Hindawi
Journal of Mathematics
Volume 2018, Article ID 2391697, 11 pages
https://doi.org/10.1155/2018/2391697

Research Article

Around Chaotic Disturbance and Irregularity for Higher Order Traveling Waves

Emile F. Doungmo Goufo [1] and Ignace Tchangou Toudjeu [2]

[1] *Department of Mathematical Sciences, University of South Africa, Florida 0003, South Africa*
[2] *Department of Electrical Engineering, Tshwane University of Technology, Pretoria 0183, South Africa*

Correspondence should be addressed to Emile F. Doungmo Goufo; franckemile2006@yahoo.ca

Received 27 February 2018; Accepted 29 March 2018; Published 3 June 2018

Academic Editor: S. K. Q. Al-Omari

Many unknown features in the theory of wave motion are still captivating the global scientific community. In this paper, we consider a model of seventh order Korteweg–de Vries (KdV) equation with one perturbation level, expressed with the recently introduced derivative with nonsingular kernel, Caputo-Fabrizio derivative (CFFD). Existence and uniqueness of the solution to the model are established and proven to be continuous. The model is solved numerically, to exhibit the shape of related solitary waves and perform some graphical simulations. As expected, the solitary wave solution to the model without higher order perturbation term is shown via its related homoclinic orbit to lie on a curved surface. Unlike models with conventional derivative ($\gamma = 1$) where regular behaviors are noticed, the wave motions of models with the nonsingular kernel derivative are characterized by irregular behaviors in the pure factional cases ($\gamma < 1$). Hence, the regularity of a soliton can be perturbed by this nonsingular kernel derivative, which, combined with the perturbation parameter ζ of the seventh order KdV equation, simply causes more accentuated irregularities (close to chaos) due to small irregular deviations.

1. Introduction

In the last decade, a great number of researchers have paid a particular attention to the study of solitary wave equations that undergo the influence of external perturbations. Most of physical dynamics related to the movement of liquids and waves are governed by Korteweg–de Vries (KdV) equation and its variants. Hence, for this particular equation, Cao et al. [1] as well as Grimshaw and Tian [2] have recently shown that a force combined together with dissipation can provoke a chaotic behavior usually detectable by other analysis like phase plane analysis or nonnegative Liapunov exponents. KdV equation and its variants are of infinite dimension and their use to address traveling waves or chaotic dynamics of low dimension is facilitated by numerical approximations, which have proven that correlation dimension established via Grassberger-Procaccia technique and information dimension obtained from formula of Kaplan-Yorke are both between two and three for steady traveling waves [3].

However, many authors (like, e.g., [4–7]) preferred to use numerical approach to analyze the KdV equation or its variants, especially the one with many levels of perturbations. Hence, it was shown in [4] that there is no periodic waves for the autonomous Korteweg–de Vries–Burgers equation of dimension two. We follow, in this paper, the same trend of numerical approach by making use of the recently developed fractional derivative with nonsingular kernel [8–13], to express a seventh order Korteweg–de Vries (KdV) equation with one perturbation level. This is the first instance where such a model is extended to the scope of fractional differentiation and fully investigated. We prove existence and uniqueness of a continuous solution. Before that, we shall give in the following section a brief review of the recent developments done in the theory of fractional differentiation.

2. Around the Nonsingular Kernel Differentiation with Fractional Order

The concept of fractional order derivative is seen by many authors as a great endeavor to ameliorate nonlinear mathematical models, widen their analysis, and expand their interpretation. Today's literature of the concept has been enriched with many innovative definitions more related to the complexity and diversity of natural phenomena surrounding us. There are fractional order derivatives of local type and also of nonlocal type [8, 14–17]. The Caputo's definition

$$D_t^{\gamma}(u(t)) = \frac{1}{\Gamma(n-\gamma)}\int_0^t (t-\tau)^{n-\gamma-1}\left(\frac{d}{d\tau}\right)^n u(\tau)\,d\tau, \quad (1)$$

$n-1<\gamma\leq n$. remains the most commonly used in the applied science, followed by Riemann-Liouville's version given by

$$D_t^{\gamma}(u(t)) = \frac{1}{\Gamma(n-\gamma)}\left(\frac{d}{dt}\right)^n\int_0^t (t-\tau)^{n-\gamma-1} u(\tau)\,d\tau, \quad (2)$$

$n-1<\gamma\leq n$.

Recent observations by Caputo and Fabrizio [8] stated that the two definitions above better describe physical processes, related to fatigue, damage, and electromagnetic hysteresis, but do not genuinely depict some behavior taking place in multiscale systems and in materials with massive heterogeneities. Hence, the same authors introduced the following new version of fractional order derivative with no singular kernel:

Definition 1 (Caputo-Fabrizio fractional order derivative (CFFD)). Let u be a function in $H^1(a;b)$; $b>a$; $\gamma\in[0;1]$; then, the Caputo-Fabrizio fractional order derivative (CFFD) is defined as

$$^{\mathrm{cf}}D_t^{\gamma}u(t) = \frac{M(\gamma)}{(1-\gamma)}\int_0^t \dot{u}(\tau)\exp\left(-\frac{\gamma(t-\tau)}{1-\gamma}\right)d\tau, \quad (3)$$

where $M(\gamma)$ is a normalization function such that $M(0)=M(1)=1$.

Remark 2. Caputo and Fabrizio [8] substituted the kernel $1/(t-\tau)^{\gamma}$ appearing in (1) when $n=1$ by the function $\exp(-\gamma(t-\tau)/(1-\gamma))$ and $1/\Gamma(1-\gamma)$ by $M(\gamma)/(1-\gamma)$. This immediately removes the singularity at $t=\tau$ that exists in the previous Caputo's expression.

For the function that does not belong to $H^1(a;b)$, the CFFD is given by

$$\begin{aligned}&^{\mathrm{cf}}D_t^{\gamma}u(t)\\ &\quad= \frac{\gamma M(\gamma)}{(1-\gamma)}\int_0^t (u(t)-u(\tau))\exp\left(-\frac{\gamma(t-\tau)}{1-\gamma}\right)d\tau.\end{aligned} \quad (4)$$

Losada and Nieto [13] upgraded this definition of CFFD by proposing the following:

$$\begin{aligned}&^{\mathrm{cf}}D_t^{\gamma}u(t)\\ &\quad= \frac{(2-\gamma)M(\gamma)}{2(1-\gamma)}\int_0^t \dot{u}(\tau)\exp\left(-\frac{\gamma(t-\tau)}{1-\gamma}\right)d\tau.\end{aligned} \quad (5)$$

Unlike the classical version of Caputo fractional order derivative [14, 18], the CFFD with no singular kernel appears to be easier to handle. Furthermore, the CFFD verifies the following equalities:

$$\lim_{\gamma\to1}{}^{\mathrm{cf}}D_t^{\gamma}u(t) = \dot{u}(t), \quad (6)$$

$$\lim_{\gamma\to0}{}^{\mathrm{cf}}D_t^{\gamma}u(t) = u(t)-u(a), \quad (7)$$

with u any suitable function and a the starting point of the integrodifferentiation. The fractional integral related to the CFFD and proposed by Losada and Nieto reads as

$$\begin{aligned}{}^{\mathrm{cf}}I_t^{\gamma}u(t) &= \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}u(t)\\ &\quad+\frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t u(\tau)\,d\tau,\end{aligned} \quad (8)$$

$\gamma\in[0,1]$ $t\geq0$. This antiderivative represents sort of average between the function u and its integral of order one. The Laplace transform of the CFFD reads

$$\mathscr{L}(D_t^{\gamma}u(t),s) = \frac{s\tilde{u}(x,s)-u_0(x)}{s+\gamma(1-s)}, \quad (9)$$

where $\tilde{u}(x,s)$ is the Laplace transform $\mathscr{L}(u(x,t),s)$ of $u(x,t)$.

