

Research Article

On the Relationship between the Invariance and Conservation Laws of Differential Equations

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In this paper, we highlight the complimentary nature of the results of Anco & Bluman and Ibragimov in the construction of conservation laws that whilst the former establishes the role of multipliers, the latter presents a formal procedure to determine the flows. Secondly, we show that there is an underlying relationship between the symmetries and conservation laws in a general setting, extending the results of Kara & Mahomed. The results take apparently differently forms for point symmetry generators and higher-order symmetries. Similarities exist, to some extent, with a previously established result relating symmetries and multipliers of a differential equation. A number of examples are presented.

1. Introduction

The role and methods associated with conservation laws are now well established and there have been some momentous works in these areas in recent times building on the contributions made by Noether which generally dealt with variational problems those that admit variational symmetries. It is not surprising then that much of the recent works focused on generalisations as far as constructions of conservation laws go, possibly nonvariational and preferably independent of a knowledge of symmetries. A vast amount and extensively cited works are due to Anco & Bluman in [1, 2], inter alia, Anderson [3, 4], and Kara & Mahomed [5], and a useful in-depth treatise is presented in the work of Olver [6] which goes a long way in discussing the concept of “recursion operators”. The first of these deals extensively with the notion of “multipliers” that if a differential equation times a factor (differential function) is a total divergence, then the Euler operator annihilates this product so that finding conserved flows amounts to finding the factors. It turns out that the multipliers are solutions of the adjoint equation. Of course, one still needs to determine the corresponding conserved flows using, amongst others, homotopy formulae [7]. A large amount of software to construct the various components of conserved vectors is available; see [8, 9].

Since conservation laws seem to be tied in with invariance properties, the intention to avoid the symmetry route can prove to be difficult. This is partly due to the amount of work required to construct conserved flows; it can be cumbersome and tedious when dealing with the large systems of differential equations that arise in physics, cosmology, and engineering. For example, constructing conservation directly from the definition may be straightforward for simple scalar ordinary differential equations but the more complex the differential equation, as they are in fluids, cosmology, and the various systems of Schrödinger equations that is abundant in the literature (to name a few), the greater the task. The popularity of Noether’s theorem lies in the existence of a formula. Trying to mimic this formula even in the nonvariational case has been tempting and partly successful; see [10]. In particular, the recent work of Ibragimov [11] develops a procedure to construct conserved vectors using the Noether operator, a symmetry of the differential equation solutions of the adjoint equation.

An in-depth study into the results due to Anco & Bluman in [1, 2] and Ibragimov [11] suggests that similarities are abundant; see [12]. However, it also shows that since the methods employed are largely different, there are some intrinsic differences and what is presented here is an attempt to show that these differences, in fact, allow these works to

complement each other. For example, the underlying aspect in the multiplier approach is primarily to construct multipliers that leads to the differential equation being conserved. These multipliers can be chosen with a specific order (in derivatives) in mind and then one may choose from a number of methods to construct the conserved vectors. In [11], the particular method appeals to the Noether operator after having knowledge of a symmetry and a solution of the adjoint equation. It will be shown that the total divergence of the conserved flow has a form dependent on whether the symmetry used is a point symmetry or an evolutionary/canonical symmetry; the general result in the latter case would include generalised symmetries.

2. Notations and Preliminaries

What follows is a summary of the definitions, concepts, and notations that will be utilised in the sequel.

Consider an k th-order system of partial differential equations (pdes) of n independent variables $\underline{x} = (x_1, x_2, \dots, x_n)$ and p dependent variables $w = (w_1, w_2, \dots, w_p)$, namely,

$$\begin{aligned} E(\underline{x}, w, w_{(1)}, \dots, w_{(r)}) &= 0, \\ u &= 1, \dots, \tilde{p}, \end{aligned} \quad (1)$$

where a locally analytic function $f(\underline{s}, w, w_1, \dots, w_k)$ of a finite number of dependent variables w, w_1, \dots, w_b denotes the collections of all first, second, ..., b th-order partial derivatives, and s is a multivariable; that is,

$$\begin{aligned} w_i^\alpha &= D_i(w^\alpha), \\ w_{ij}^\alpha &= D_j D_i(w^\alpha), \dots \end{aligned} \quad (2)$$

respectively, with the total differentiation operator with respect to x^i given by

$$D_i = \frac{\partial}{\partial x^i} + w_i^\alpha \frac{\partial}{\partial w^\alpha} + w_{ij}^\alpha \frac{\partial}{\partial w_j^\alpha} + \dots \quad i = 1, \dots, p. \quad (3)$$

