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Research Article

The Improved Generalized tanh-coth Method Applied to Sixth-Order Solitary Wave Equation

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The improved generalized tanh-coth method is used in nonlinear sixth-order solitary wave equation. This method is a powerful and advantageous mathematical tool for establishing abundant new traveling wave solutions of nonlinear partial differential equations. The new exact solutions consisted of trigonometric functions solutions, hyperbolic functions solutions, exponential functions solutions, and rational functions solutions. The numerical results were obtained with the aid of Maple.

1. Introduction

Nonlinear evolution equations (NLEEs) play an important role in various branches of scientific disciplines, such as fluid mechanics, optical fibers, plasma physics, chemical physics, biology, solid state physics, oceans engineering, and many other scientific applications. The solitary wave was introduced by Russell more than a century ago [1]. In the past years, many powerful methods for finding exact solutions of NLEEs have been proposed, such as the generalized (G'/G)-expansion method [2], the tanh-coth method [3], the modified sine-cosine method [4], the generalized unified method [5], the improved F-expansion method [6], the generalized Kudryashov method [7], the generalized Riccati equation mapping method [8], the modified Kudryashov method [9], the $\exp(-\phi(\xi))$ method [10], the lie symmetry analysis method [11], the first integral method [12], and the consistent Riccati expansion [13].

Another powerful method has been presented by Malfliet [14], who had customized the tanh technique and called the tanh method. In 2002, Fan and Hona [15] extended the tanh method which is called the extended tanh method, by using $U(\xi) = \sum_{k=0}^{M} a_k Y^k$ as traveling wave solutions. In 2007, Wazwaz [3] extended and improved this method which is called the tanh-coth method. In this method $U(\xi) = \sum_{k=0}^{M} a_k Y^k + \sum_{k=1}^{M} b_k Y^{-k}$ is used as traveling wave solutions. In 2008 Gómez and Salas [16] improved and generalized this method which is called the improved generalized tanh-coth method, by using $U(\xi) = \sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i \phi(\xi)^{M-i}$, where ϕ is the solution of the generalized Riccati equation. Afterwards, several researchers applied this method to obtain new exact solutions for nonlinear PDEs [17–20].

In 2017, Christou [21] studies solitons occurring in electrical nonlinear transmission lines; there are called electrical solitons. The problem is applied to Ohm's law of solid state physics by using Taylor-series expansions.

In this paper, we focus on using the improved generalized tanh-coth method for finding exact solutions of the sixth-order solitary wave equation:

$$V_{tt} = \left[aV + bV^2 - cV^3 + gV_{xx} + hVxxxx\right]_{xx}, \qquad (1)$$

which was proposed by Christou [21] and

$$a = \frac{h^2}{C_0 L}$$

$$b = \frac{1}{2F_0}$$

$$c = \frac{1}{3F_0^2}$$

$$g = \frac{(12\delta + 1)\, h^4}{12C_0 L}$$
$$h = \frac{(60\delta + 1)\, h^6}{360C_0 L}. \tag{2}$$

In Section 2, we briefly describe the improved generalized tanh-coth method; in Section 3, the improved generalized tanh-coth method is applied to the sixth-order solitary wave equations. The last section is short summary and discussion.

2. The Improved Generalized tanh-coth Method

Consider the nonlinear partial differential equation in the variables x and t

$$P_1\left(u, u_x, u_t, u_{x,t}, u_{x,x}, u_{t,t}, \ldots\right) = 0. \tag{3}$$

The traveling wave transformation is given by

$$u(x,t) = U(\xi), \quad \xi = x - \lambda t + \xi_0, \tag{4}$$

where λ is the wave speed. We can reduce (3) to the ordinary differential equation

$$P_2\left(U, U', U'', \ldots\right) = 0. \tag{5}$$

According to the improved generalized tanh-coth method, we seek the exact solution of (3) that can be expressed in the following form:

$$U(\xi) = \sum_{i=0}^{M} a_i \phi(\xi)^i + \sum_{i=M+1}^{2M} a_i \phi(\xi)^{M-i}, \tag{6}$$

where M is a positive integer that will be determined by balancing the highest order derivative term with the highest order nonlinear term. The coefficients a_i are constants ($a_M \neq 0$ and $a_{-M} \neq 0$) that are determined later while the new variable $\phi(\xi)$ is the solution to the generalized Riccati equation

$$\phi'(\xi) = \alpha + \beta\phi(\xi) + \gamma\left(\phi(\xi)\right)^2, \tag{7}$$

where α, β, and γ are constants. The solutions of generalized Riccati equation are given by [18].

Case 1 (exponential function solutions). When $\alpha = 0$

$$\phi(\xi) = \frac{\beta}{-\gamma + \beta e^{-\beta\xi}}. \tag{8}$$

Case 2 (trigonometric and hyperbolic function solution). When $\beta = 0$,

$$\phi(\xi) = \begin{cases} \dfrac{\sqrt{\alpha\gamma}}{\gamma}\tan\left(\sqrt{\alpha\gamma}\xi\right), & \alpha > 0,\ \gamma > 0, \\ \dfrac{\sqrt{\alpha\gamma}}{\gamma}\tanh\left(\sqrt{\alpha\gamma}\xi\right), & \alpha > 0,\ \gamma < 0, \\ \dfrac{\sqrt{-\alpha\gamma}}{\gamma}\tanh\left(-\sqrt{-\alpha\gamma}\xi\right), & \alpha < 0,\ \gamma > 0, \\ \dfrac{\sqrt{\alpha\gamma}}{\gamma}\tan\left(-\sqrt{\alpha\gamma}\xi\right), & \alpha < 0,\ \gamma < 0. \end{cases} \tag{9}$$