Definition 3 (New Riemann-Liouville fractional order derivative (NRLFD)). As a response to the CFFD and being aware of the conflicting situations that exist between the classical Riemann-Liouville and Caputo derivatives, the classical Riemann-Liouville definition was modified [9, 10] in order to propose another definition known as the new Riemann-Liouville fractional derivative (NRLFD) without singular kernel and expressed for $\gamma\in[0,1]$ as

$$\begin{aligned}&{}_a\mathfrak{D}_t^{\gamma}u(t)\\ &\quad= \frac{M(\gamma)}{1-\gamma}\frac{d}{dt}\int_a^t u(\tau)\exp\left(-\frac{\gamma}{1-\gamma}(t-\tau)\right)d\tau.\end{aligned} \quad (10)$$

Again, the NRLFD is without any singularity at $t=\tau$ compared to the classical Riemann-Liouville fractional order derivative and also verifies

$$\begin{aligned}\lim_{\gamma\to1}{}_a\mathfrak{D}_t^{\gamma}u(t) &= \dot{u}(t),\\ \lim_{\gamma\to0}{}_a\mathfrak{D}_t^{\gamma}u(t) &= u(t).\end{aligned} \quad (11)$$

Compared to (7), we note here the exact correspondence with u at $\gamma\to0$. The Laplace transform of the NRLFD reads as [9, 10]

$$\mathscr{L}\left(\mathfrak{D}_t^{-\gamma}u(t),s\right) = \frac{sM(\gamma)}{s+\gamma(1-s)}\mathscr{L}(u(t),s). \quad (12)$$

Other versions and innovative definitions of fractional derivatives have since been introduced. This paper however uses the CFFD, so for more details about those recent definitions, please feel free to consult the articles and works mentioned above and also the references mentioned therein.

3. Existence and Uniqueness

In this section we prove the existence and uniqueness results for the seventh order Korteweg–de Vries equation (KdV) with one perturbation level, expressed with the CFFD and given by

$$ {}^{\mathrm{cf}}D_t^{\gamma} u(x,t) = -6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}, \quad (13) $$

assumed to satisfy the initial condition

$$ u(x,0) = f(x), \quad (14) $$

where ζ is the perturbation parameter and ${}^{\mathrm{cf}}D_t^{\gamma}$ is the Caputo-Fabrizio fractional order derivative (CFFD) given in (5). Existence results for the model (13)-(14) here above are established by making use of the expression of the antiderivative (8). This yields

$$ \begin{aligned} & u(x,t) - u(x,0) \\ &\quad = {}^{\mathrm{cf}}I_t^{\gamma}\left(-6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}\right). \end{aligned} \quad (15) $$

This can be rewritten as

$$ \begin{aligned} u(x,t) - u(x,0) &= \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(-6uu_x - u_{xxx}\right. \\ &\quad \left. + u_{xxxxx} - \zeta u_{xxxxxxx}\right) + \frac{2\gamma}{(2-\gamma)M(\gamma)} \\ &\quad \cdot \int_0^t \left(-6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}\right) d\tau. \end{aligned} \quad (16) $$

Set now

$$ \mathscr{Y}(x,t,u,\zeta) = -6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}. \quad (17) $$

The next step is to look for a real constant $K \geq 0$ such that

$$ \left\|\mathscr{Y}(x,t,u,\zeta) - \mathscr{Y}(x,t,v,\zeta)\right\| \leq K\|u - v\|. \quad (18) $$

In fact

$$ \begin{aligned} & \mathscr{Y}(x,t,u,\zeta) - \mathscr{Y}(x,t,v,\zeta) \\ &\quad = \left(-6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}\right) \\ &\qquad - \left(-6vv_x - v_{xxx} + v_{xxxxx} - \zeta v_{xxxxxxx}\right) \\ &\quad = 6\left(vv_x - uu_x\right) + \left(v_{xxx} - u_{xxx}\right) \\ &\qquad + \left(u_{xxxxx} - v_{xxxxx}\right) + \zeta\left(v_{xxxxxxx} - u_{xxxxxxx}\right). \end{aligned} \quad (19) $$

Well known properties for the norms give

$$ \begin{aligned} & \left\|\mathscr{Y}(x,t,u,\zeta) - \mathscr{Y}(x,t,v,\zeta)\right\| = \left\|6\left(vv_x - uu_x\right)\right. \\ &\quad + \left(v_{xxx} - u_{xxx}\right) + \left(u_{xxxxx} - v_{xxxxx}\right) \\ &\quad \left. + \zeta\left(v_{xxxxxxx} - u_{xxxxxxx}\right)\right\| \leq 6\left\|vv_x - uu_x\right\| \\ &\quad + \left\|v_{xxx} - u_{xxx}\right\| + \left\|u_{xxxxx} - v_{xxxxx}\right\| + \zeta\left\|v_{xxxxxxx}\right. \\ &\quad \left. - u_{xxxxxxx}\right\| \leq 6\left\|\partial_x\left(v^2 - u^2\right)\right\| + \left\|\partial_{xxx}(v - u)\right\| \\ &\quad + \left\|\partial_{xxxxx}(u - v)\right\| + \zeta\left\|\partial_{xxxxxxx}(v - u)\right\|. \end{aligned} \quad (20) $$

Keeping in mind that u and v are bounded functions, then there exists real numbers $r_1 > 0$ and $r_2 > 0$ such that

$$ \begin{aligned} \|u\| &\leq r_1, \\ \|v\| &\leq r_2. \end{aligned} \quad (21) $$

Set $r = \max(r_1, r_2)$; hence,

$$ \begin{aligned} \|u\| &\leq r, \\ \|v\| &\leq r. \end{aligned} \quad (22) $$

Therefore, the Lipschitz condition holds for the partial derivatives $\partial_x u$ and $\partial_x v$ and there exists a real constant $K_1 \geq 0$ such that

$$ \begin{aligned} & \left\|\mathscr{Y}(x,t,u,\zeta) - \mathscr{Y}(x,t,v,\zeta)\right\| \\ &\quad \leq 6K_1\left\|v^2 - u^2\right\| + K_1^3\|v - u\| + K_1^5\|u - v\| \\ &\qquad + \zeta K_1^7\|v - u\| \\ &\quad \leq 6K_1\|u + v\| \cdot \|u - v\| + K_1^3\|v - u\| + K_1^5\|u - v\| \\ &\qquad + \zeta K_1^7\|v - u\| \\ &\quad \leq \left[12rK_1 + K_1^3 + K_1^5 + \zeta K_1^7\right]\|u - v\|, \end{aligned} \quad (23) $$

where the bounded condition (14) has been exploited, whence

$$ \left\|\mathscr{Y}(x,t,u,\zeta) - \mathscr{Y}(x,t,v,\zeta)\right\| \leq K\|u - v\| \quad (24) $$

with

$$ K = 12rK_1 + K_1^3 + K_1^5 + \zeta K_1^7. \quad (25) $$

This proves that $\mathscr{Y}$ satisfies the Lipschitz condition and then, it allows us to state the following proposition.

