over a diffeomorphism f of X admits the r-order jet prolongation to a morphism of r-order jet manifolds

$$J^r\Phi : J^rY \ni j^r_x s \longrightarrow j^r_{f(x)}\left(\Phi\circ s\circ f^{-1}\right) \in J^rY'. \tag{B.6}$$

Every section s of a fibre bundle $Y \to X$ has the r-order jet prolongation to a section $(J^rs)(x) = j^r_x s$ of a jet bundle $J^rY \to X$.

The surjections (B.5) form the inverse sequence of finite order jet manifolds

$$Y \xleftarrow{\pi} J^1Y \longleftarrow \cdots J^{r-1}Y \xleftarrow{\pi^r_{r-1}} J^rY \longleftarrow \cdots. \tag{B.7}$$

Its inverse limit $J^\infty Y$ is defined as a minimal set such that there exist surjections

$$\begin{aligned} \pi^\infty &: J^\infty Y \longrightarrow X, \\ \pi^\infty_0 &: J^\infty Y \longrightarrow Y, \\ \pi^\infty_k &: J^\infty Y \longrightarrow J^kY, \end{aligned} \tag{B.8}$$

obeying the relations $\pi^\infty_r = \pi^k_r\circ\pi^\infty_k$ for all admissible k and $r < k$. It consists of those elements

$$(\ldots, z_r, \ldots, z_k, \ldots), \quad z_r \in J^rY,\ z_k \in J^kY, \tag{B.9}$$

of the Cartesian product $\prod_k J^kY$ which satisfy the relations $z_r = \pi^k_r(z_k)$ for all $k > r$. One can think of elements of $J^\infty Y$ as being infinite order jets of sections of $Y \to X$ identified by their Taylor series at points of X. A set $J^\infty Y$ is provided with the inverse limit topology. This is the coarsest topology such that the surjections π^∞_r (B.8) are continuous. Its base consists of inverse images of open subsets of J^rY, $r = 0, \ldots$, under the maps π^∞_r. With the inverse limit topology, $J^\infty Y$ is a paracompact Fréchet manifold [16, 24, 25]. One can show that surjections π^∞_r are open maps admitting local sections; that is, $J^\infty Y \to J^rY$ are continuous bundles. A bundle coordinate atlas $\{U_Y, (x^\lambda, y^i)\}$ of $Y \to X$ provides $J^\infty Y$ with a manifold coordinate atlas

$$\begin{aligned} &\left\{(\pi^\infty_0)^{-1}(U_Y), \left(x^\lambda, y^i_\Lambda\right)\right\}_{0\leq|\Lambda|}, \\ &y'^i_{\lambda+\Lambda} = \frac{\partial x^\mu}{\partial x'^\lambda} d_\mu y'^i_\Lambda. \end{aligned} \tag{B.10}$$

Definition B.1. One calls $J^\infty Y$ the infinite order jet manifold.

Disclosure

One of the authors G. Sardanashvily has passed away on September 1 this year. It is a great loss to his grateful colleagues and students.

Competing Interests

The authors declare that they have no competing interests.

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