Case 3 (exponential function solutions). When $\gamma = 0$,

$$\phi(\xi) = \frac{-\alpha + \beta e^{\beta\xi}}{\beta}. \tag{10}$$

Case 4 (rational function solution). When $\alpha = \beta = 0$,

$$\phi(\xi) = -\frac{1}{\gamma\xi}. \tag{11}$$

Case 5 (rational function solution). When $\beta^2 \neq 0$ and $\beta^2 = 4\alpha\gamma$,

$$\phi(\xi) = -\frac{2\alpha(\beta\xi + 2)}{\beta^2\xi}. \tag{12}$$

Case 6 (trigonometric function solution). When $\beta^2 < 4\alpha\gamma$ and $\gamma \neq 0$,

$$\phi(\xi) = \frac{\sqrt{4\alpha\gamma - \beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma - \beta^2}\xi\right) - \beta}{2\gamma}. \tag{13}$$

Case 7 (hyperbolic function solution). When $\beta^2 > 4\alpha\gamma$ and $\gamma \neq 0$,

$$\phi(\xi) = \frac{\sqrt{\beta^2 - 4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2 - 4\alpha\gamma}\xi\right) - \beta}{2\gamma}. \tag{14}$$

We substitute (6) into (5) and collect all terms with the same order of $\phi^j(\xi)$; we get a polynomial in $\phi(\xi)$. Equating each coefficient of the polynomial to zero, we will give a system of algebraic equations involving the parameters a_i, α, γ, and β. Solving the system, we can construct a variety of exact solutions of (5).

3. The Improved Generalized tanh-coth Method Applied to Sixth-Order Solitary Wave Equation

We use the wave transformations $V(x,t) = V(\xi)$, $\xi = x - \lambda t + \xi_0$, to reduce (1) to the following ODE:

$$\begin{aligned} &\left(\lambda^2 - a\right)V''(\xi) - b\left(V^2(\xi)\right)'' + c\left(V^3(\xi)\right)'' \\ &\quad + gV''''(\xi) - hV''''''(\xi) = 0. \end{aligned} \tag{15}$$

Balancing the highest order term V'''''' with the highest order nonlinear term $(V^3)''''$ in (13), we have

$$M + 6 = 3M + 2; \tag{16}$$

then $M = 2$. Consequently, we set

$$V(\xi) = a_0 + a_1\phi(\xi) + a_2(\phi(\xi))^2 + a_3(\phi(\xi))^{-1} + a_4(\phi(\xi))^{-2}. \tag{17}$$

Using (6) and (14) in (13) and equating all the coefficients of power of $\phi(\xi)$ to be zero, we obtain a system of algebraic equations in the unknowns a_0, a_1, a_2, a_3, a_4, λ, α, β, and γ.

