Research Article

A Family of Measures of Noncompactness in the Locally Sobolev Spaces and Its Applications to Some Nonlinear Volterra Integrodifferential Equations

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We study a new family of measures of noncompactness in the locally Sobolev space $W_{\text{loc}}^{n,1}(\mathbb{R}_+^N)$, equipped with a suitable topology. As an application of the technique associated with this family of measures of noncompactness, we study the existence of solutions for a class of nonlinear Volterra integrodifferential equations. Further, we give an illustrative example to verify the effectiveness and applicability of our results.

1. Introduction

A locally integrable function is a function which is integrable on every compact subset of its domain of definition. The importance of such functions lies in the fact that their function space is similar to L^p spaces, but its members are not required to satisfy any growth restriction on their behavior at infinity: in other words, locally integrable functions can grow arbitrarily fast at infinity but are still manageable in a way similar to ordinary integrable functions.

Sobolev spaces [1], that is, the class of functions with derivatives in L^p , play an important role in the analysis. In the recent times, there have been considerable efforts to study these spaces. The significant impact of them arises from the fact that solutions of partial differential equations are naturally found in Sobolev spaces. They also can be applied to approximation theory, calculus of variations, differential geometry, spectral theory, and so on.

The degree of noncompactness of a set is measured by means of functions called measures of noncompactness.

Measures of noncompactness are very useful tools in Banach spaces. They are widely used in fixed point theory, differential equations, functional equations, integral and integrodifferential equations, optimization, and so on [2–8].

In this paper, we define two topological structures on the locally Sobolev space $W_{\text{loc}}^{n,1}(\mathbb{R}_+^N)$: the Fréchet metric topology $\tau_{\text{loc}}^{\mathcal{F}}$ given by a sequence of seminorms and the topology $\tau_{\text{loc}}^{\omega}$ generated by the family of projections,

$$\pi_T : W_{\text{loc}}^{n,1}(\mathbb{R}_+^N) \longrightarrow W^{n,1}([0, T]^N), \quad T \geq 0, \quad (1)$$

where the spaces $W^{n,1}([0, T]^N)$ are furnished with weak topologies. Next, we introduce a weak measure of noncompactness in $W^{n,1}([0, T]^N)$. Thereafter, we define a family of measures of noncompactness $\{\mu_T\}_{T \geq 0}$ in $W_{\text{loc}}^{n,1}(\mathbb{R}_+^N)$ with the topology $\tau_{\text{loc}}^{\omega}$ and we investigate the basic properties of $\{\mu_T\}_{T \geq 0}$.

As an application, we study the existence of solutions for a class of nonlinear Volterra integrodifferential equations

$$x(t_1, \dots, t_N) = f\left((t_1, \dots, t_N), \int_0^{t_1} \dots \int_0^{t_N} \vartheta\left((t_1, \dots, t_N), (s_1, \dots, s_N), x(\xi(s_1, \dots, s_N)), \frac{\partial x}{\partial s_1}(\xi(s_1, \dots, s_N)), \dots, \frac{\partial x}{\partial s_N}(\xi(s_1, \dots, s_N)), T_0 x(s_1, \dots, s_N)\right) ds_1 \dots ds_N\right). \quad (2)$$

Finally, we give an illustrative example to verify the effectiveness and applicability of our results.

2. Preliminaries

Let $m(\mathcal{D})$ denote the Lebesgue measure of a Lebesgue measurable subset $\mathcal{D} \subset \mathbb{R}_+^N$. For subset $\mathcal{D} \subset [0, T]^N$ of a fixed N -cell $[0, T]^N$, we write $\mathcal{D}' = [0, T]^N \setminus \mathcal{D}$.

Suppose that $n \in \mathbb{N}$. We recall that the Sobolev space $W^{n,1}([0, T]^N)$ ($T \geq 0$) is defined to consist of those real measurable functions f which, together with all their distributional derivatives $D^\alpha f$ of order $|\alpha| \leq n$, belong to $L^1([0, T]^N)$. Here $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index; that is, each α_j is a nonnegative integer, $|\alpha| = \alpha_1 + \dots + \alpha_n$, and $D^\alpha = \partial^{|\alpha|} / (\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n})$.

If $0 \leq S \leq T$, then the symbol π_S^T stands for the operator of restriction

$$\begin{aligned} \pi_S^T : W^{n,1}([0, T]^N) &\longrightarrow W^{n,1}([0, S]^N), \\ \pi_S^T(f) &:= f|_{[0, S]^N}, \end{aligned} \quad (3)$$

where $f|_{[0, S]^N}$ is the restriction of the function $f \in W^{n,1}([0, T]^N)$ to the N -cell $[0, S]^N$.