In order to determine conserved densities and fluxes, we resort to the invariance and multiplier approach based on the well-known result that the Euler-Lagrange operator annihilates a total divergence. Firstly, if (T^{x1}, T^{x2}, \dots) is a conserved vector corresponding to a conservation law, then

$$D_{x1} T^{x1} + D_{x2} T^{x2} + \dots = 0 \quad (4)$$

along the solutions of the differential equation $E(\underline{x}, w, w_{(1)}, \dots, w_{(k)}) = 0$.

Moreover, if there exists a nontrivial differential function Q , called a ‘‘multiplier’’, such that

$$\begin{aligned} Q(\underline{x}, w, w_{(1)} \dots) E(\underline{x}, w, w_{(1)}, \dots, w_{(r)}) \\ = D_{x1} T^{x1} + D_{x2} T^{x2} + \dots, \end{aligned} \quad (5)$$

for some (conserved) vector (T^{x1}, T^{x2}, \dots) , then

$$\frac{\delta}{\delta u} [Q(\underline{x}, w, w_{(1)} \dots) E(\underline{s}, w, w_{(1)}, \dots, w_{(r)})] = 0, \quad (6)$$

where $\delta/\delta u$ is the Euler operator. Hence, one may determine the multipliers, using (6), and then construct the corresponding conserved vectors; several approaches for this exist of which the better known one is the ‘‘homotopy’’ approach [13].

If a pde is variational, then the conservation laws may be constructed from Noether’s theorem. It can be shown that Lie point symmetries that leave the system of differential equations invariant contain the algebra of Noether/divergent/variational symmetries [6, 11].

Conservation laws may be expressed as conserved forms [4]. For example, if $\underline{x} = (t, s)$, the conserved form would be

$$\omega = T^t ds - T^s dt \quad (7)$$

(where (T^t, T^s) is the conserved vector such that $D_t T^t + D_s T^s = 0$ on the solutions of the pde $E(s, t, w, w_{(1)}, \dots, w_{(r)}) = 0$). Here, $T^t ds$ leads to the ‘‘conserved density’’ if t and s are time and space, respectively.

3. Conservation Laws

3.1. In the first case, we consider the relationship between the conserved flows and the respective point symmetry generators of the differential equation.

Example 1. We, firstly, utilise the heat equation $u_t - u_{xx} = 0$ as an illustrative example. The final result is presented in a proposition. The multipliers $Q_1 = -x$ and $Q_2 = -e^t \sin x$ are discussed in [11] to construct conserved vectors; there referred to as solutions of the adjoint equation $v_t + v_{xx} = 0$. Thus, $(\delta/\delta u)[Q_1(u_t - u_{xx})] = 0$. In general, then, $(\delta/\delta u)[v(u_t - u_{xx})] = 0$ so that, by [11], the conserved flow via the point symmetry $X = 2t\partial_x - xu\partial_u$ is

$$\begin{aligned} T^t &= -v(xu + 2tu_x), \\ T^x &= v(2tu_t + u + xu_x) - v_x(xu + 2tu_x) \end{aligned} \quad (8)$$