$$\begin{aligned}
\phi(\xi)^{-8}:\ & -5040\alpha^6 z a_4 + 42\alpha^2 c a_4^3 = 0,\\
\phi(\xi)^{-7}:\ & -720\alpha^6 z a_3 - 19440\alpha^5\beta z a_4 + 90\alpha^2 c a_3 a_4^2\\
& + 8\alpha\beta c a_4^3 = 0,\\
\phi(\xi)^{-6}:\ & 165 c a_3 a_4^2\alpha\beta + 72 c a_4^3\alpha\gamma + 60 c a_0 a_4^2\alpha^2\\
& + 60 c a_3^2\alpha^2 a_4 - 29400 z a_4\alpha^4\beta^2 - 13440 z a_4\alpha^5\gamma\\
& - 2520 z a_3\alpha^5\beta + 36 c a_4^3\beta^2 - 20 b a_4^2\alpha^2\\
& - 120 g a_4\alpha^4 = 0,\\
\phi(\xi)^{-5}:\ & 75 c a_3 a_4^2\beta^2 + 66 c a_4^3\beta\gamma + 72 c a_0 a_3 a_4\alpha^2\\
& + 108 c a_0 a_4^2\alpha\beta + 150 c a_3 a_4^2\alpha\gamma + 108 c a_3^2 a_4\alpha\beta\\
& - 38640 z a_4\alpha^4\beta\gamma - 24 b a_3 a_4\alpha^2 - 36 b a_4^2\alpha\beta\\
& + 36 c a_1 a_4^2\alpha^2 - 21840 z a_4\alpha^3\beta^3 - 3360 z a_3\alpha^4\beta^2\\
& - 1680 z a_3\alpha^5\gamma - 336 g a_4\alpha^3\beta + 12 c a_3^3\alpha^2\\
& - 24 g a_3\alpha^4 = 0,\\
\phi(\xi)^{-4}:\ & 96 c a_0 a_4^2\alpha\gamma + 36 c a_1 a_3 a_4\alpha^2 + 135 c a_3 a_4^2\beta\gamma\\
& - 4200 z a_3\alpha^4\beta\gamma - 40152 z a_4\alpha^3\beta^2\gamma - 42 b a_3 a_4\alpha\beta\\
& + 96 c a_3^2 a_4\alpha\gamma + 63 c a_1 a_4^2\alpha\beta - 12096 z a_4\alpha^4\gamma^2\\
& - 8106 z a_4\alpha^2\beta^4 - 2100 z a_3\alpha^3\beta^3 - 240 g a_4\alpha^3\gamma\\
& - 330 g a_4\alpha^2\beta^2 - 60 g a_3\alpha^3\beta + 21 c a_3^3\alpha\beta + 48 c a_3^2 a_4\\
& - 12 b a_0 a_4\alpha^2 - 32 b a_4^2\alpha\gamma + 18 c a_0^2 a_4\alpha^2 + 18 c a_0 a_3^2\alpha^2\\
& + 48 c a_0 a_4^2\beta^2 + 18 c a_2 a_4^2\alpha^2 - 6 a a_4\alpha^2 + 6\lambda^2 a_4\alpha^2\\
& + 30 c a_4^3\gamma^2 - 6 b a_3^2\alpha^2 - 16 b a_4^2\beta^2\\
& + 126 c a_0 a_3 a_4\alpha\beta = 0,\\
\phi(\xi)^{-3}:\ & 54 c a_0 a_3 a_4\beta^2 + 84 c a_0 a_4^2\beta\gamma + 12 c a_2 a_3 a_4\alpha^2\\
& + 30 c a_2 a_4^2\alpha\beta - 20 b a_0 a_4\alpha\beta - 440 g a_4\alpha^2\beta\gamma\\
& - 3584 z a_3\alpha^3\beta^2\gamma - 17920 z a_4\alpha^2\beta^3\gamma\\
& - 22960 z a_4\alpha^3\beta\gamma^2 - 36 b a_3 a_4\alpha\gamma + 30 c a_0^2 a_4\alpha\beta\\
& + 54 c a_1 a_4^2\alpha\gamma + 84 c a_3^2 a_4\beta\gamma + 12 c a_0 a_1 a_4\alpha^2\\
& + 30 c a_0 a_3^2\alpha\beta - 1330 z a_4\beta^5\alpha - 1232 z a_3\alpha^4\gamma^2\\
& - 602 z a_3\alpha^2\beta^4 - 130 g a_4\alpha\beta^3 - 40 g a_3\alpha^3\gamma\\
& - 50 g a_3\alpha^2\beta^2 + 18 c a_3^3\alpha\gamma + 60 c a_3 a_4^2\gamma^2 + 10 a_4\lambda^2\alpha\beta\\
& - 4 b a_0 a_3\alpha^2 - 10 a a_4\alpha\beta - 4 b a_1 a_4\alpha^2 - 10 b a_3^2\alpha\beta\\
& - 18 b a_3 a_4\beta^2 - 28 b a_4^2\beta\gamma + 6 c a_0^2 a_3\alpha^2 + 2 c a_1 a_4^2\beta^2\\
& + 6 c a_1 a_3^2\alpha^2 - 2 a a_3\alpha^2 + 2 a_3\lambda^2\alpha^2 + 9 c a_3^3\beta^2\\
& + 108 c a_0 a_3 a_4\alpha\gamma + 60 c a_1 a_3 a_4\alpha\beta = 0,\\
\phi(\xi)^{-2}:\ & 24 c a_0 a_3^2\alpha\gamma + 9 c a_1 a_3^2\alpha\beta + 24 c a_1 a_3 a_4\beta^2\\
& + 24 c a_2 a_4^2\alpha\gamma - 6 b a_0 a_3\alpha\beta - 16 b a_0 a_4\alpha\gamma\\
& - 232 g a_4\alpha\gamma\beta^2 - 60 g a_3\alpha^2\beta\gamma - 116 z a_3\alpha^2\beta\gamma\\
& - 1848 z a_3\alpha^3\beta\gamma^2 - 13320 z a_4\alpha^2\beta^2\gamma^2\\
& - 3096 z a_4\alpha\beta^2\gamma - 30 b a_3 a_4\beta\gamma - 6 b a_1 a_4\alpha\beta\\
& + 9 c a_0^2 a_3\alpha\beta + 24 c a_0^2 a_4\alpha\gamma + 45 c a_1 a_4^2\beta\gamma\\
& - 3968 z a_4\alpha^3\gamma^3 - 63 z a_3\beta^5\alpha - 136 g a_4\alpha^2\gamma^2\\
& - 15 g a_3\alpha\beta^3 + 15 c a_3^3\beta\gamma + 36 c a_3^2 a_4\gamma^2 + 3 a_3\lambda^2\alpha\beta\\
& + 8 a_4\lambda^2\alpha\gamma - 8 b a_0 a_4\beta^2 - 3 a a_3\alpha\beta - 8 a a_4\alpha\gamma\\
& - 8 b a_3^2\alpha\gamma + 12 c a_0^2 a_4\beta^2 + 12 c a_0 a_3^2\beta^2 + 36 c a_0 a_4^2\gamma^2\\
& + 12 c a_2 a_4^2\beta^2 - 4 a a_4\beta^2 + 4 a_4\lambda^2\beta^2 - 4 a_3^2 b\beta^2\\
& - 12 b a_4^2\gamma^2 - 64 z a_4\beta^6 - 16 g a_4\beta^4 + 90 c a_0 a_3 a_4\beta\gamma\\
& + 18 c a_0 a_1 a_4\alpha\beta + 48 c a_1 a_3 a_4\alpha\gamma + 18 c a_2 a_3 a_4\alpha\beta = 0,\\
\phi(\xi)^{-1}:\ & 18 c a_0 a_3^2\beta\gamma + 36 c a_0 a_3 a_4\gamma^2 + 6 c a_1 a_3^2\alpha\gamma\\
& + 6 c a_2 a_3 a_4\beta^2 + 18 c a_2 a_4^2\beta\gamma - 4 b a_0 a_3\alpha\gamma\\
& - 12 b a_0 a_4\beta\gamma - 120 g a_4\alpha\beta\gamma^2 - 22 g a_3\alpha\beta^2\gamma\\
& - 720 z a_3\alpha^2\beta^2\gamma^2 - 114 z a_3\alpha\beta^4\gamma - 3696 z a_4\alpha^2\beta\gamma^3\\
& - 2352 z a_4\alpha\beta^3\gamma^2 - 4 b a_1 a_4\alpha\gamma + 6 c a_0^2 a_3\alpha\gamma
\end{aligned}$$