The locally Sobolev space $W_{\text{loc}}^{n,1}(\mathbb{R}_+^N)$ (shortly $W_{\text{loc}}^{n,1}$) is defined to consist of those real measurable functions f which, together with all their distributional derivatives $D^\alpha f$ of order $|\alpha| \leq n$, are locally integrable on $\mathbb{R}_+^N$; that is, $\|f\|_{n,1}^T < \infty$ for each $T \geq 0$, where

$$\|f\|_{n,1}^T = \max_{|\alpha| \leq n} \int_{[0, T]^N} |D^\alpha f| d\mu. \quad (4)$$

Furthermore, we denote by π_T the operator of restriction

$$\begin{aligned} \pi_T : W_{\text{loc}}^{n,1}(\mathbb{R}_+^N) &\longrightarrow W^{n,1}([0, T]^N), \\ \pi_T(f) &:= f|_{[0, T]^N}. \end{aligned} \quad (5)$$

We consider the space $W_{\text{loc}}^{n,1}$ with the Fréchet metric topology $\tau_{\text{loc}}^{\mathcal{F}}$.

Definition 1 (see [9]). The metrizable topology induced in $W_{\text{loc}}^{n,1}$ by the family of seminorms (4), that is, the topology defined by the distance

$$d(f, g) := \sum_{T=1}^{\infty} \frac{1}{2^T} \frac{\|f - g\|_{n,1}^T}{1 + \|f - g\|_{n,1}^T}, \quad (6)$$

or equivalently

$$d_1(f, g) := \sup \{2^{-T} \|f - g\|_{n,1}^T, T \geq 0\}, \quad (7)$$

will be called Fréchet metric topology in $W_{\text{loc}}^{n,1}$ and it will be denoted by $\tau_{\text{loc}}^{\mathcal{F}}$.

The convergence and compactness in the topology $\tau_{\text{loc}}^{\mathcal{F}}$ are characterized in the following proposition.

Proposition 2. A sequence $\{f_k\} \subset W_{\text{loc}}^{n,1}$ is convergent to $f \in W_{\text{loc}}^{n,1}$ in the topology $\tau_{\text{loc}}^{\mathcal{F}}$ if and only if $\lim_{k \rightarrow \infty} \|f_k - f\|_{n,1}^T = 0$ for each $T \geq 0$. A set $X \subset W_{\text{loc}}^{n,1}$ is relatively compact in the topology $\tau_{\text{loc}}^{\mathcal{F}}$ if and only if $\pi_T(X)$ is relatively compact in the Banach space $W^{n,1}([0, T]^N)$ for $T \geq 0$.

Proof. The proof is easy. $\square$

We conclude this section by recalling Dunford-Pettis Theorem (see [10]) which will be used in proving our results in next section.

Theorem 3. Let F be a bounded set in $L^1([0, T]^N)$. Then F has compact closure in the weak topology τ^ω if and only if F is equi-integrable; that is,

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that} \quad \int_{\mathcal{D}} |f| d\mu < \varepsilon \quad (8)$$

for all measurable sets $\mathcal{D} \subset [0, T]^N$ with $m(\mathcal{D}) < \delta$ and all $f \in F$.

3. Main Results

In this section, we introduce a weak measure of noncompactness in $W^{n,1}([0, T]^N)$. For this purpose, we need the following result.

Proposition 4. Assume that F is a bounded set in $W^{n,1}([0, T]^N)$. Then F has compact closure with the weak topology τ^ω if and only if

$$\forall \varepsilon > 0, \exists \delta > 0 \quad \text{such that} \quad \max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f| d\mu < \varepsilon \quad (9)$$

for all measurable sets $\mathcal{D} \subset [0, T]^N$ with $m(\mathcal{D}) < \delta$ and all $f \in F$.

Proof. Take $k = \sum_{i=0}^n N^i$ and let $(L^1([0, T]^N))^k$ be a Banach space with norm

$$\|(f_1, \dots, f_k)\|_{\max} = \max \{\|f_i\|_1^T, i = 1, \dots, k\}. \quad (10)$$

Define $\Phi : W^{n,1}([0, T]^N) \rightarrow (L^1([0, T]^N))^k$ ($\{\alpha : |\alpha| \leq n\} = \{\beta_1, \beta_2, \dots, \beta_{k-1}\}$) by

$$\Phi(f) = (f, D^{\beta_1} f, D^{\beta_2} f, \dots, D^{\beta_{k-1}} f). \quad (11)$$

Obviously Φ is one-to-one, continuous linear operator and $\Phi(W^{n,1}([0, T]^N))$ is closed in $(L^1([0, T]^N))^k$. Thus, $W^{n,1}([0, T]^N)$ and $\Phi(W^{n,1}([0, T]^N))$ are isometrically isomorphic. The proof is completed by Dunford-Pettis Theorem. $\square$

Consider $\mathfrak{N}_{W^{n,1}([0, T]^N)}^\omega$ the subfamily of $W^{n,1}([0, T]^N)$ consisting of relatively compact sets in the topology τ^ω and $\mathfrak{M}_{W^{n,1}([0, T]^N)}$ the family of all nonempty and bounded subsets (in norm) of $W^{n,1}([0, T]^N)$. Now, we introduce a weak measure of noncompactness in $W^{n,1}([0, T]^N)$.

Let X be a bounded subset of the space $W^{n,1}([0, T]^N)$. For $f \in X$ and $\varepsilon > 0$ let us denote

$$\begin{aligned} C^T(f, \varepsilon) &= \sup \left\{ \max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f| d\mu : \mathcal{D} \right. \\ &\quad \left. \subset [0, T]^N : m(\mathcal{D}) < \varepsilon \right\}, \\ C^T(X, \varepsilon) &= \sup \{ C^T(f, \varepsilon) : f \in X \}, \\ \mu(X) &:= C^T(X) = \lim_{\varepsilon \rightarrow 0} C^T(X, \varepsilon). \end{aligned} \quad (12)$$

The essential properties of this measure are contained in the following lemma.