Proposition 4. *If the condition* $1 > ((2(1-\gamma)/(2-\gamma)M(\gamma))K + 2tK\gamma/(2-\gamma)M(\gamma))$ *holds, then, there exists a unique and continuous solution to the seventh order Korteweg–de Vries equation with one perturbation level expressed with the CFFD given in (5):*

$$ \begin{aligned} {}^{cf}D_t^{\gamma} u(x,t) &= -6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}, \\ u(x,0) &= f(x). \end{aligned} \quad (26) $$

Proof. Let us go back to the model (16) rewritten as

$$ \begin{aligned} & u(x,t) - u(x,0) \\ &\quad = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(\mathscr{Y}(x,t,u,\zeta)\right) \\ &\qquad + \frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t \left(\mathscr{Y}(x,\tau,u,\zeta)\right) d\tau \end{aligned} \quad (27) $$

which yields the recurrence formulation given as follows:

$$u_0(x,t) = u(x,0),$$
$$u_n(x,t) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}(\mathscr{Y}(x,t,u_{n-1},\zeta)) + \frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t (\mathscr{Y}(x,\tau,u_{n-1},\zeta))\,d\tau. \tag{28}$$

Let

$$\underline{u}(x,t) = \lim_{n\to\infty} u_n(x,t), \tag{29}$$

and it can be shown that $\underline{u}(x,t) = u(x,t)$ is a continuous solution. Indeed, if we take

$$B_n(x,t) = u_n(x,t) - u_{n-1}(x,t) \tag{30}$$

then, it is straightforward to see that

$$u_n(x,t) = \sum_{p=0}^{n} B_p(x,t). \tag{31}$$

More explicitly, we have

$$B_n(x,t) = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}[\mathscr{Y}(x,t,u_{n-1},\zeta) - \mathscr{Y}(x,t,u_{n-2},\zeta)] + \frac{2\gamma}{(2-\gamma)M(\gamma)} \cdot \int_0^t (\mathscr{Y}(x,\tau,u_{n-1},\zeta) - \mathscr{Y}(x,\tau,u_{n-2},\zeta))\,d\tau. \tag{32}$$

Passing this equation to the norm yields

$$\begin{aligned} \|B_n(x,t)\| &= \|u_n(x,t) - u_{n-1}(x,t)\| \\ &\le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\|\mathscr{Y}(x,t,u_{n-1},\zeta) - \mathscr{Y}(x,t,u_{n-2},\zeta)\| + \frac{2\gamma}{(2-\gamma)M(\gamma)}\left\|\int_0^t [\mathscr{Y}(x,\tau,u_{n-1},\zeta) - \mathscr{Y}(x,\tau,u_{n-2},\zeta)]\,d\tau\right\| \\ &\le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\|\mathscr{Y}(x,t,u_{n-1},\zeta) - \mathscr{Y}(x,t,u_{n-2},\zeta)\| + \frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t \|\mathscr{Y}(x,\tau,u_{n-1},\zeta) - \mathscr{Y}(x,\tau,u_{n-2},\zeta)\|\,d\tau. \end{aligned} \tag{33}$$

Applying the Lipschitz condition to $\mathscr{Y}$ gives

$$\|B_n(x,t)\| \le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K\|u_{n-1} - u_{n-2}\| + \frac{2K\gamma}{(2-\gamma)M(\gamma)}\int_0^t \|u_{n-1} - u_{n-2}\|\,d\tau \tag{34}$$

which can be rewritten as

$$\|B_n(x,t)\| \le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K\|B_{n-1}\| + \frac{2K\gamma}{(2-\gamma)M(\gamma)}\int_0^t \|B_{n-1}\|\,d\tau. \tag{35}$$

After integrating, we make use of well known properties of the recursive technique from (35) to have

$$\|B_n(x,t)\| \le \left[\left(\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K\right)^n + \left(\frac{2tK\gamma}{(2-\gamma)M(\gamma)}\right)^n\right] \cdot f(x), \tag{36}$$

with $f(x) = u(x,0)$, which explicitly shows the existence of the solution and that it is continuous.

The step forward is to prove that the solution of the model (26) is given by the function

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{37}$$

For that, let

$$Q_n(x,t) = \underline{u}(x,t) - u_n(x,t) \quad \text{with } n \in \mathbb{N}. \tag{38}$$

Making use of (29), we should have $\lim_{n\to\infty} Q_n = 0$. In other terms, the gap that exists between $\underline{u}(x,t)$ and $u_n(x,t)$ should vanish as $n \to \infty$. Consider

$$\underline{u} - u_{n-1} = \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}[\mathscr{Y}(x,t,u,\zeta) - \mathscr{Y}(x,t,u_n,\zeta)] + \frac{2\gamma}{(2-\gamma)M(\gamma)} \cdot \int_0^t (\mathscr{Y}(x,\tau,u,\zeta) - \mathscr{Y}(x,\tau,u_n,\zeta))\,d\tau, \tag{39}$$

giving

$$\begin{aligned} \|\underline{u}(x,t) - u_{n+1}\| &\le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\|\mathscr{Y}(x,t,u,\zeta) - \mathscr{Y}(x,t,u_n,\zeta)\| + \frac{2\gamma}{(2-\gamma)M(\gamma)} \cdot \int_0^t \|(\mathscr{Y}(x,\tau,u,\zeta) - \mathscr{Y}(x,\tau,u_n,\zeta))\|\,d\tau \\ &\le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K\|u - u_n\| + \frac{2K\gamma}{(2-\gamma)M(\gamma)} \cdot \int_0^t \|u - u_n\|\,d\tau \le \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K\|Q_n\| + \frac{2K\gamma}{(2-\gamma)M(\gamma)}\int_0^t \|Q_n\|\,d\tau. \end{aligned} \tag{40}$$