The total divergence is

$$\begin{aligned} D_t T^t + D_x T^x &= -v_t(xu + 2tu_x) \\ &\quad - v(xu_t + 2u_x + 2tu_{xt}) \\ &\quad + v_x(2tu_t + u + xu_x) \\ &\quad + v(2tu_{xt} + 2u_x + xu_{xx}) \\ &\quad - v_x(u + xu + x + 2tu_{xx}) \\ &\quad - (xu + 2tu_x)v_{xx} \\ &= -(xu + 2tu_x)(v_t + v_{xx}) \\ &\quad - (2tv_x - xv)(u_t - u_{xx}) \end{aligned} \quad (9)$$

That is, the total divergence takes the form

$$\begin{aligned} D_t T^t + D_x T^x &= -(xu + 2tu_x)(v_t + v_{xx}) \\ &\quad + (-xv + 2tv_x)(u_t - u_{xx}) \end{aligned}$$

$$\begin{aligned}
&= W(v_t + v_{xx}) + (\mathcal{X}v)(u_t - u_{xx}) \\
&= (\mathcal{X}v)(u_t - u_{xx})
\end{aligned} \tag{10}$$

where $\mathcal{X} = 2tD_x - x$. If $v = 1$, $W = -(xu + 2tu_x)$ is the characteristic of X . If, in $Q = -xv + 2tv_x$,

(i) $v = 1$, then $D_t T^t + D_x T^x = -x(u_t - u_{xx})$ which leads to the multiplier $Q_1 = -x$;

(ii) $v = x$, then $D_t T^t + D_x T^x = (-x^2 + 2t)(u_t - u_{xx})$ leading to the multiplier $Q = -x^2 + 2t$.

Example 2. Consider the one-dimensional wave equation $u_{tt} - u_{xx} = 0$ and the Lorentz rotation symmetry $Y = t\partial_x + x\partial_t$ with characteristic $W = -tu_x - xu_t$ and adjoint equation $-v_{tt} + v_{xx} = 0$. The detailed calculation using the results in [11] leads to

$$\begin{aligned}
T^t &= xv(u_{tt} - u_{xx}) + v_t(xu_t + tu_x) + v(-xu_{tt} - u_x \\
&\quad - tu_{xt}), \\
T^x &= tv(u_{tt} - u_{xx}) - v_x(xu_t + tu_x) - v(-tu_{xx} - u_t \\
&\quad - xu_{xt})
\end{aligned} \tag{11}$$

so that

$$\begin{aligned}
D_t T^t + D_x T^x &= -(xu_t + tu_x)(-v_{tt} + v_{xx}) \\
&\quad + (xv_t + tv_x)(u_{tt} - u_{xx}) \\
&= W(-v_{tt} + v_{xx}) + (\mathcal{Y}v)(u_{tt} - u_{xx}) \\
&= (\mathcal{Y}v)(u_{tt} - u_{xx}),
\end{aligned} \tag{12}$$

where $\mathcal{Y} = tD_x + xD_t$.

The following proposition that defines the relationship between point symmetries, multipliers, and conservation laws constructed via the Noether operator in [11] can be easily proved.

Proposition 3. *If $Z = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \phi(x, t, u)\partial_u$ (characteristic $W = \phi - \xi u_x - \tau u_t$) is a Lie point symmetry generator of a second-order partial differential equation (pde) $E(x, t, u, u_{(x)}, u_t, u_{tt}, \dots) = 0$ (whose adjoint equation is $F(x, t, v, v_{(x)}, v_t, \dots) = 0$), $\mathbf{L} = vE$ and*

$$\begin{aligned}
T^t &= \tau\mathbf{L} + W\left(\frac{\partial\mathbf{L}}{\partial u_t} - D_t\frac{\partial\mathbf{L}}{\partial u_{tt}} - D_x\frac{\partial\mathbf{L}}{\partial u_{tx}}\right) \\
&\quad + D_t W\frac{\partial\mathbf{L}}{\partial u_{tt}} + D_x\frac{\partial\mathbf{L}}{\partial u_{tx}}, \\
T^x &= \xi\mathbf{L} + W\left(\frac{\partial\mathbf{L}}{\partial u_x} - D_t\frac{\partial\mathbf{L}}{\partial u_{xt}} - D_x\frac{\partial\mathbf{L}}{\partial u_{xx}}\right) \\
&\quad + D_t W\frac{\partial\mathbf{L}}{\partial u_{xt}} + D_x\frac{\partial\mathbf{L}}{\partial u_{xx}},
\end{aligned} \tag{13}$$