Lemma 5. *The weak measure of noncompactness $\mu : \mathfrak{M}_{W^{n,1}([0, T]^N)} \rightarrow \mathbb{R}_+$ satisfies the following conditions:*

- (1) $\ker \mu = \mathfrak{N}_{W^{n,1}([0, T]^N)}^\omega$.
- (2) If $X \subset Y$, then $\mu(X) \leq \mu(Y)$.
- (3) $\mu(\lambda X) = |\lambda| \mu(X)$.
- (4) $\mu(X + Y) \leq \mu(X) + \mu(Y)$.
- (5) $\mu(X) = \mu(\overline{X}^\omega) = \mu(\overline{\text{conv} X})$ where $\overline{X}^\omega$ is the weak closure of a set X and $\overline{\text{conv} X}$ is the closed convex hull (with respect to the norm topology) of X .
- (6) If $\{X_n\}$ is a sequence of nonempty, bounded (in norm), weakly closed subsets of $W^{n,1}([0, T]^N)$ and $X_1 \supset X_2 \supset \dots$ with $\lim_{n \rightarrow \infty} \mu(X_n) = 0$, then the intersection set $X_\infty := \bigcap_{n=1}^\infty X_n$ is nonempty and weakly compact in $W^{n,1}([0, T]^N)$.

Proof. Take $X \in \mathfrak{M}_{W^{n,1}([0, T]^N)}$ such that $\mu(X) = 0$. Then $\lim_{\varepsilon \rightarrow 0} C^T(X, \varepsilon) = 0$. Therefore, for any $\varrho > 0$ there exists $\delta > 0$ such that

$$\max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f| d\mu < \varrho \quad (13)$$

for all $\mathcal{D} \subset [0, T]^N$ with $m(\mathcal{D}) < \delta$ and all $f \in X$. So, using Proposition 4 X is relatively compact in the weak topology and $\ker \mu \subseteq \mathfrak{N}_{W^{n,1}([0, T]^N)}^\omega$. In order to verify $\ker \mu \supset \mathfrak{N}_{W^{n,1}([0, T]^N)}^\omega$, suppose that $X \in \mathfrak{N}_{W^{n,1}([0, T]^N)}^\omega$. Applying Proposition 4 for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f| d\mu < \varepsilon \quad (14)$$

for all $f \in X$ and $\mathcal{D} \subset [0, T]^N$ such that $m(\mathcal{D}) < \delta$. Now, for all $f \in X$ we have

$$\begin{aligned} C^T(f, \delta) &= \sup \left\{ \max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f| d\mu : \mathcal{D} \right. \\ &\quad \left. \subset [0, T]^N, m(\mathcal{D}) < \delta \right\} < \varepsilon. \end{aligned} \quad (15)$$

It enforces that $C^T(X, \delta) < \varepsilon$ and thus we get $\lim_{\delta \rightarrow 0} C^T(X, \delta) = 0$. It implies that $\mu(X) = 0$, and so $\mathfrak{N}_{W^{n,1}([0, T]^N)}^\omega \subset \ker \mu$. We conclude that (1) holds.

The fact that μ satisfies conditions (2)–(4) is obvious.

Now, we check that condition (5) holds. For this purpose, suppose that $X \in \mathfrak{M}_{W^{n,1}([0, T]^N)}$ and $f \in \overline{X}^\omega$. Thus, there is a sequence such that $f_m \rightarrow f$ weakly in $W^{n,1}([0, T]^N)$. On the other hand, by the definition of $C^T(X, \varepsilon)$ we have

$$\max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f_m| d\mu \leq C^T(X, \varepsilon) \quad (16)$$

for any $m \in \mathbb{N}$ and $\mathcal{D} \subset [0, T]^N$ with $m(\mathcal{D}) < \varepsilon$.

Letting $m \rightarrow \infty$ and using Fatou's Lemma we yield

$$\max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f| d\mu \leq C^T(X, \varepsilon) \quad (17)$$

for any $\mathcal{D} \subset [0, T]^N$ with $m(\mathcal{D}) < \varepsilon$.

Hence

$$\begin{aligned} C^T(\overline{X}^\omega, \varepsilon) &\leq C^T(X, \varepsilon), \\ \lim_{\varepsilon \rightarrow 0} C^T(\overline{X}^\omega, \varepsilon) &\leq \lim_{\varepsilon \rightarrow 0} C^T(X, \varepsilon). \end{aligned} \quad (18)$$

This implies that

$$\mu(\overline{X}^\omega) \leq \mu(X). \quad (19)$$

Moreover, we obtain by condition (2) that $\mu(\overline{X}^\omega) = \mu(X)$. Particularly, $\mu(\overline{\text{conv} X}^\omega) = \mu(\text{conv} X)$. By [11, Theorem 3.12] we have that

$$\mu(X) = \mu(\text{conv} X) = \mu(\overline{\text{conv} X}^\omega) = \mu(\overline{\text{conv} X}). \quad (20)$$