Hence, $\lim_{n\to\infty} Q_n = 0$ and from the right hand side, we have

$$\lim_{n\to\infty} u_n = \underline{u}. \tag{41}$$

Just take $u(x,t) = \underline{u}(x,t)$ as the solution of (26) that is continuous. Moreover, the Lipschitz condition for $\mathscr{Y}$ yields

$$\begin{aligned} u(x,t) &- \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\mathscr{Y}(x,t,u,\zeta) - \frac{2\gamma}{(2-\gamma)M(\gamma)} \\ &\cdot \int_0^t \mathscr{Y}(x,t,u,\zeta)\,d\tau = B_n(x,t) \\ &+ \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\left(\mathscr{Y}(x,\tau,u_{n-1},\zeta) - \mathscr{Y}(x,t,u,\zeta)\right) \\ &+ \frac{2\gamma}{(2-\gamma)M(\gamma)} \\ &\cdot \int_0^t \left(\mathscr{Y}(x,\tau,u_{n-1},\zeta) - \mathscr{Y}(x,t,u,\zeta)\right)d\tau. \end{aligned} \tag{42}$$

This yields

$$\begin{aligned} &\left\| u(x,t) - \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\mathscr{Y}(x,t,u,\zeta) \right. \\ &\left. - \frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t \mathscr{Y}(x,t,u,\zeta)\,d\tau \right\| = \left\|B_n(x,t)\right\| \\ &+ \left(\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)} + \frac{2\gamma}{(2-\gamma)M(\gamma)}\right)\left\|B_{n-1}(x,t)\right\| \end{aligned} \tag{43}$$

considering the initial condition and taking the limit as $n \to 0$ gives

$$\begin{aligned} u(x,t) = f(x) &+ \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}\mathscr{Y}(x,t,u,\zeta) \\ &+ \frac{2\gamma}{(2-\gamma)M(\gamma)}\int_0^t \mathscr{Y}(x,t,u,\zeta)\,d\tau. \end{aligned} \tag{44}$$

Uniqueness. To prove that the solution is unique, we take two different functions u and v that satisfy the model (26); then,

$$\begin{aligned} \|u-v\| \le &\frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K\|u-v\| \\ &+ \frac{2tK\gamma}{(2-\gamma)M(\gamma)}\|u-v\|, \end{aligned} \tag{45}$$

equivalently

$$\|u-v\|\left(1 - \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K - \frac{2tK\gamma}{(2-\gamma)M(\gamma)}\right) \le 0. \tag{46}$$

This yields $u = v$ if

$$1 - \frac{2(1-\gamma)}{(2-\gamma)M(\gamma)}K - \frac{2tK\gamma}{(2-\gamma)M(\gamma)} > 0, \tag{47}$$

where we have used the Lipschitz condition for $\mathscr{Y}$ and this ends the proof. □

4. Shape of Solitary Waves via Numerical Approximations

4.1. Shape of Solitary Waves for the Lower-Order Approximation. In this section, we are interested in waves traveling to a specific direction, and then, we consider solutions to the seventh order KdV equation

$${}^{\mathrm{cf}}D_t^{\gamma}u(x,t) = -6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}. \tag{48}$$

We start by considering the conventional case where $\gamma = 1$ to have, using ${}^{\mathrm{cf}}D_t^{1} f(t) = f'(t)$, the following model:

$$\frac{\partial u(x,t)}{\partial t} = -6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}. \tag{49}$$

We investigate the traveling waves taking the form $u(x,t) = u(x+ct) = u(\eta)$, where c is the speed of the wave. We assume that u does not depend on x independently from t but rather depends on the combined variable $\eta = x+ct$. We also assume that the wave dies at infinity, meaning

$$\begin{aligned} \lim_{\eta\to\pm\infty} u(\eta) &= \lim_{\eta\to\pm\infty} u_\eta(\eta) = \cdots = \lim_{\eta\to\pm\infty} u_{\eta\eta\eta\eta\eta\eta\eta}(\eta) \\ &= 0. \end{aligned} \tag{50}$$

Now, it is possible to transform the seventh order KdV equation (49) into an ordinary differential equation (ODE) by making use of the basic properties of differentiation. Then, $u_x = u_\eta \cdot \eta_x = u_\eta$ and $u_t = u_\eta \cdot \eta_t = cu_\eta$ which yield the following ODE:

$$-cu_\eta - 6uu_\eta - u_{\eta\eta\eta} + u_{\eta\eta\eta\eta\eta} - \zeta u_{\eta\eta\eta\eta\eta\eta\eta} = 0. \tag{51}$$

The η-integration of this equation once gives

$$-cu - 3u^2 - u_{\eta\eta} + u_{\eta\eta\eta\eta} - \zeta u_{\eta\eta\eta\eta\eta\eta} = 0, \tag{52}$$

where we have ignored the constant of η-integration that is null due to boundary conditions (50). If the higher order perturbation term $\zeta u_{\eta\eta\eta\eta\eta\eta}$ is ignored, then we have

$$-cu - 3u^2 - u_{\eta\eta} + u_{\eta\eta\eta\eta} = 0. \tag{53}$$

Then we can solve numerically this equation by transforming it into a system of four ODEs of order one as follows:

$$\begin{aligned} u_\eta &= w, \\ u_{\eta\eta} &= w_\eta = y, \\ u_{\eta\eta\eta} &= y_\eta = z, \\ u_{\eta\eta\eta\eta} &= z_\eta = cu + 3u^2 + y. \end{aligned} \tag{54}$$

Numerical simulations are done in the phase-space $(u, u_\eta, u_{\eta\eta})$ as shown in Figure 1. Let us now come back to the full model (48) with the nonsingular kernel derivative CFFD (given in (5)) and with no higher order perturbation parameter ζ, given as