$$+ 18ca_0^2a_4\beta\gamma + 6ca_0a_1a_4\beta^2 - 126za_4\beta^5\gamma$$
$$- 22za_3\alpha^3\gamma^3 - 30ga_4\beta^3\gamma - 16ga_3\alpha^2\gamma^2 + 2a_3\lambda^2\alpha\gamma$$
$$+ 6a_4\lambda^2\beta\gamma - 2ba_0a_3\beta^2 - 2aa_3\alpha\gamma - 6aa_4\beta\gamma$$
$$- 2ba_1a_4\beta^2 - 6ba_3^2\beta\gamma - 12ba_3a_4\gamma^2 + 3ca_0^2a_3\beta^2$$
$$+ 18ca_1a_4^2\gamma^2 + 3ca_1a_3^2\beta^2 - aa_3\beta^2 + a_3\lambda^2\beta^2$$
$$+ 6ca_3^3\gamma^2 - za_3\beta^6 - ga_3\beta^4 + 12ca_0a_1a_4\alpha\gamma$$
$$+ 36ca_1a_3a_4\beta\gamma + 12ca_2a_3a_4\alpha\gamma = 0,$$

$$\phi(\xi)^0 : a_3\lambda^2\beta\gamma + a_1\lambda^2\beta\alpha + 6ca_2^2a_4\alpha^2 + 6ca_2a_4^2\gamma^2$$
$$+ 6ca_0a_1^2\alpha^2 + 6ca_0a_3^2\gamma^2 + 6ca_0^2a_4\gamma^2 + 6ca_0^2a_2\alpha^2$$
$$- 62za_2\beta^4\alpha^2 - 272za_2\gamma^2\alpha^4 - za_1\beta^5\alpha - 62za_4\beta^4\gamma^2$$
$$- 22za_4\alpha^2\gamma^4 - za_3\beta^5\gamma - 16ga_4\alpha\gamma^3 - 14ga_4\beta^2\gamma^2$$
$$- ga_3\beta^3\gamma - ga_1\beta^3\alpha - 14ga_2\beta^2\alpha^2 - 16ga_2\gamma\alpha^3$$
$$- 4ba_0a_4\gamma^2 - 4ba_0a_2\alpha^2 - aa_3\beta\gamma - aa_1\beta\alpha$$
$$- 584za_2\beta^2\alpha^3\gamma - 52za_1\beta^3\gamma\alpha^2 - 136za_1\beta\gamma^2\alpha^3$$
$$- 136za_3\alpha^2\beta\gamma^3 - 52za_3\alpha\beta^3\gamma^2 - 584za_4\alpha\beta^2\gamma^3$$
$$- 8ga_1\beta\gamma\alpha^2 - 8ga_3\alpha\beta\gamma^2 - 2ba_0a_3\beta\gamma - 2ba_0a_1\beta\alpha$$
$$- 2ba_1a_4\beta\gamma - 2ba_2a_3\alpha\beta + 3ca_0^2a_3\beta\gamma + 3ca_0^2a_1\beta\alpha$$
$$+ 12ca_1a_3a_4\gamma^2 + 3ca_1^2a_3\alpha\beta + 12ca_1a_2a_3\alpha^2$$
$$+ 3ca_1a_3^2\beta\gamma - 2ba_1^2\alpha^2 - 2ba_3^2\gamma^2 - 2aa_4\gamma^2 - 2aa_2\alpha^2$$
$$+ 2a_4\lambda^2\gamma^2 + 2a_2\lambda^2\alpha^2 + 6ca_0a_1a_4\beta\gamma + 6ca_0a_2a_3\alpha\beta$$
$$+ 6ca_1a_2a_4\alpha\beta + 6ca_2a_3a_4\beta\gamma = 0,$$

$$\phi(\xi) : 18ca_2^2a_4\alpha\beta - 4ba_2a_3\alpha\gamma - 4ba_0a_1\gamma\alpha + 6ca_0^2a_1\gamma\alpha$$
$$+ 18ca_0^2a_2\alpha\beta + 6ca_0a_2a_3\beta^2 + 18ca_0a_1^2\alpha\beta$$
$$+ 36ca_0a_1a_2\alpha^2 + 6ca_1^2a_3\alpha\gamma + 6ca_1a_2a_4\beta^2$$
$$- 12ba_0a_2\alpha\beta - 22ga_1\beta^2\gamma\alpha - 120ga_2\beta\alpha^2\gamma$$
$$- 114za_1\beta^4\gamma\alpha - 720za_1\beta^2\gamma^2\alpha^2 - 2352za_2\beta^3\gamma\alpha^2$$
$$- 3696za_2\beta\gamma^2\alpha^3 - 6ba_1^2\alpha\beta - 12ba_1a_2\alpha^2$$
$$- 2ba_2a_3\beta^2 + 3ca_0^2a_1\beta^2 + 18ca_2^2a_3\alpha^2 + 3ca_1^2a_3\beta^2$$
$$+ 2a_1\lambda^2\gamma\alpha + 6a_2\lambda^2\alpha\beta - 126za_2\beta^5\alpha - 272za_1\gamma^3\alpha^3$$
$$- 16ga_1\gamma^2\alpha^2 - 30ga_2\beta^3\alpha - 2ba_0a_1\beta^2 - 2aa_1\gamma\alpha$$
$$- 6aa_2\alpha\beta + 6ca_1^3\alpha^2 - za_1\beta^6 - ga_1\beta^4 - aa_1\beta^2$$
$$+ a_1\lambda^2\beta^2 + 12ca_0a_2a_3\alpha\gamma + 36ca_1a_2a_3\alpha\beta$$
$$+ 12ca_1a_2a_4\alpha\gamma = 0,$$