It remains to prove (6); suppose that $\{X_m\}$ is a sequence of the weakly closed and nonempty sets of $\mathfrak{M}_{W^{n,1}([0, T]^N)}$ such that $X_{m+1} \subset X_m$ for $m = 1, 2, \dots$, and $\lim_{m \rightarrow \infty} \mu(X_m) = 0$. Now, for any $m \in \mathbb{N}$, take $f_m \in X_m$. We claim that $\mathcal{B} = \{f_m\}^\omega$ is a weakly compact set in $W^{n,1}([0, T]^N)$. To prove this assertion, we need to check that $\mu(\mathcal{B}) = 0$. Let $\varepsilon > 0$ be fixed. Since $\lim_{m \rightarrow \infty} \mu(X_m) = 0$, then there exists $m_1 \in \mathbb{N}$ such that $\mu(X_{m_1}) < \varepsilon$.

Hence, we can find $\delta_1 > 0$ small enough that

$$C^T(X_{m_1}, \delta_1) < \varepsilon. \quad (21)$$

Thus, for all $m \geq m_1$, we have

$$\begin{aligned} \sup \left\{ \max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f_m| d\mu : \mathcal{D} \subset [0, T]^N, m(\mathcal{D}) < \delta_1 \right\} \\ < \varepsilon. \end{aligned} \quad (22)$$

As we know that the set $\{f_1, \dots, f_{m_1-1}\}$ is weakly compact. By Proposition 4, there exists $\delta_2 > 0$ such that

$$\max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f_i| d\mu < \varepsilon \quad (23)$$

for all $i = 1, \dots, m_1 - 1$ and $\mathcal{D} \subset [0, T]^N$ such that $m(\mathcal{D}) < \delta_2$. Therefore

$$\sup \left\{ \max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f_i| d\mu : i = 1, \dots, m_1 - 1, \mathcal{D} \subset [0, T]^N, m(\mathcal{D}) < \delta_2 \right\} < \varepsilon. \quad (24)$$

Using (22) and (24) we obtain $C^T(\mathcal{B}, \delta) < \varepsilon$ for $\delta = \min\{\delta_1, \delta_2\}$. It means that $\mu(\mathcal{B}) = C^T(\mathcal{B}) = 0$.

Thus, there is a subsequence such that $f_{m_j} \rightarrow f_0$ weakly in $\mathcal{B}$. Since $f_m \in X_m$, $X_{m+1} \subset X_m$, and X_m is weakly closed for all $m \in \mathbb{N}$, we get

$$f_0 \in \bigcap_{j=1}^{\infty} X_{m_j} = \bigcap_{m=1}^{\infty} X_m = X_{\infty}, \quad (25)$$

which completes the proof of (6). $\square$

4. The Topology τ_{loc}^ω and the Family of Measures of Noncompactness

In this section, we will define the topology τ_{loc}^ω in $W_{\text{loc}}^{n,1}$. Furthermore, we will introduce the family of measures of

noncompactness $\{\mu_T\}_{T \geq 0}$ in $W_{\text{loc}}^{n,1}$ with the topology τ_{loc}^ω and we will describe its properties.

Definition 6. The topology τ_{loc}^ω is the weakest topology on $W_{\text{loc}}^{n,1}$ such that all operators $\pi_T : W_{\text{loc}}^{n,1}(\mathbb{R}_+^N) \rightarrow W^{n,1}([0, T]^N)$ ($T \geq 0$) defined in (5) are continuous.

A base for τ_{loc}^ω is the collection of all sets of the form $\pi_T^{-1}(G)$, where $T \geq 0$ and G is an arbitrary open subset of $W^{n,1}([0, T]^N)$ with the weak topology. Also, there is the natural topological homeomorphism h from $W_{\text{loc}}^{n,1}$ onto $h(W_{\text{loc}}^{n,1}) \subseteq \prod_{T \geq 0} W^{n,1}([0, T]^N)$, given by formula

$$h(x) := \{\pi_T(x)\}_{T \geq 0} \quad \text{for } x \in W_{\text{loc}}^{n,1}. \quad (26)$$

In this note, we say that a subset $X \subset W_{\text{loc}}^{n,1}$ is bounded, if X is bounded for every seminorm $\|\cdot\|_{n,1}^T$, $T \geq 0$. Furthermore, we denote by $\mathfrak{M}_{W_{\text{loc}}^{n,1}}$ the family of all nonempty and bounded subsets of $W_{\text{loc}}^{n,1}$ and by $\mathfrak{N}_{W_{\text{loc}}^{n,1}}$ its subfamily consisting of all relatively compact sets in the topology τ_{loc}^ω .

Now, we are in a position to introduce a family of measures of noncompactness in $W_{\text{loc}}^{n,1}$ with the topology τ_{loc}^ω .