$${}^{\mathrm{cf}}D_t^{\gamma}u(x,t) = -6uu_x - u_{xxx} + u_{xxxxx}. \tag{55}$$

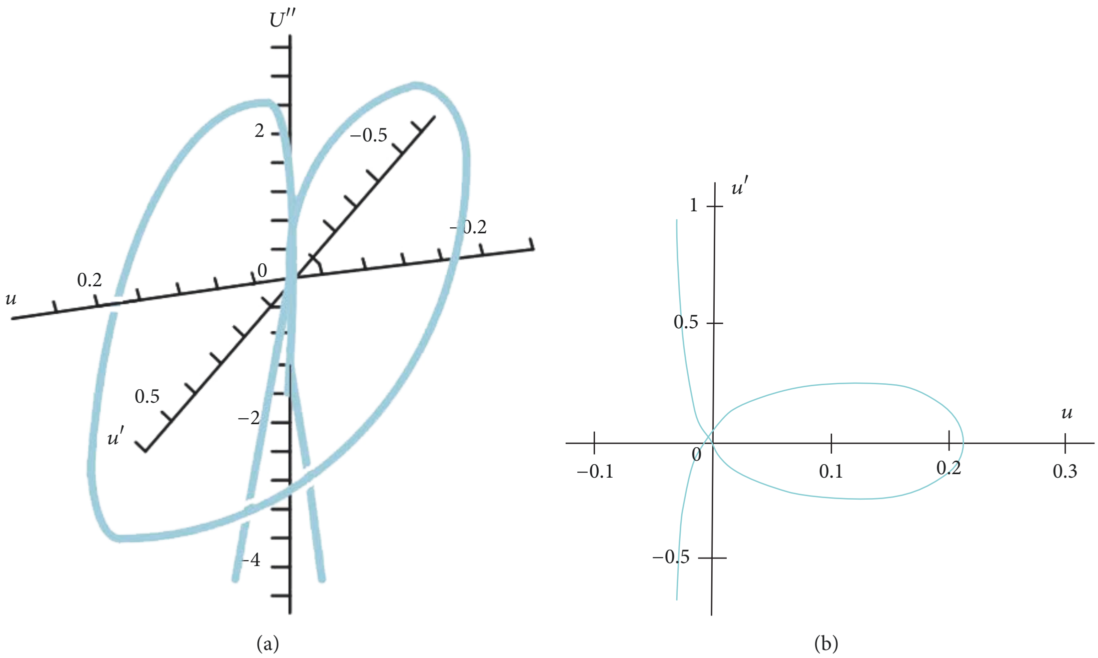

Figure 1: Numerical plot in the phase-space $(u, u_\eta, u_{\eta\eta})$ for the model KdV model (55) or model (48) with no higher order perturbation term, for $\gamma = 1$ (conventional case). As expected in (a), the soliton solution is shown via its related homoclinic orbit to lie on a curved surface. In (b), the projection on the plane (u, u_η).

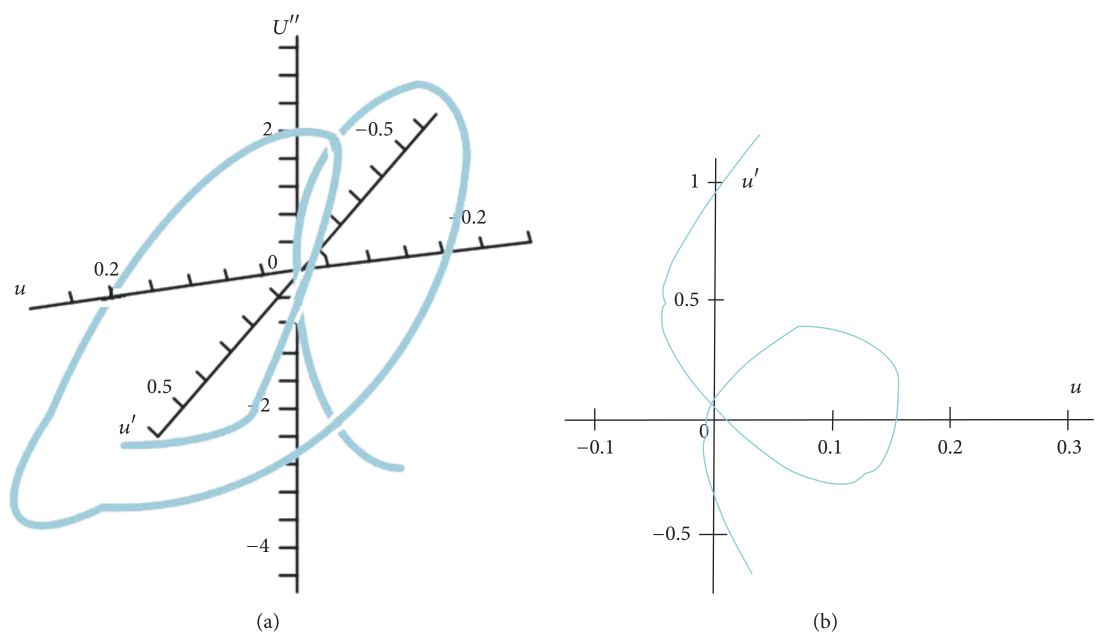

Figure 2: Numerical plot in the phase-space $(u, u_\eta, u_{\eta\eta})$ for the model KdV model (55) with no higher order perturbation term, for $\gamma = 0.75$. As expected in (a), even in the pure fractional case, the soliton solution is shown via its related homoclinic orbit to lie on a curved surface, with some irregular movements compared to Figure 1. In (b), the projection on the plane (u, u_η).

This fractional equation is solved numerically by making use of the Adams-Bashforth-Moulton type method also known as predictor-corrector (PECE) technique and is fully detailed in the article by Diethelm et al. [19]. Figures 1 and 2 represent the numerical simulations of solutions to the fractional model (55) with different values of the derivative order γ. They clearly point out relative irregularities when the derivative order of the CFFD is $\gamma < 1$.

4.2. Shape of Solitary Waves for the Higher Order Approximation. Now, the perturbation term $\zeta u_{\eta\eta\eta\eta\eta\eta\eta}$ is considered to have (48):

$$ {}^{cf}D_t^{\gamma} u(x,t) = -6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx} \quad (56)$$

subject to localized initial condition

$$u(x,0) = \sqrt{\frac{12d}{k}} \operatorname{sech}^2\left(\frac{x}{k}\right). \quad (57)$$

Although we proved existence and uniqueness for this nonlinear problem, it still faces the challenge of providing an explicit expression of exact solution or approximated solution. It is almost impossible to use some analytical methods, like for instance, integral transform methods, the Green function technique, or the technique separation of variables. Hence, a semianalytical method like the Laplace iterative method can be a valuable tool to provide a special solution to the model (56)-(57) as shown in the following lines.

We start by applying the Laplace transform (9) on both sides of (56) to obtain

$$\begin{aligned} &\frac{s\tilde{u}(x,s) - u_0(x)}{s + \gamma(1-s)} \\ &\quad = \mathscr{L}\left(-6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}, s\right). \end{aligned} \quad (58)$$

Let $\theta(\gamma,s) = s + \gamma(1-s)$; then,

$$\begin{aligned} \tilde{u}(x,s) &= \frac{u(x,0)}{s} + \frac{\theta(\gamma,s)}{s} \\ &\quad \cdot \mathscr{L}\left(-6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}, s\right). \end{aligned} \quad (59)$$

Application of inverse Laplace transform $\mathscr{L}^{-1}$ on both sides yields

$$\begin{aligned} u(x,t) &= \mathscr{L}^{-1}\left(\frac{u(x,0),t}{s}\right) + \mathscr{L}^{-1}\left(\frac{\theta(\gamma,s)}{s}\right. \\ &\quad \left.\cdot \mathscr{L}\left(-6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}, s\right), t\right). \end{aligned} \quad (60)$$

This leads to the following recursive system

$$\begin{aligned} u_{n+1}(x,t) &= u_n(x,t) + \mathscr{L}^{-1}\left(\frac{\theta(\gamma,s)}{s}\mathscr{L}\left(-6u_n(u_n)_x\right.\right. \\ &\quad \left.\left. - (u_n)_{xxx} + (u_n)_{xxxxx} - \zeta(u_n)_{xxxxxxx}, s\right), t\right), \\ u_0(x,t) &= u(x,0), \end{aligned} \quad (61)$$

then making the solution to be $u(x,t) = \lim_{n\to\infty} u_n(x,t)$. Numerical approximations are performed according to the following steps:

(i) $u_0(x,t) = u(x,0)$ is considered as initial input:

Choose j as the number of terms to compute.

Name u_{app} to be the approximate solution.