then the divergence

$$\begin{aligned}
D_t T^t + D_x T^x &= WF(x, t, v, v_{(x)}, v_t, \dots) \\
&\quad + (\xi(x, t, v)v_x + \tau(x, t, v)v_t + \phi(x, t, v)) \\
&\quad \cdot E(x, t, u, u_{(x)}, u_t, u_{tt}, \dots) \\
&= WF(x, t, v, v_{(x)}, v_t, \dots) + (\mathcal{Z}v - \lambda v) \\
&\quad \cdot E(x, t, u, u_{(x)}, u_t, u_{tt}, \dots),
\end{aligned} \tag{14}$$

where $\mathcal{Z} = \xi(x, t, v)D_x + \tau(x, t, v)D_t + \phi(x, t, v)$ and λ is determined by the conformal factor. That is, if $ZE = \mu_1 E$ and $D_t \tau + D_x \xi = \mu_2$, then $\lambda = \mu_1 + \mu_2$; λ need not be a constant. On particular solutions $v = v(x, t)$ of the adjoint equation, we have a conserved flow (T^t, T^x) with multiplier $Q = \xi(x, t, v)v_x + \tau(x, t, v)v_t + \phi(x, t, v(x, t))$.

After some cumbersome calculations, Proposition 3 is easily generalised to the multidimensional pde $E(x, t, u, u_{(x)}, u_t, \dots, u_{(r)}) = 0$.

Example 4(a). For the third-order KdV equation $u_t - u_{xxx} - u_{xxx} = 0$, the adjoint equation is $v_t - v_{xxx} - uv_x = 0$. The calculations in [11], using the point symmetry $X = -3t\partial_t - x\partial_x + 2u\partial_u$ and an extended version of (13), lead to the conserved vector components

$$\begin{aligned}
T^t &= v(3tu_{xxx} + 3tuu_x + xu_x + 2u), \\
T^x &= -v(2u^2 + xu_t + 3tuu_t + 4u_{xx} + 3tu_{txx}) \\
&\quad + v_x(3u_x + 3tu_{tx} + xu_{xx}) \\
&\quad - v_{xx}(2u + 3tu_t + xu_x),
\end{aligned} \tag{15}$$

so that, after detailed simplification we get

$$\begin{aligned}
D_t T^t + D_x T^x &= (2u + 3tu_u + xu_x)(v_t - v_{xxx} - uv_x) \\
&\quad + (-xv_x - 3tv_t + 2v - v)(u_t - uu_x - u_{xxx}) \\
&= W(v_t - v_{xxx} - uv_x) \\
&\quad + [(\mathcal{X} - 1)v](u_t - uu_x - u_{xxx}),
\end{aligned} \tag{16}$$

where $W = 2u + 3tu_u + xu_x$ and $\mathcal{X} = -3t\partial_t - x\partial_x + 2$. In Proposition 3, $\lambda = 5 - 4 = 1$.

Example 4(b). Consider the simplest Schrödinger equation with cubic nonlinearity $iu_t - u_{xx} + u|u|^2 = 0$. If we put $u = p + iq$ then define $\mathbf{L} = v[-q_t - p_{xx} + p(p^2 + q^2)] + w[p_t - q_{xx} + q(p^2 + q^2)]$ where (v, w) is the solution of the system $-w_t - v_{xx} + v(p^2 + q^2) + 2p(vp + wq) = 0$, $v_t - w_{xx} + w(p^2 + q^2) + 2q(vp + wq) = 0$, the adjoint of $-q_t - p_{xx} + p(p^2 + q^2) = 0$, $p_t - q_{xx} + q(p^2 + q^2) = 0$. The components of the conserved vector using $X = \partial_t$ are then