Definition 7. The family of mappings $\{\mu_T\}_{T \geq 0}$, $\mu_T : \mathfrak{M}_{W_{\text{loc}}^{n,1}} \rightarrow \mathbb{R}_+$, defined by formula

$$\mu_T(X) := \lim_{\varepsilon \rightarrow 0^+} \left\{ \sup_{f \in X} \left\{ \sup \left\{ \max_{|\alpha| \leq n} \int_{\mathcal{D}} |D^\alpha f| d\mu : \mathcal{D} \subset [0, T]^N, m(\mathcal{D}) \leq \varepsilon \right\} \right\} \right\}, \quad (27)$$

is said to be the family of measures of noncompactness in $W_{\text{loc}}^{n,1}$ with topology τ_{loc}^ω . Indeed,

$$\mu_T(X) = C^T(\pi_T(X)), \quad (28)$$

where C^T is defined in (12).

Theorem 8. The family of measures of noncompactness $\{\mu_T\}_{T \geq 0}$, $\mu_T : \mathfrak{M}_{W_{\text{loc}}^{n,1}} \rightarrow \mathbb{R}_+$ defined by (27), fulfills the following conditions:

- (1°) The family $\ker\{\mu_T\} := \{X \in \mathfrak{M}_{W_{\text{loc}}^{n,1}} : \mu_T(X) = 0 \text{ for } T \geq 0\}$ is nonempty and $\ker\{\mu_T\} = \mathfrak{N}_{W_{\text{loc}}^{n,1}}$.
- (2°) If $X \subset Y$, then $\mu_T(X) \leq \mu_T(Y)$ for $T \geq 0$.
- (3°) $\mu_T(\lambda X) = |\lambda| \mu_T(X)$ for $T \geq 0$.
- (4°) $\mu_T(X + Y) \leq \mu_T(X) + \mu_T(Y)$ for $T \geq 0$.
- (5°) $\mu_T(X) = \mu_T(\overline{X}^\omega) = \mu_T(\overline{\text{conv} X}^\omega)$ for $T \geq 0$ where $\overline{X}^\omega$ and $\overline{\text{conv} X}^\omega$ are the closure of the set X in the topology τ_{loc}^ω and the closed convex hull of the set X with respect to the topology τ_{loc}^ω , respectively.
- (6°) If $\{X_n\}$ is a sequence of nonempty, bounded, weakly closed subsets of $W_{\text{loc}}^{n,1}$ and $X_1 \supset X_2 \supset \dots$ with

$\lim_{n \rightarrow \infty} \mu_T(X_n) = 0$ for each $T \geq 0$ then the intersection set $X_\infty := \bigcap_{n=1}^{\infty} X_n$ is nonempty and compact in τ_{loc}^ω .

Proof. First we show that (1°) holds. To do this, take $X \in \ker\{\mu_T\}$. Therefore for each $T \geq 0$, $\mu_T(X) = C^T(\pi_T(X)) = 0$. Now, by using Lemma 5, $\overline{\pi_T(X)}^\omega$ is weakly compact in $W^{n,1}([0, T]^N)$ for $T \geq 0$. Applying Tychonoff's Theorem we conclude that $\prod_{T \geq 0} \overline{\pi_T(X)}^\omega$ is weakly compact in $\prod_{T \geq 0} W^{n,1}([0, T]^N)$.

On the other hand, it is easy to see that $\prod_{T \geq 0} (\pi_T(X)) = h(X)$. It in turns implies that

$$\prod_{T \geq 0} \overline{\pi_T(X)}^\omega = \overline{\prod_{T \geq 0} \pi_T(X)}^\omega = \overline{h(X)}^\omega = h(\overline{X}^\omega). \quad (29)$$

Now, since h is a homeomorphism, it follows that $\overline{X}^\omega$ is compact in τ_{loc}^ω and $X \in \mathfrak{N}_{W_{\text{loc}}^{n,1}}$.

Conversely, suppose that $X \in \mathfrak{N}_{W_{\text{loc}}^{n,1}}$. Thus, $\overline{X}^\omega$ is compact in τ_{loc}^ω , which implies that $\pi_T(\overline{X}^\omega)$ is weakly compact in $W^{n,1}([0, T]^N)$.

By Lemma 5 we have

$$\mu_T(X) = C^T(\pi_T(X)) \leq C^T(\pi_T(\overline{X}^\omega))$$

$$= \mu(\pi_T(\bar{X}^\omega)) = 0, \quad T \geq 0; \quad (30)$$

that is, $X \in \ker\{\mu_T\}$.

Parts (2°)–(4°) are easy to verify. To prove (5°), suppose that $X \subset W_{\text{loc}}^{n,1}$. Using again Lemma 5 and inclusion $\pi_T(\bar{X}^\omega) \subset \overline{\pi_T(X)}^\omega$, we get

$$\begin{aligned} C^T(\pi_T(X)) &= C^T(\overline{\pi_T(X)}^\omega) \geq C^T(\pi_T(\bar{X}^\omega)) \\ &\geq C^T(\pi_T(X)). \end{aligned} \quad (31)$$

Hence, $C^T(\pi_T(X)) = C^T(\pi_T(\bar{X}^\omega))$ and $\mu_T(X) = \mu_T(\bar{X}^\omega)$.