Set $u_{\text{app}} = u(x,0) = \sqrt{12d/k}\operatorname{sech}^2(x/k)$ and $u_{\text{app}} = u_{\text{app}}$.

(ii) For the other terms use

$u_{n+1}(x,t) = u_n(x,t) + \mathscr{L}^{-1}((\theta(\gamma,s)/s)\mathscr{L}(-6u_n(u_n)_x - (u_n)_{xxx} + (u_n)_{xxxxx} - \zeta(u_n)_{xxxxxxx}, s), t)$.

Compute $X_n(x,t) = X_{n-1}(x,t) + u_{\text{app}}$.

to finally get $u_{\text{app}}(x,t) = X_n(x,t) + u_{\text{app}}$

where

$$\begin{aligned} X_n(x,t) &= \mathscr{L}^{-1}\left(\frac{\theta(\gamma,s)}{s}\mathscr{L}\left(-6u_n(u_n)_x - (u_n)_{xxx}\right.\right. \\ &\quad \left.\left. + (u_n)_{xxxxx} - \zeta(u_n)_{xxxxxxx}, s\right), t\right). \end{aligned} \quad (62)$$

Note we can use $\theta(\gamma,s)$, to obtain following results and some numerical approximations with related absolute errors are summarized in Tables 1 and 2 (which present some numerical approximations with related absolute errors as given by the sequences (63) and (65), respectively, representing the pure fractional case ($\gamma = 0.8$) and the standard case ($\gamma = 1$)).

$$\begin{aligned} u(x,0) &= \sqrt{\frac{12d}{k}}\operatorname{sech}^2\left(\frac{x}{k}\right), \\ u_1(x,t) &= u(x,0) + \frac{\left(1.900\times10^{-3}\right)t\sqrt{d}}{k^{3/2}}\left\{1.233\frac{\sqrt{d}}{k^{1/2}}\right. \\ &\quad - \frac{3.467}{k^2} - \frac{293}{k^4} - \frac{98933\zeta}{k^6} \\ &\quad + \left(-\frac{2.257}{k^2} - \frac{135.200}{k^4} + \frac{99631\zeta}{k^6}\right)\cosh\left(\frac{2x}{k}\right) \\ &\quad + \left(-\frac{0.102}{k^2} + \frac{173.700}{k^4} + \frac{14681\zeta}{k^6}\right)\cosh\left(\frac{4x}{k}\right) \\ &\quad \left. + \left(\frac{0.572}{k^2} - \frac{13.730}{k^4} + \frac{496\zeta}{k^6}\right)\cosh\left(\frac{6x}{k}\right)\right\} \\ &\quad \cdot \operatorname{sech}^{12}\left(\frac{x}{k}\right)\tanh\left(\frac{x}{k}\right) \\ &\quad + \frac{(1-\gamma)\Gamma(\gamma)t^{\gamma-1}\sqrt{d}}{k}\left\{0.225\frac{\sqrt{d}}{k} - \frac{3.830}{k^{1/4}}\right. \\ &\quad - \frac{352.303\zeta}{k^{1/6}} \\ &\quad + \left(-\frac{0.057}{k^{1/2}} - \frac{3.520}{k^{1/4}} + \frac{685\zeta}{k^{1/6}}\right)\cosh\left(\frac{2x}{k}\right) \\ &\quad + \left(-\frac{0.030}{k^{1/2}} + \frac{3.820}{k^{1/4}} + \frac{641\zeta}{k^{1/6}}\right)\cosh\left(\frac{4x}{k}\right) \end{aligned}$$

Table 1: Some values for numerical and exact solutions to the seventh order KdV equation expressed with the CFFD given in (5) at $k = 2.5$, $d = 4.0$, and $\gamma = 0.80$.

		For $\gamma = 0.80$		
Time (t)	Spatial (x)	Exact value	Numerical value	Error made
0.5	−15.0	0.000098126	0.000224130	1.26×10^{-4}
	−7.0	0.056620000	0.056514000	1.06×10^{-4}
	0.0	0.100372300	0.100470000	1.00×10^{-4}
	+7.0	0.070555000	0.069355000	1.20×10^{-3}
	+15.0	0.000118180	0.000120180	2.00×10^{-6}
1.0	−15.0	0.000091592	0.000284080	1.20×10^{-4}
	−7.0	0.054779000	0.054779000	00
	0.0	0.100390000	0.100380000	1.00×10^{-5}
	+7.0	0.075546000	0.071755000	1.20×10^{-4}
	+15.0	0.000126620	0.000073380	2.00×10^{-4}
2.0	−15.0	0.000081235	0.000181240	1.00×10^{-4}
	−7.0	0.048620000	0.069620000	2.10×10^{-2}
	0.0	0.100410000	0.100510000	1.01×10^{-4}
	+7.0	0.085083000	0.084954000	1.29×10^{-4}
	+15.0	0.000142760	0.000242760	1.00×10^{-4}
4.0	−15.0	0.000065920	0.000132590	1.26×10^{-3}
	−7.0	0.039495000	0.039495000	00
	0.0	0.100430000	0.090430000	1.00×10^{-2}
	+7.0	0.104610000	0.102610000	2.00×10^{-3}
	+15.0	0.000175920	0.000175920	00
8.0	−15.0	0.000045819	0.000224130	1.26×10^{-3}
	−7.0	0.027489000	0.058768000	1.6×10^{-3}
	0.0	0.090400000	0.100400000	1×10^{-2}
	+7.0	0.149720000	0.071755000	1.2×10^{-3}
	+15.0	0.000253100	0.000120180	1.01×10^{-5}
10.0	−15.0	0.000038771	0.000224130	1.26×10^{-4}
	−7.0	0.023272000	0.058768000	1.06×10^{-3}
	0.0	0.090402000	0.091402000	1×10^{-3}
	+7.0	0.176380000	0.176380000	00
	+15.0	0.000299110	0.00012018	2.1×10^{-5}

$$
\begin{aligned}
&+ \left(\frac{0.072}{k^{1/2}} - \frac{0.770}{k^{1/4}} + \frac{9.650\zeta}{k^{1/6}} \right) \cosh\left(\frac{6x}{k}\right) \Bigg\} \\
&\cdot \frac{\operatorname{sech}^6 (x/k) \tanh (x/k)}{2^\gamma - \gamma}, \\
&u_2 (x, t) = u_1 (x, 0) + \cdots, \\
&\vdots
\end{aligned}
\tag{63}
$$

Note that, for $\gamma = 1$, we recover for the conventional model (49)

$$
\frac{\partial u (x, t)}{\partial t} = -6uu_x - u_{xxx} + u_{xxxxx} - \zeta u_{xxxxxxx}, \tag{64}
$$

the following well known result

$$
\begin{aligned}
u (x, 0) &= \sqrt{\frac{12d}{k}} \operatorname{sech}^2 \left(\frac{x}{k}\right), \\
u_1 (x, t) &= u (x, 0) + \frac{\left(1.900 \times 10^{-3}\right) t \sqrt{d}}{k^{3/2}} \Bigg\{ 1.233 \frac{\sqrt{d}}{k^{1/2}} \\
&\quad - \frac{3.467}{k^2} - \frac{293}{k^4} - \frac{98933\zeta}{k^6} \\
&\quad + \left(-\frac{2.257}{k^2} - \frac{135.200}{k^4} + \frac{99631\zeta}{k^6} \right) \cosh\left(\frac{2x}{k}\right) \\
&\quad + \left(-\frac{0.102}{k^2} + \frac{173.700}{k^4} + \frac{14681\zeta}{k^6} \right) \cosh\left(\frac{4x}{k}\right)
\end{aligned}
$$

Table 2: Some values for numerical and exact solutions to the seventh order KdV equation expressed with the CFFD given in (5) at $k = 2.5$, $d = 4.0$, and $\gamma = 1$.