$$\begin{aligned}
T^t &= \mathbf{L} - p_t w + vq_t, \\
T^x &= -p_t v_x + p_{xt} v - q_t w_x + wq_{xt}
\end{aligned} \tag{17}$$

so that, after some manipulation,

$$\begin{aligned}
& D_t T^t + D_x T^x \\
&= p_t \left[-w_t - v_{xx} + v(p^2 + q^2) + 2p(vp + wq) \right] \\
&\quad + q_t \left[v_t - w_{xx} + w(p^2 + q^2) + 2q(vp + wq) \right] \\
&\quad + v_t \left[-q_t - p_{xx} + p(p^2 + q^2) \right] \\
&\quad + w_t \left[p_t - q_{xx} + q(p^2 + q^2) \right] \tag{18} \\
&= -W^1 \left[-w_t - v_{xx} + v(p^2 + q^2) + 2p(vp + wq) \right] \\
&\quad - W^2 \left[v_t - w_{xx} + w(p^2 + q^2) + 2q(vp + wq) \right] \\
&\quad - (\mathcal{X}v) \left[-q_t - p_{xx} + p(p^2 + q^2) \right] \\
&\quad - (\mathcal{X}w) \left[p_t - q_{xx} + q(p^2 + q^2) \right],
\end{aligned}$$

where $W^1 = -p_t$, $W^2 = -q_t$, and $\mathcal{X} = -D_t$. When $v = p$ and $w = q$, we get the well-known energy conservation via (17) using the multiplier (p_t, q_t) .

3.2. We now consider the connection between generalised symmetries, higher-order symmetries, and evolutionary/canonical symmetries and associated conservation laws. Again, we suppose $\mathbf{L} = v(x, t)E$ [11].

Example 5. In this example, we revisit the heat equation $u_t - u_{xx} = 0$ with its evolutionary symmetry $X_1 = (tu_x + (1/2)xu)\partial_u$ (from the point symmetry $-t\partial_x + (1/2)xu\partial_u$) and higher symmetries $X_2 = u_{xx}\partial_u$ and $X_3 = (2tu_{xxx} + xu_{xx})\partial_u$ used to construct conserved flows (T^t, T^x) .

(i) With X_1 , we obtain the components of the conserved vector to be

$$\begin{aligned}
T^t &= v \left(tu_x + \frac{1}{2}xu \right), \\
T^x &= -v \left(tu_{xx} + \frac{1}{2}u + \frac{1}{2}xu_x \right), \tag{19}
\end{aligned}$$

so that

$$\begin{aligned}
D_t T^t + D_x T^x &= v_t \left(tu_x + \frac{1}{2}xu \right) \\
&\quad + v \left(u_x + tu_{xt} + \frac{1}{2}xu_t \right) \\
&\quad + -v_x \left(tu_{xx} + \frac{1}{2}u + \frac{1}{2}xu_x \right) \\
&\quad - v \left(tu_{xxx} + u_x + \frac{1}{2}xu_{xx} \right) \tag{20} \\
&= (v_t + v_{xx}) \left(tu_x + \frac{1}{2}xu \right) \\
&\quad + v \left(tu_{xt} - tu_{xxx} + \frac{1}{2}xu_t - \frac{1}{2}xu_{xx} \right) \\
&= W(v_t + v_{xx}) + v\mathcal{R}_1(u_t - u_{xx}),
\end{aligned}$$

where $W = tu_x + (1/2)xu$ and $\mathcal{R}_1 = tD_x + (1/2)x$ are the recursion operator associated with X_1 .