We know that $\pi_T(\bar{Y}^\omega) \subset \overline{\pi_T(Y)}$ for $Y \in W_{\text{loc}}^{n,1}$; therefore we have

$$\begin{aligned} C^T(\pi_T(X)) &\leq C^T(\pi_T(\text{conv} X)) \\ &\leq C^T(\pi_T(\overline{\text{conv} X}^\omega)) \\ &\leq C^T(\overline{\pi_T(\text{conv} X)}) = C^T(\overline{\text{conv} \pi_T(X)}) \\ &= C^T(\pi_T(X)). \end{aligned} \quad (32)$$

This means that $\mu_T(X) = \mu_T(\bar{X}^\omega) = \mu_T(\overline{\text{conv} X}^\omega)$.

It remains to prove (6°). Suppose that $\{X_n\}$ is a sequence of nonempty and weakly closed sets of $\mathfrak{M}_{W_{\text{loc}}^{n,1}}$ such that $X_{n+1} \subset X_n$ for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \mu_T(X_n) = 0$ for $T \geq 0$. By applying Lemma 5, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} C^T(\pi_T(X_n)) &= \lim_{n \rightarrow \infty} C^T(\pi_T(X_n)) \\ &= \lim_{n \rightarrow \infty} \mu_T(X_n) = 0. \end{aligned} \quad (33)$$

Therefore, by Lemma 5, we have that $\bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega$ is nonempty and weakly compact in $W^{n,1}([0, T]^N)$ for $T \geq 0$. It implies that $\prod_{T \geq 0} (\bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega)$ is nonempty and weakly compact.

It is routine to see that $\bigcap_{n=1}^\infty h(X_n) = \prod_{T \geq 0} (\bigcap_{n=1}^\infty \pi_T(X_n))$. Now, suppose that $\hat{\pi}_T : h(W_{n,1}^{\text{loc}}) \rightarrow W^{n,1}([0, T]^N)$ is a mapping defined by

$$\hat{\pi}_T(h(x)) = \pi_T(x). \quad (34)$$

By [12, Proposition 2.5.5], the family of all sets $\hat{\pi}_T^{-1}(U_T)$, where U_T is a neighborhood of $\pi_T(x)$ in the space $W^{n,1}([0, T]^N)$, is a base for the subspace $h(W_{n,1}^{\text{loc}})$ at the point $h(x)$. Also, we know that if f is a continuous mapping between two topological spaces, then

$$f\left(\bigcap_{n=1}^\infty \overline{B_n}\right) \subseteq \bigcap_{n=1}^\infty \overline{f(B_n)} \quad (35)$$

for all subsets B_n , $n = 1, 2, \dots$ contained in the domain of f .

Take $f = \hat{\pi}_T$ and $B_n = h(X_n)$, $n = 1, 2, \dots$. Then from (35) we have

$$\hat{\pi}_T\left(\bigcap_{n=1}^\infty \overline{h(X_n)}^\omega\right) \subseteq \bigcap_{n=1}^\infty \overline{\hat{\pi}_T(h(X_n))}^\omega = \bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega, \quad (36)$$

which implies that

$$\bigcap_{n=1}^\infty \overline{h(X_n)}^\omega \subseteq \prod_{T \geq 0} \bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega. \quad (37)$$

Conversely, let $\{f_T\} \in \prod_{T \geq 0} \bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega$. Then for each $T \geq 0$, $f_T \in \bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega = \bigcap_{n=1}^\infty \overline{\hat{\pi}_T(h(X_n))}^\omega$. Hence, for each $n \in \mathbb{N}$, $f_T \in \hat{\pi}_T(h(X_n))^\omega$ ($T \geq 0$).

Suppose that U_T is a neighborhood of f_T in the space $W^{n,1}([0, T]^N)$. Then

$$U_T \cap \hat{\pi}_T(h(X_n)) \neq \emptyset. \quad (38)$$

This implies that $\hat{\pi}_T(U_T) \cap h(X_n) \neq \emptyset$ and $\{f_T\} \in \overline{h(X_n)}^\omega$. Since this is true for each $n \in \mathbb{N}$, then

$$\{f_T\} \in \bigcap_{n=1}^\infty \overline{h(X_n)}^\omega. \quad (39)$$

Hence

$$\prod_{T \geq 0} \bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega \subseteq \bigcap_{n=1}^\infty \overline{h(X_n)}^\omega. \quad (40)$$

Therefore using (37) and (40) we get $\bigcap_{n=1}^\infty \overline{h(X_n)}^\omega = \prod_{T \geq 0} \bigcap_{n=1}^\infty \overline{\pi_T(X_n)}^\omega$ and so $\bigcap_{n=1}^\infty \overline{h(X_n)}^\omega$ is nonempty and weakly compact. Since h is a homeomorphism, we get

$$\bigcap_{n=1}^\infty \overline{h(X_n)}^\omega = \bigcap_{n=1}^\infty \overline{h(X_n)}^\omega = h\left(\bigcap_{n=1}^\infty \overline{X_n}^\omega\right); \quad (41)$$

hence, $\bigcap_{n=1}^\infty \overline{X_n}^\omega = \bigcap_{n=1}^\infty X_n = X_\infty$ is nonempty and compact in $W_{\text{loc}}^{n,1}$. $\square$

As an immediate consequence of Theorem 8(1°), we have the following corollary.