		For $\gamma = 1$		
Time (t)	Spatial (x)	Exact value	Numerical value	Error made
	−15.0	0.000099315	0.0001 094 20	1.01×10^{-5}
	−7.0	0.593670000	0.593670000	00
0.5	0.0	0.151107000	0.151107000	00
	+7.0	0.06971 8000	0.06971 8000	00
	+15.0	0.000 167 700	0.00009 6770	2.00×10^{-5}
	−15.0	0.00009 159 2	0.00008 857 2	3.02×10^{-6}
	−7.0	0.05477 9000	0.05477 9000	00
1.0	0.0	0.151907000	0.151907000	00
	+7.0	0.07 554 6000	0.07 554 6000	00
	+15.0	0.0001 266 20	0.00002 6620	1.00×10^{-4}
	−15.0	0.00007790 1	0.000237900	1.60×10^{-4}
	−7.0	0.04 6635000	0.04663 5000	00
2.0	0.0	0.152007000	0.152140000	1.31×10^{-4}
	+7.0	0.08 868 8000	0.08 868 8000	00
	+15.0	0.0001 488 70	0.000159170	1.03×10^{-5}
	−15.0	0.00005 635 2	0.001316 400	1.26×10^{-3}
	−7.0	0.03 378 5000	0.03 378 5000	00
4.0	0.0	0.152060000	0.162060000	1.00×10^{-2}
	+7.0	0.122 130000	0.120 120000	2.01×10^{-3}
	+15.0	0.0002057 90	0.0002057 90	00
	−15.0	0.00002 948 8	0.001 289 500	1.26×10^{-3}
	−7.0	0.01 771 1000	0.01 931 1000	1.6×10^{-3}
8.0	0.0	0.152060000	0.162060000	1×10^{-2}
	+7.0	0.230 430000	0.231 630000	1.2×10^{-3}
	+15.0	0.0003 932 70	0.0004 033 70	1.01×10^{-5}
	−15.0	0.00002 133 1	0.00002 007 1	1.26×10^{-6}
	−7.0	0.01 281 9000	0.01283 0000	1.06×10^{-5}
10.0	0.0	0.161760000	0.162760000	1.00×10^{-3}
	+7.0	0.315 320000	0.315 320000	00
	+15.0	0.0005 436 40	0.0005 436 40	00

$$+\left(\frac{0.572}{k^2}-\frac{13.730}{k^4}+\frac{496\zeta}{k^6}\right)\cosh\left(\frac{6x}{k}\right)\Bigg\}$$
$$\cdot\operatorname{sech}^6\left(\frac{x}{k}\right)\tanh\left(\frac{x}{k}\right). \tag{65}$$

This result corresponds to the similar one obtained in [20, 21] or in [22] via Adomian decomposition method. The other terms were computed following the same iterative approach and then, the functions $u(x,t)$ admits the closed form found to be

$$u(x,t)=\sqrt{\frac{12d}{k}}\operatorname{sech}^2\left(\frac{1}{k}\left(x-\frac{\sqrt{d}}{k^{5/2}}t\right)\right). \tag{66}$$

The shape of numerical approximations is depicted in Figures 3–6, plotted for different values of the derivative order γ and showing a spot of irregular perturbed motion in the pure fractional case ($\gamma = 0.80$) compared to the conventional case ($\gamma = 1$). Before going to conclusions, we can summarize the physical aspects of this work (issued from the mathematical analysis and simulations) as follows: analysis performed on the KdV model with no higher order perturbation term has shown the soliton solution via its related homoclinic orbit lying on a curved surface (Figures 1 and 2), and this remains true in the conventional case as well as the pure fractional case. However, irregular movements are more important in the latter case. Analysis performed on the seventh order KdV equation with one perturbations level has shown the shape of solitary waves characterized by motions with more irregular behaviors tending to become chaotic. The chaos is more likely to happen in pure fractional case compared to the conventional case (Figures 3–6). Hence, the main results of this paper, reflected by the six figures, insinuate that the

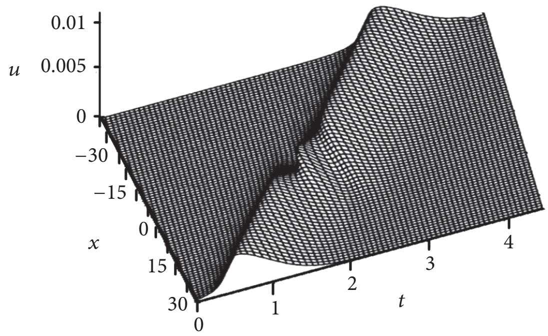

Figure 3: The shape of approximated solution $u_{\text{app}} = u_3$ in the conventional case ($\gamma = 1$). It is plotted with respect to different fixed values of time and space at $k = 2.5$, $d = 4.0$, $\zeta = 0.5$. The motion is quite standard and regular.

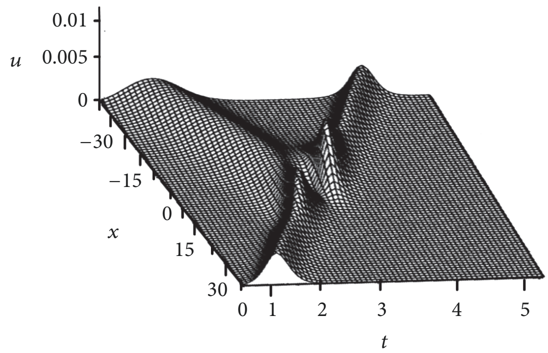

Figure 4: The shape of approximated solution $u_{\text{app}} = u_3$ in the pure fractional case ($\gamma = 0.80$), plotted with respect to different fixed values of time and space at $k = 2.5$, $d = 4.0$, $\zeta = 0.5$. The motion is shown to behave irregularly compared to the conventional case and the dynamic tending to a chaotic one.

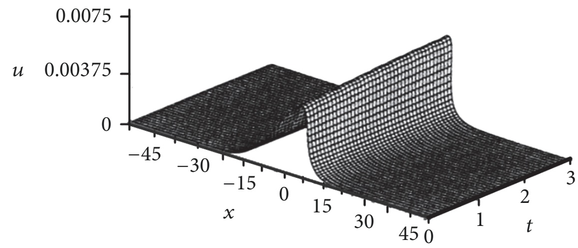

Figure 5: The shape of approximated solution $u_{\text{app}} = u_3$ in the conventional case ($\gamma = 1$). It is plotted with respect to different fixed values of time and space at $k = 2.5$, $d = 4.0$, $\zeta = 0.01$. The motion is quite standard and regular.