(ii) Using X_2 , we get

$$\begin{aligned}
T^t &= vu_{xx}, \\
T^x &= v_x u_{xx} - vu_{xxx}, \tag{21}
\end{aligned}$$

so that

$$\begin{aligned}
D_t T^t + D_x T^x &= u_{xx}(v_t + v_{xx}) + vD_x D_x(u_t - u_{xx}) \\
&= W(v_t + v_{xx}) + v\mathcal{R}_2(u_t - u_{xx}). \tag{22}
\end{aligned}$$

(iii) With X_3 , we get

$$\begin{aligned}
T^t &= v(xu_{xx} + 2tu_{xxx}), \\
T^x &= v_x(xu_{xx} + 2tu_{xxx}) \\
&\quad - v(u_{xx} + xu_{xxx} + 2tu_{xxxx}), \tag{23}
\end{aligned}$$

so that after some simplifications the total divergence is

$$\begin{aligned}
D_t T^t + D_x T^x &= (v_t + v_{xx})(xu_{xx} + 2tu_{xxx}) \\
&\quad + v(2t[u_{txxx} - uxxxxx] + x[u_{txx} - uxxxxx]) \\
&= W(v_t + v_{xx}) + v\mathcal{R}_3(u_t - u_{xx}), \tag{24}
\end{aligned}$$

where $W = 2tu_{xxx} + xu_{xx}$ and $\mathcal{R}_3 = xD_x D_x + 2tD_x D_x D_x$ are the respective recursion operator.

Example 6. It is well known that, for the wave equation $u_{tt} - u_{xx} = 0$ and any variational equation, ‘‘multipliers’’ or, equivalently, solutions of the adjoint equation are symmetries of the equation so that in the simple case of the evolutionary vector field $Y = (tu_x + xu_t)\partial_u$ is a generalised symmetry and $Q = tu_x + xu_t$ is multiplier. Applying Noether’s theorem is clearly the efficient route to construct a conservation law. Alternatively, if we assume $L = v(x, t)(u_{tt} - u_{xx})$ using the procedure in [11], we get

$$\begin{aligned}
T^t &= v_x(tu_x + xu_t) - v(tu_{xx} + u_t + xu_{xt}), \\
T^x &= -v_t(tu_x + xu_t) + v(u_x + xu_{tt} + tu_{xt}), \tag{25}
\end{aligned}$$

so that

$$\begin{aligned}
D_t T^t + D_x T^x &= (-v_{tt} + v_{xx})(tu_x + xu_t) \\
&\quad + v(t[u_{ttt} - uxxx] + x[u_{ttt} - utxx]) \\
&= W(-v_{tt} + v_{xx}) + v\mathcal{R}(u_{tt} - u_{xx}), \tag{26}
\end{aligned}$$

where $W = tu_x + xu_t$ and $\mathcal{R} = tD_x + xD_t$.

Proposition 7. In Proposition 3, if Z is a generalised symmetry or evolutionary/canonical vector field such that $ZE = (\mathcal{R} + \lambda)E$, where \mathcal{R} is the recursion operator associated with Z , then

$$\begin{aligned} D_t T^t + D_x T^x &= WF(x, t, v, v_{(x)}, v_t, \dots) \\ &+ v(\mathcal{R} + \lambda)E(x, t, u, u_{(x)}, u_t, u_{tt}, \dots). \end{aligned} \quad (27)$$

Again, the proposition can be generalised to the multidimensional case.

Example 8. We revisit the KdV equation with its evolutionary vector field $X = (xu_x + 3tu_t + 2u)\partial_u$. It can be shown that

$$\begin{aligned} T^t &= v(uu_x + 3tu_t + 2u), \\ T^x &= (uu_x + 3tu_t + 2u)(-uv - v_{xx}) \\ &+ v_x(3u_x + 3tu_{xt} + xu_{xx}) \\ &- v(4u_{xx} + 3tu_{xxt} + xu_{xxx}) \end{aligned} \quad (28)$$

so that

$$\begin{aligned} D_t T^t + D_x T^x &= (uu_x + 3tu_t + 2u)(v_t - v_{xxx} - uv_x) \\ &+ v^5(u_t - uu_x - u_{xxx}) \\ &+ v(3tD_t + xD_x)(u_t - uu_x - u_{xxx}) \\ &= W(v_t - v_{xxx} - uv_x) \\ &+ v(3 + \mathcal{R})(u_t - uu_x - u_{xxx}), \end{aligned} \quad (29)$$

where $\mathcal{R} = 3tD_t + xD_x + 2$, $W = uu_x + 3tu_t + 2u$, and we note that $X(u_t - uu_x - u_{xxx}) = (3 + \mathcal{R})(u_t - uu_x - u_{xxx})$.