$$\phi(\xi)^2 : 24ca_1a_2a_3\beta^2 + 9ca_1^2a_3\beta\gamma + 24ca_2^2a_4\alpha\gamma$$
$$- 16ba_0a_2\alpha\gamma - 60ga_1\beta\alpha\gamma^2 - 232ga_2\beta^2\alpha\gamma$$
$$- 1848za_1\beta\gamma^3\alpha^2 - 1176za_1\beta^3\alpha\gamma^2 - 3096za_2\beta^4\gamma\alpha$$
$$- 13320za_2\beta^2\gamma^2\alpha^2 - 30ba_1a_2\alpha\beta - 6ba_2a_3\beta\gamma$$
$$- 6ba_0a_1\beta\gamma + 9ca_0^2a_1\beta\gamma + 24ca_0^2a_2\alpha\gamma + 45ca_2^2a_3\alpha\beta$$
$$+ 24ca_0a_1^2\alpha\gamma - 3968za_2\gamma^3\alpha^3 - 63za_1\beta^5\gamma$$
$$- 15ga_1\beta^3\gamma - 136ga_2\gamma^2\alpha^2 + 3a_1\lambda^2\beta\gamma + 8a_2\lambda^2\alpha\gamma$$
$$- 8ba_0a_2\beta^2 - 3aa_1\beta\gamma - 8aa_2\alpha\gamma - 8ba_1^2\alpha\gamma$$
$$+ 12ca_0^2a_2\beta^2 + 36ca_0a_2^2\alpha^2 + 12ca_0a_1^2\beta^2 + 15ca_1^3\alpha\beta$$
$$+ 36ca_1^2a_2\alpha^2 + 12ca_2^2a_4\beta^2 - 4aa_2\beta^2 + 4a_2\lambda^2\beta^2$$
$$- 12ba_2^2\alpha^2 - 4ba_1^2\beta^2 - 64za_2\beta^6 - 16ga_2\beta^4$$
$$+ 90ca_0a_1a_2\alpha\beta + 18ca_0a_2a_3\beta\gamma + 48ca_1a_2a_3\alpha\gamma$$
$$+ 18ca_1a_2a_4\beta\gamma = 0,$$

$$\phi(\xi)^3 : 6ca_1^2a_3\gamma^2 + 60ca_1a_2^2\alpha^2 + 10a_2\lambda^2\beta\gamma$$
$$- 1330za_2\beta^5\gamma - 602za_1\beta^4\gamma^2 - 1232za_1\gamma^4\alpha^2$$
$$- 50ga_1\beta^2\gamma^2 - 40ga_1\alpha\gamma^3 - 130ga_2\beta^3\gamma - 4ba_0a_1\gamma$$
$$- 10aa_2\beta\gamma - 10ba_1^2\beta\gamma - 18ba_1a_2\beta^2 - 28ba_2^2\alpha\beta$$
$$- 4ba_2a_3\gamma^2 + 6ca_0^2a_1\gamma^2 + 27ca_2^2a_3\beta^2 + 18ca_1^3\alpha\gamma$$
$$+ 108ca_0a_1a_2\alpha\gamma + 60ca_1a_2a_3\beta\gamma - 20ba_0a_2\beta\gamma$$
$$- 440ga_2\beta\alpha\gamma^2 - 3584za_1\beta^2\alpha\gamma^3 - 22960za_2\beta\gamma^3\alpha^2$$
$$+ 12ca_1a_2a_4\gamma^2 + 84ca_1^2a_2\alpha\beta + 30ca_2^2a_4\beta\gamma$$
$$- 17920za_2\beta^3\alpha\gamma^2 + 54ca_2^2a_3\alpha\gamma - 36ba_1a_2\alpha\gamma$$
$$+ 30ca_0^2a_2\beta\gamma + 30ca_0a_1^2\beta\gamma + 54ca_0a_1a_2\beta^2$$
$$+ 84ca_0a_2^2\alpha\beta + 12ca_0a_2a_3\gamma^2 - 2aa_1\gamma^2 + 2a_1\lambda^2\gamma^2$$
$$+ 9ca_1^3\beta^2 = 0,$$

$$\phi(\xi)^4 : 135ca_1a_2^2\alpha\beta + 36ca_1a_2a_3\gamma^2 + 96ca_1^2a_2\alpha\gamma$$
$$- 4200za_1\beta\gamma^4\alpha - 40152za_2\beta^2\alpha\gamma^3 + 63ca_2^2a_3\beta\gamma$$

$$
\begin{aligned}
&-42ba_1a_2\beta\gamma+96ca_0a_2^2\alpha\gamma-8106za_2\beta^4\gamma^2\\
&-12096za_2\gamma^4\alpha^2-2100za_1\beta^3\gamma^3-60ga_1\beta\gamma^3\\
&-330ga_2\beta^2\gamma^2-240ga_2\gamma^3\alpha-12ba_0a_2\gamma^2\\
&-32ba_2^2\alpha\gamma+18ca_0^2a_2\gamma^2+18ca_0a_1^2\gamma^2+48ca_0a_2^2\beta^2\\
&+21ca_1^3\beta\gamma+48ca_1^2a_2\beta^2+18ca_2^2a_4\gamma^2-6aa_2\gamma^2\\
&-6aa_2\gamma^2+6a_2\lambda^2\gamma^2+30ca_2^3\alpha^2-6ba_1^2\gamma^2\\
&-16ba_2^2\beta^2+126ca_0a_1a_2\beta\gamma=0,\\
\phi(\xi)^5:\ &-336ga_2\beta\gamma^3-24ba_1a_2\gamma^2-36ba_2^2\beta\gamma\\
&+36ca_2^2a_3\gamma^2+75ca_1a_2^2\beta^2+66ca_2^3\alpha\beta\\
&-21840za_2\beta^3\gamma^3-3360za_1\beta^2\gamma^4-1680za_1\gamma^5\alpha\\
&+72ca_0a_1a_2\gamma^2+108ca_0a_2^2\beta\gamma+150ca_1a_2^2\alpha\gamma\\
&+108ca_1^2a_2\beta\gamma-38640za_2\beta\gamma^4\alpha+12ca_1^3\gamma^2\\
&-24ga_1\gamma^4=0,\\
\phi(\xi)^6:\ &60ca_0a_2^2\gamma^2+60ca_1^2a_2\gamma^2+72ca_2^3\alpha\gamma\\
&-29400za_2\beta^2\gamma^2-13440za_2\gamma^5\alpha-2520za_1\beta\gamma^5\\
&+165ca_1a_2^2\beta\gamma+36ca_2^3\beta^2-20ba_2^2\gamma^2\\
&-120ga_2\gamma^4=0,\\
\phi(\xi)^7:\ &-19440\beta\gamma^5za_2-720a_1z\gamma^6+8ca_2^3\beta\gamma\\
&+90ca_1a_2^2\gamma^2=0,\\
\phi(\xi)^8:\ &-5040a_2z\gamma^6+42ca_2^3\gamma^2=0.
\end{aligned}
\tag{18}
$$