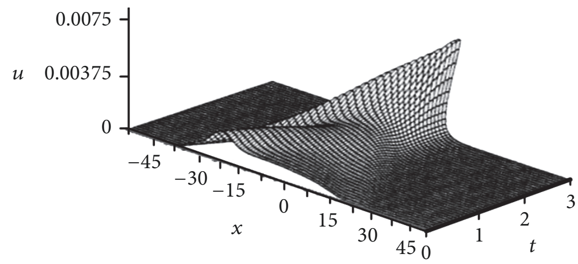

Figure 6: The shape of approximated solution $u_{\text{app}} = u_3$ in the pure fractional case ($\gamma = 0.80$), plotted with respect to different fixed values of time and space at $k = 2.5$, $d = 4.0$, $\zeta = 0.01$. The motion is shown to behave irregularly compared to the conventional case and the dynamic tending to a chaotic one.

regularity of a soliton can be perturbed by the nonsingular kernel derivative, which, combined with the perturbation parameter of the KdV model, may lead to chaos.

5. Concluding Remarks

We have made use of the recent version of derivative with nonsingular kernel to prove existence and uniqueness results for a model of seventh order Korteweg–de Vries (KdV) equation with one perturbation level. The unique solution is continuous. We then use numerical approximations to evaluate the behavior of the solution under the influence of external factors. It happened that when applied to equations of wave motion like the seventh order KdV equation, the new derivative acts as one of those external factors, which, combined with the perturbation term ζ of the model, causes the solution to be more irregular and unpredictable. This is the first instance where such a model is fully investigated and such result is exposed. This work differs from the previous ones within introduction of the nonsingular kernel derivative into a powerful model like Korteweg–de Vries's, which reveals another interesting feature that exists in the domain of wave motion as well as chaos theory.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was partially supported by the National Research Foundation (NRF) of South Africa, Grant no. 105932.

References

[1] Q. Cao, K. Djidjeli, W. G. Price, and E. H. Twizell, "Periodic and chaotic behaviour in a reduced form of the perturbed generalized Korteweg-de Vries and KADomtsev-Petviashvili equations," *Physica D: Nonlinear Phenomena*, vol. 125, no. 3-4, pp. 201–221, 1999.

[2] R. Grimshaw and X. Tian, "Periodic and chaotic behaviour in a reduction of the perturbed Korteweg-de Vries equation," *Proceedings of the Royal Society A Mathematical, Physical and Engineering Sciences*, vol. 445, no. 1923, pp. 1–21, 1994.

[3] X. Tian and R. H. Grimshaw, "Low-dimensional chaos in a perturbed Korteweg-de Vries equation," *International Journal of Bifurcation and Chaos*, vol. 5, no. 4, pp. 1221–1233, 1995.

[4] Z. Feng and R. Knobel, "Traveling waves to a Burgers-Korteweg-de Vries-type equation with higher-order nonlinearities," *Journal of Mathematical Analysis and Applications*, vol. 328, no. 2, pp. 1435–1450, 2007.

[5] H. Grad and P. N. Hu, "Unified shock profile in a plasma," *Physics of Fluids*, vol. 10, no. 12, pp. 2596–2602, 1967.

[6] P. N. Hu, "Collisional theory of shock and nonlinear waves in a plasma," *Physics of Fluids*, vol. 15, no. 5, pp. 854–864, 1972.

[7] R. S. Johnson, "A non-linear equation incorporating damping and dispersion," *Journal of Fluid Mechanics*, vol. 42, pp. 49–60, 1970.

[8] M. Caputo and M. Fabrizio, "A new definition of fractional derivative without singular kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 1–13, 2015.

[9] E. F. Doungmo Goufo, "Chaotic processes using the two-parameter derivative with non-singular and non-local kernel: basic theory and applications," *Chaos: An Interdisciplinary Journal of Nonlinear Science*, vol. 26, no. 8, Article ID 084305, 10 pages, 2016.

[10] E. F. D. Goufo, "Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg-de Vries-Burgers equation," *Mathematical Modelling and Analysis*, vol. 21, no. 2, pp. 188–198, 2016.

[11] Y. Khan, S. Panjeh Ali Beik, K. Sayevand, and A. Shayganmanesh, "A numerical scheme for solving differential equations with space and time-fractional coordinate derivatives," *Quaestiones Mathematicae*, vol. 38, no. 1, pp. 41–55, 2015.

[12] R. Kandasemy, S. Raut, D. Verma, and G. There, "Design of solar tricycle for handicapped person," *Journal of Mechanical and Civil Engineering*, vol. 25, no. 2, pp. 11–24, 2013.

[13] J. Losada and J. J. Nieto, "Properties of the new fractional derivative without singular Kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 87–92, 2015.

[14] M. Caputo, "Linear models of dissipation whose *Q* is almost frequency independent-II," *The Geophysical Journal of the Royal Astronomical Society*, vol. 13, no. 5, pp. 529–539, 1967.

[15] E. F. Doungmo Goufo, "A biomathematical view on the fractional dynamics of cellulose degradation," *Fractional Calculus and Applied Analysis*, vol. 18, no. 3, pp. 554–564, 2015.

[16] E. F. Doungmo Goufo, "Solvability of chaotic fractional systems with 3D four-scroll attractors," *Chaos, Solitons & Fractals*, vol. 104, pp. 443–451, 2017.

[17] Y. Khan, K. Sayevand, M. Fardi, and M. Ghasemi, "A novel computing multi-parametric homotopy approach for system of linear and nonlinear Fredholm integral equations," *Applied Mathematics and Computation*, vol. 249, pp. 229–236, 2014.

[18] J. Prüss, *Evolutionary Integral Equations and Applications*, vol. 87 of *Monographs in Mathematics*, Birkhäuser, Basel, Switzerland, 1993.

[19] K. Diethelm, N. J. Ford, A. D. Freed, and Y. Luchko, "Algorithms for the fractional calculus: a selection of numerical methods," *Computer Methods Applied Mechanics and Engineering*, vol. 194, no. 6-8, pp. 743–773, 2005.

[20] B. R. Duffy and E. J. Parkes, "Travelling solitary wave solutions to a seventh-order generalized KdV equation," *Physics Letters A*, vol. 214, no. 5-6, pp. 271-272, 1996.

[21] R.-X. Yao and Z.-B. Li, "On t/x-dependent conservation laws of the generalized nth-order KdV equation," *Chinese Journal of Physics*, vol. 42, no. 4, pp. 315–322, 2014.

[22] S. M. El-Sayed and D. Kaya, "An application of the ADM to seven-order Sawada-Kotara equations," *Applied Mathematics and Computation*, vol. 157, no. 1, pp. 93–101, 2004.