4. Discussion

It is clear that, in each case, the conserved flows (T^t, T^x) are nontrivial since $D_t T^t + D_x T^x$ do not vanish identically but, rather, on the solutions of the differential equation. The dependence of this method on solutions of the adjoint equation is equivalent to the multiplier approach since multipliers are solutions of the adjoint equation. Thus, as mentioned before, the two approaches in [1, 11] complement each other and the latter has a formal procedure to construct the conserved flows using symmetries of the differential equation. Moreover, we showed that the total divergence, quite explicitly, displays a relationship between symmetries (point or generalised) and conservation laws in a general setting; compare this to the results in [5]. Also, the main results of this paper mimic, to some extent, the results established on the relationship between symmetries and multipliers of a differential equation as discussed in [14].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

References

- [1] S. C. Anco and G. Bluman, "Direct construction method for conservation laws of partial differential equations Part I: Examples of conservation law classifications," *European Journal of Applied Mathematics*, vol. 13, no. 5, pp. 545–566, 2002.
- [2] S. C. Anco and G. Bluman, "Direct construction method for conservation laws of partial differential equations part II: general treatment," *European Journal of Applied Mathematics*, vol. 13, no. 5, pp. 567–585, 2002.
- [3] I. M. Anderson and T. E. Duchamp, "Variational principles for second-order quasilinear scalar equations," *Journal of Differential Equations*, vol. 51, no. 1, pp. 1–47, 1984.
- [4] I. M. Anderson and J. Pohjanpelto, "The cohomology of invariant variational bicomplexes," *Acta Applicandae Mathematicae*, vol. 41, no. 1–3, pp. 3–19, 1995.
- [5] A. H. Kara and F. M. Mahomed, "Relationship between symmetries and conservation laws," *International Journal of Theoretical Physics*, vol. 39, no. 1, pp. 23–40, 2000.
- [6] P. J. Olver, *Applications of Lie Groups to Differential Equations*, Springer, New York, NY, USA, 1986.
- [7] D. Poole and W. Hereman, "The homotopy operator method for symbolic integration by parts and inversion of divergences with applications," *Applicable Analysis: An International Journal*, vol. 89, no. 4, pp. 433–455, 2010.
- [8] W. Hereman and A. Nuseir, "Symbolic methods to construct exact solutions of nonlinear partial differential equations," *Mathematics and Computers in Simulation*, vol. 43, no. 1, pp. 13–27, 1997.
- [9] A. F. Cheviakov, "GeM software package for computation of symmetries and conservation laws of differential equations," *Computer Physics Communications*, vol. 176, no. 1, pp. 48–61, 2007.
- [10] A. H. Kara and F. M. Mahomed, "Noether-type symmetries and conservation laws via partial Lagrangians," *Nonlinear Dynamics*, vol. 45, no. 3–4, pp. 367–383, 2006.
- [11] N. H. Ibragimov, "A new conservation theorem," *Journal of Mathematical Analysis and Applications*, vol. 333, no. 1, pp. 311–328, 2007.
- [12] S. Anco, "On the Incompleteness of Ibragimov's Conservation Law Theorem and Its Equivalence to a Standard Formula Using Symmetries and Adjoint-Symmetries," *Symmetry*, vol. 9, no. 3, p. 33, 2017.
- [13] A. Kara, A. Biswas, Q. Zhou, L. Moraru, S. P. Moshokoa, and M. Belic, "Conservation laws for optical solitons with Chen–Lee–Liu equation," *Optik - International Journal for Light and Electron Optics*, vol. 174, pp. 195–198, 2018.
- [14] S. C. Anco and A. H. Kara, "Symmetry-invariant conservation laws of partial differential equations," *European Journal of Applied Mathematics*, vol. 29, no. 1, pp. 78–117, 2018.