Solving the system of algebraic equations with the aid of Maple, using (18), we obtain the following results.

First Set

$$
\begin{aligned}
a_0&=\pm\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right),\\
a_1&=0,\\
a_2&=0,\\
a_3&=\pm\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c},\\
a_4&=\pm\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c},\\
\lambda&=\pm\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-360\alpha\gamma\beta^2ch^2+45\beta^4ch^2-30ach-10b^2h+3cg^2\right)}}{ch}.
\end{aligned}
\tag{19}
$$

Case 1. When $\beta=0$, $\alpha>0$, and $\gamma>0$, $\phi(\xi)=(\sqrt{\alpha\gamma}/\gamma)\tan(\sqrt{\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$
\begin{aligned}
V_1(x,t)&=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)\\
&\quad+\frac{2\sqrt{30}\sqrt{hc}\gamma\alpha^2}{c}\cot^2\left(\sqrt{\alpha\gamma}\left(x+\lambda t+\xi_0\right)\right),\\
V_2(x,t)&=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)\\
&\quad+\frac{2\sqrt{30}\sqrt{hc}\gamma\alpha^2}{c}\cot^2\left(\sqrt{\alpha\gamma}\left(x-\lambda t+\xi_0\right)\right),
\end{aligned}
\tag{20}
$$

where

$$
\begin{aligned}
\lambda&=\frac{1}{30}\\
&\quad\cdot\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-30ach-10b^2h+3cg^2\right)}}{ch}.
\end{aligned}
\tag{21}
$$

Case 2. When $\beta=0$, $\alpha<0$, and $\gamma>0$, $\phi(\xi)=(\sqrt{-\alpha\gamma}/\gamma)\tanh(-\sqrt{-\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$
\begin{aligned}
V_3(x,t)&=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)\\
&\quad-\frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\coth^2\left(-\sqrt{-\alpha\gamma}\left(x+\lambda t+\xi_0\right)\right),
\end{aligned}
$$

$$V_4(x,t) = \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\coth^2\left(-\sqrt{-\alpha\gamma}\left(x-\lambda t+\xi_0\right)\right), \tag{22}$$

where

$$\lambda = \frac{1}{30} \cdot \frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{23}$$

Case 3. When $\gamma = 0$, $\phi(\xi) = (-\alpha + \beta e^{\beta\xi})/\beta$, the combined formal single kink solutions of (1) are

$$\begin{aligned} V_5(x,t) &= \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha}{c\left(-\alpha+\beta e^{\beta(x+\lambda t+\xi_0)}\right)} + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha^2}{c\left(-\alpha+\beta e^{\beta(x+\lambda t+\xi_0)}\right)^2}, \\ V_6(x,t) &= \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha}{c\left(-\alpha+\beta e^{\beta(x-\lambda t+\xi_0)}\right)} + \frac{2\beta^2\sqrt{30}\sqrt{hc}\alpha^2}{c\left(-\alpha+\beta e^{\beta(x-\lambda t+\xi_0)}\right)^2}, \end{aligned} \tag{24}$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{25}$$

Case 4. When $\beta^2 \neq 0$ and $\beta^2 = 4\alpha\gamma$, $\phi(\xi) = -2\alpha(\beta\xi+2)/\beta^2\xi$, the rational solutions of (1) are

$$\begin{aligned} V_7(x,t) &= \frac{1}{30}\left(\frac{15\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{ch}\right) - \frac{\beta^3\sqrt{30}\sqrt{hc}\left(x-\lambda t+\xi_0\right)}{c\left(\beta\left(x-\lambda t+\xi_0\right)+2\right)} + \frac{\beta^5\sqrt{30}\sqrt{hc}\left(x-\lambda t+\xi_0\right)^2}{2c\left(\beta\left(x-\lambda t+\xi_0\right)+2\right)^2}, \\ V_8(x,t) &= \frac{1}{30}\left(\frac{15\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{ch}\right) - \frac{\beta^3\sqrt{30}\sqrt{hc}\left(x+\lambda t+\xi_0\right)}{c\left(\beta\left(x+\lambda t+\xi_0\right)+2\right)} + \frac{\beta^5\sqrt{30}\sqrt{hc}\left(x+\lambda t+\xi_0\right)^2}{2c\left(\beta\left(x+\lambda t+\xi_0\right)+2\right)^2}, \end{aligned} \tag{26}$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(-30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{27}$$