Corollary 9. *A subset $X \subset W_{\text{loc}}^{1,1}$ is relatively compact in the topology τ_{loc}^ω if and only if $\pi_T(X)$ is relatively weakly compact in $W^{1,1}([0, T]^N)$ for each $T > 0$.*

We recall that a subset $X \subset W^{1,1}([0, T]^N)$ is a.e. equicontinuous on $[0, T]^N$, if, for each $\varepsilon > 0$, there exists a closed subset $\mathcal{D}_\varepsilon \subset [0, T]^N$ such that $m(\mathcal{D}_\varepsilon') \leq \varepsilon$ and the restriction of all functions from X to the set $\mathcal{D}_\varepsilon$ form an equicontinuous family of functions. Furthermore, X is said to be countable a.e. equicontinuous on $[0, T]^N$ if every sequence $\{x_n\} \subset X$ has a subsequence $\{\bar{x}_n\}$, which is a.e. equicontinuous on $[0, T]^N$.

Using some ideas from [8] we have the following results.

Lemma 10. *Suppose that a subset $X \subset W^{1,1}([0, T]^N)$ is bounded in norm. Then X is relatively compact if and only if X*

is relatively weakly compact and countable a.e. equicontinuous on $[0, T]^N$.

Proof. We first suppose that X is a relatively compact subset of the Banach space $W^{1,1}([0, T]^N)$. Since the set X is also relatively weakly compact, then it suffices to show that X is countable a.e. equicontinuous. Regarding the proof of Corollary 9 of [13], X is totally bounded in $W^{1,1}([0, T]^N)$ if and only if X and $D^1 X = \{\partial f / \partial x_i : i = 1, \dots, N, f \in X\}$ are totally bounded in $L^1([0, T]^N)$. Then X and $D^1 X$ are relatively compact in $L^1([0, T]^N)$. From [8, Lemma 3.7], it follows that X and $D^1 X$ are countable a.e. equicontinuous on $[0, T]^N$ with respect to $\|\cdot\|_1^T$. It can be easily shown that X is countable a.e. equicontinuous on $[0, T]^N$ with respect to $\|\cdot\|_{1,1}^T$. Conversely, suppose that X is relatively weakly compact in $W^{1,1}([0, T]^N)$ and countable a.e. equicontinuous on $[0, T]^N$. Obviously, X and $D^1 X$ are countable a.e. equicontinuous on $[0, T]^N$ with respect to $\|\cdot\|_1^T$. Using Proposition 4 and Theorem 3, it yields that X and $D^1 X$ are relatively weakly compact in $L^1([0, T]^N)$. Applying again [8, Lemma 3.7], it obtains that X and $D^1 X$ are relatively compact in $L^1([0, T]^N)$ and so X is relatively compact in $W^{1,1}([0, T]^N)$. $\square$

Lemma 11. Suppose that a subset $X \subset W_{loc}^{1,1}$ is bounded. Then X is relatively compact in the topology $\tau_{loc}^{\mathcal{F}}$ if and only if X is relatively compact in the topology τ_{loc}^{ω} and X is countable a.e. equicontinuous on N -cells of $\mathbb{R}_+^N$.

$$x(t_1, \dots, t_N) = f\left((t_1, \dots, t_N), \int_0^{t_1} \dots \int_0^{t_N} \vartheta\left((t_1, \dots, t_N), (s_1, \dots, s_N), x(\xi(s_1, \dots, s_N)), \frac{\partial x}{\partial s_1}(\xi(s_1, \dots, s_N)), \dots, \frac{\partial x}{\partial s_N}(\xi(s_1, \dots, s_N)), T_0 x(s_1, \dots, s_N)\right) ds_1 \dots ds_N\right). \quad (42)$$

We start with some preliminary facts which will be needed further on.

Suppose that $J \subset \mathbb{R}^N$ is a given measurable subset and S is a metric space. We say that a function $f((t_1, \dots, t_N), x(t_1, \dots, t_N)) = f : J \times S \rightarrow \mathbb{R}$ satisfies Carathéodory conditions if it is measurable in $(t_1, \dots, t_N)$ for any $x \in S$ and is continuous in x for almost all $(t_1, \dots, t_N) \in J$.

Definition 13. Assume that $J \subset \mathbb{R}_+^N$ is a given set and $f : J \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a function satisfying the Carathéodory conditions. Then, to every positive real-valued function $x = x(t_1, \dots, t_N)$ which is measurable on J , we may assign the function $(Fx)(t_1, \dots, t_N) = f((t_1, \dots, t_N), x(t_1, \dots, t_N))$, $(t_1, \dots, t_N) \in J$. The operator F defined in such a way is said to be the superposition (or Nemytskii) operator generated by the function f . It is well known that the function Fx is also measurable on J .

Definition 14. We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is locally absolutely continuous if for each nonempty compact interval $[a, b]$, the restriction of f to $[a, b]$ is absolutely continuous.

Proof. Suppose that X is relatively compact in the topology $\tau_{loc}^{\mathcal{F}}$. According to Proposition 2, $\pi_T(X)$ is relatively compact in $W^{1,1}([0, T]^N)$ for each $T > 0$. Applying Lemma 10, we obtain $\pi_T(X)$ is relatively weakly compact for all $T > 0$ and X is countable a.e. equicontinuous on N -cells of $\mathbb{R}_+^N$. From Corollary 9, we conclude that X is relatively compact in the topology τ_{loc}^{ω} .