Case 5. When $\beta^2 < 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{4\alpha\gamma - \beta^2}\tan((1/2)\sqrt{4\alpha\gamma - \beta^2}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$\begin{aligned} V_9(x,t) &= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{4\alpha\gamma-\beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma-\beta^2}\left(x+\lambda t+\xi_0\right)\right)-\beta}\right) \\ &\quad + \frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma-\beta^2}\left(x+\lambda t+\xi_0\right)\right)-\beta\right)^2}\right), \\ V_{10}(x,t) &= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) \end{aligned}$$

$$+\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{4\alpha\gamma-\beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma-\beta^2}\left(x-\lambda t+\xi_0\right)\right)-\beta}\right)$$
$$+\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left((1/2)\sqrt{4\alpha\gamma-\beta^2}\left(x-\lambda t+\xi_0\right)\right)-\beta\right)^2}\right), \tag{28}$$

where

$$\lambda=\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-360\alpha\gamma\beta^2ch^2+45\beta^4ch^2-30ach-10b^2h+3cg^2\right)}}{ch}. \tag{29}$$

Case 6. When $\beta^2 > 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{\beta^2-4\alpha\gamma}\tanh((1/2)\sqrt{\beta^2-4\alpha\gamma}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$V_{11}(x,t)=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)$$
$$+\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x+\lambda t+\xi_0\right)\right)-\beta}\right)$$
$$+\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x+\lambda t+\xi_0\right)\right)-\beta\right)^2}\right),$$
$$V_{12}(x,t)=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)$$
$$+\frac{2\beta\sqrt{30}\sqrt{hc}\alpha}{c}\left(\frac{2\gamma}{\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x-\lambda t+\xi_0\right)\right)-\beta}\right)$$
$$+\frac{2\sqrt{30}\sqrt{hc}\alpha^2}{c}\left(\frac{4\gamma^2}{\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left((1/2)\sqrt{\beta^2-4\alpha\gamma}\left(x-\lambda t+\xi_0\right)\right)-\beta\right)^2}\right), \tag{30}$$

where

$$\lambda=\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-360\alpha\gamma\beta^2ch^2+45\beta^4ch^2-30ach-10b^2h+3cg^2\right)}}{ch}. \tag{31}$$

Second Set

$$\begin{aligned}
a_0 &= \pm\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2 h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right),\\
a_1 &= \pm\frac{2\beta\sqrt{30}\sqrt{hc}\gamma}{c},\\
a_2 &= \pm\frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c},\\
a_3 &= 0,\\
a_4 &= 0,\\
\lambda &= \pm\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 360\alpha\gamma\beta^2ch^2 + 45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}.
\end{aligned} \tag{32}$$

Case 1. When $\alpha = 0$, $\phi(\xi) = \beta/(-\gamma + \beta e^{-\beta\xi})$, the combined formal single kink solutions of (1) are

$$\begin{aligned}
V_{13}(x,t) &= \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2 h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma}{c\left(-\gamma + \beta e^{-\beta(x+\lambda t+\xi_0)}\right)}\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(-\gamma + \beta e^{-\beta(x+\lambda t+\xi_0)}\right)^2},\\
V_{14}(x,t) &= \frac{1}{30}\left(\frac{5\sqrt{30}\sqrt{hc}\beta^2 h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma}{c\left(-\gamma + \beta e^{-\beta(x-\lambda t+\xi_0)}\right)}\\
&\quad + \frac{2\beta^2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(-\gamma + \beta e^{-\beta(x-\lambda t+\xi_0)}\right)^2},
\end{aligned} \tag{33}$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(45\beta^4ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{34}$$

Case 2. When $\beta = 0$, $\alpha > 0$, and $\gamma > 0$, $\phi(\xi) = (\sqrt{\alpha\gamma}/\gamma)\tan(\sqrt{\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$\begin{aligned}
V_{15}(x,t) &= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\tan^2\left(\sqrt{\alpha\gamma}\left(x + \lambda t + \xi_0\right)\right),\\
V_{16}(x,t) &= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\alpha\gamma}{c}\tan^2\left(\sqrt{\alpha\gamma}\left(x - \lambda t + \xi_0\right)\right),
\end{aligned} \tag{35}$$

where

$$\begin{aligned}
\lambda &= \frac{1}{30}\\
&\quad \cdot \frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}.
\end{aligned} \tag{36}$$

Case 3. When $\beta = 0$, $\alpha < 0$, and $\gamma > 0$, $\phi(\xi) = (\sqrt{-\alpha\gamma}/\gamma)\tanh(-\sqrt{-\alpha\gamma}\xi)$, the periodic solutions of (1) are

$$\begin{aligned}
V_{17}(x,t) &= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\left(\sqrt{-\alpha\gamma}\right)^2}{c}\\
&\quad \cdot \tanh^2\left(-\sqrt{-\alpha\gamma}\left(x + \lambda t + \xi_0\right)\right),\\
V_{18}(x,t) &= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&\quad + \frac{2\sqrt{30}\sqrt{hc}\left(\sqrt{-\alpha\gamma}\right)^2}{c}\\
&\quad \cdot \tanh^2\left(-\sqrt{-\alpha\gamma}\left(x - \lambda t + \xi_0\right)\right),
\end{aligned} \tag{37}$$

where

$$\lambda = \frac{1}{30} \cdot \frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2 - 30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{38}$$

Case 4. When $\alpha = \beta = 0, \phi(\xi) = -1/\gamma\xi$, the rational solutions of (1) are

$$\begin{aligned}
V_{19}(x,t) &= \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x+\lambda t+\xi_0\right)\right)^2},\\
V_{20}(x,t) &= \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x-\lambda t+\xi_0\right)\right)^2},\\
V_{21}(x,t) &= \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x+\lambda t+\xi_0\right)\right)^2},\\
V_{22}(x,t) &= \frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x-\lambda t+\xi_0\right)\right)^2},\\
V_{23}(x,t) &= -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x+\lambda t+\xi_0\right)\right)^2},\\
V_{24}(x,t) &= -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) + \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x-\lambda t+\xi_0\right)\right)^2},\\
V_{25}(x,t) &= -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x+\lambda t+\xi_0\right)\right)^2},\\
V_{26}(x,t) &= -\frac{1}{30}\left(\frac{\sqrt{30}\sqrt{hc}g + 10bh}{hc}\right) - \frac{2\sqrt{30}\sqrt{hc}\gamma^2}{c\left(\gamma\left(x-\lambda t+\xi_0\right)\right)^2},
\end{aligned} \tag{39}$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(-30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{40}$$

Case 5. When $\beta^2 \neq 0$ and $\beta^2 = 4\alpha\gamma$, $\phi(\xi) = -2\alpha(\beta\xi+2)/\beta^2\xi$, the rational solutions of (1) are

$$\begin{aligned}
V_{27}(x,t) &= \frac{1}{30}\frac{15\sqrt{30}\sqrt{hc}\beta^2h + 10bh + \sqrt{30}\sqrt{ch}g}{ch} - \frac{\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x-\lambda t+\xi_0\right)+2\right)}{c\left(x-\lambda t+\xi_0\right)} + \frac{4\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x-\lambda t+\xi_0\right)+2\right)^2}{c\left(x-\lambda t+\xi_0\right)^2},\\
V_{28}(x,t) &= \frac{1}{30}\frac{15\sqrt{30}\sqrt{hc}\beta^2h + 10bh + \sqrt{30}\sqrt{ch}g}{ch} - \frac{\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x+\lambda t+\xi_0\right)+2\right)}{c\left(x+\lambda t+\xi_0\right)} + \frac{4\beta\sqrt{30}\sqrt{ch}\left(\beta\left(x+\lambda t+\xi_0\right)+2\right)^2}{c\left(x+\lambda t+\xi_0\right)^2},
\end{aligned} \tag{41}$$

where

$$\lambda = \frac{1}{30}\frac{\sqrt{-30ch\left(-30ach - 10b^2h + 3cg^2\right)}}{ch}. \tag{42}$$

Case 6. When $\beta^2 < 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{4\alpha\gamma - \beta^2}\tan((1/2)\sqrt{4\alpha\gamma - \beta^2}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$\begin{aligned}
&V_{29}(x,t)\\
&= \frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h + 5\sqrt{30}\sqrt{hc}\beta^2h + \sqrt{30}\sqrt{hc}g + 10bh}{hc}\right)\\
&+ \frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{4\alpha\gamma - \beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma - \beta^2}\left(x+\lambda t+\xi_0\right)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-\beta\Big)\\
&+\frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma-\beta^2}\,(x+\lambda t+\xi_0)\right)\right.\\
&\left.-\beta\right)^2,\\
V_{30}(x,t)&\\
&=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)\\
&+\frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma-\beta^2}\,(x-\lambda t+\xi_0)\right)\right.\\
&\left.-\beta\right)\\
&+\frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{4\alpha\gamma-\beta^2}\tan\left(\frac{1}{2}\sqrt{4\alpha\gamma-\beta^2}\,(x-\lambda t+\xi_0)\right)\right.\\
&\left.-\beta\right)^2,
\end{aligned}
\tag{43}
$$

where

$$
\lambda=\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-360\alpha\gamma\beta^2ch^2+45\beta^4ch^2-30ach-10b^2h+3cg^2\right)}}{ch}. \tag{44}
$$

Case 7. When $\beta^2 > 4\alpha\gamma$ and $\gamma \neq 0$, $\phi(\xi) = (\sqrt{\beta^2-4\alpha\gamma}\tanh((1/2)\sqrt{\beta^2-4\alpha\gamma}\xi) - \beta)/2\gamma$, the periodic solutions of (1) are

$$
\begin{aligned}
V_{31}(x,t)&\\
&=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)\\
&+\frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x+\lambda t+\xi_0)\right)\right.\\
&\left.-\beta\right)\\
&+\frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x+\lambda t+\xi_0)\right)\right.\\
&\left.-\beta\right)^2,\\
V_{32}(x,t)&\\
&=\frac{1}{30}\left(\frac{40\sqrt{30}\sqrt{hc}\gamma\alpha h+5\sqrt{30}\sqrt{hc}\beta^2h+\sqrt{30}\sqrt{hc}g+10bh}{hc}\right)\\
&+\frac{\beta\sqrt{30}\sqrt{hc}}{c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x-\lambda t+\xi_0)\right)\right.\\
&\left.-\beta\right)\\
&+\frac{\sqrt{30}\sqrt{hc}}{2c}\left(\sqrt{\beta^2-4\alpha\gamma}\tanh\left(\frac{1}{2}\sqrt{\beta^2-4\alpha\gamma}\,(x-\lambda t+\xi_0)\right)\right.\\
&\left.-\beta\right)^2,
\end{aligned}
\tag{45}
$$

where

$$
\lambda=\frac{1}{30}\frac{\sqrt{-30ch\left(720\alpha^2\gamma^2ch^2-360\alpha\gamma\beta^2ch^2+45\beta^4ch^2-30ach-10b^2h+3cg^2\right)}}{ch}. \tag{46}
$$

4. Summary and Discussion

In conclusion, the improved generalized tanh-coth method is proposed to obtain more general exact solutions of the sixth-order solitary wave equations. It can be observed that the method used is a powerful and more general tool for finding the exact solutions. By using the method, we have obtained new exact solutions in terms of the hyperbolic functions, the trigonometric functions, the exponential functions, and the rational functions. The results reveal that the improved generalized tanh-coth method has significant effects on the wave behavior and can also be applied to other NLEEs in mathematical physics.

The solutions of generalized Riccati equation (7): there are more than seven cases. We can extend the improved generalized tanh-coth method by considering expanding solutions of generalized Riccati equation in this method.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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