Conversely, suppose that X is relatively compact in the topology τ_{loc}^{ω} and X is countable a.e. equicontinuous on N -cells of $\mathbb{R}_+^N$. From Corollary 9, we deduce that $\pi_T(X)$ is relatively weakly compact for each $T > 0$. Hence from Lemma 10 and Proposition 2, we find that X is relatively compact in the topology $\tau_{loc}^{\mathcal{F}}$. $\square$

Lemma 12 (see [5]). Assume that (S, d) is a complete metric space. Let $X, X_n, n = 1, 2, \dots$ be nonempty subsets of S such that X_n are relatively compact for $n = 1, 2, \dots$ and $\lim_{n \rightarrow \infty} \mathbb{D}(X, X_n) = 0$, where $\mathbb{D}(\cdot, \cdot)$ denotes the Hausdorff distance between sets. Then the set X is also relatively compact.

5. Example of Applications of the Family of Measures of Noncompactness $\{\mu_T\}_{T \geq 0}$

In this section, we study the applicability of the family of measures of noncompactness $\{\mu_T\}_{T \geq 0}$ in the locally Sobolev space $W_{loc}^{1,1}$ with the topology τ_{loc}^{ω} . For this purpose, we investigate the existence solutions to a special nonlinear Volterra integrodifferential equation of the form

Now, let $\Omega \subset \mathbb{R}^N$. Recall that the space $AC = AC(\Omega)$ consists of all real-valued measurable functions f on Ω such that, for every line τ parallel to one of the s_i -coordinate axes $\{(s_1, \dots, s_N) : s_1 = \dots = s_{i-1} = s_{i+1} = \dots = s_N = 0\}$, f is locally absolutely continuous on $\Omega \cap \tau$.

Given a space X of measurable functions over Ω , we denote by $\overline{X}$ the space of all measurable functions f such that there exists $\bar{f} \in X$ with $f = \bar{f}$ almost everywhere in Ω . In particular, for $f \in \overline{AC}$ we denote by $(\partial/\partial t_i)f$ the derivative $(\partial/\partial t_i)\bar{f}$ where $\bar{f} \in AC$ and $f = \bar{f}$ almost everywhere in Ω .

Following Marcus and Mizel [14] we call a function f on $\Omega \times \mathbb{R}$ a locally absolutely continuous Carathéodory function if $f(\cdot, u)$ belongs to AC for every $u \in \mathbb{R}$, and $f(s, \cdot)$ is continuous and locally absolutely continuous for almost all $s \in \Omega$.

By using some facts from [15] we can prove the following theorem.

Theorem 15. Let $N > 1$ be a natural number and let $f : \mathbb{R}_+^N \times \mathbb{R} \rightarrow \mathbb{R}$ be a locally absolutely continuous Carathéodory

function. Then the superposition operator F generated by $f((t_1, \dots, t_N), x)$ maps the space $W_{loc}^{1,1}$ into itself and it is bounded and continuous with the Fréchet topology $\tau_{loc}^{\mathcal{F}}$ if and only if the partial derivative $(\partial f / \partial t_i)$ ($i = 1, \dots, N$) generates superposition operator from $L^{N/(N-1)}$ into L^1 , and the partial derivative $\partial f / \partial x$ generates a superposition operator from $L^{N/(N-1)}$ into L^∞ ; that is, for each $x \in W_{loc}^{1,1}$ and $i = 1, \dots, N$, we have

$$\left| \frac{\partial}{\partial t_i} f((t_1, \dots, t_N), x(t_1, \dots, t_N)) \right| \leq a(t_1, \dots, t_N) + b |x(t_1, \dots, t_N)|^{N/(N-1)}, \quad (43)$$

$$\left| \frac{\partial f}{\partial x}((t_1, \dots, t_N), x(t_1, \dots, t_N)) \right| \leq c(t_1, \dots, t_N) + d,$$

where $(t_1, \dots, t_N) \in \mathbb{R}_+^N$, $a \in L_{loc}^1(\mathbb{R}_+^N)$, $c \in L^\infty(\mathbb{R}_+^N)$, and $b, d \geq 0$.

Proof. See [15, Theorem 9.5] and comment after it (page 185) and apply it for the special case $p = q = 1$. $\square$

Finally, we proceed by establishing some facts concerning the linear Volterra integral operator in $W_{loc}^{1,1}$.

We denote by Δ the area $\Delta = \{(t_1, \dots, t_N), (s_1, \dots, s_N) : 0 \leq s_i \leq t_i\}$ and assume $\kappa : \Delta \rightarrow \mathbb{R}_+$ is a given function which is measurable with respect to both variables. For an arbitrary function $x \in W_{loc}^{n,1}$ put

$$\begin{aligned} Kx(t_1, \dots, t_N) &:= \int_0^{t_1} \cdots \int_0^{t_N} \kappa((t_1, \dots, t_N), (s_1, \dots, s_N)) \\ &\quad \cdot x(s_1, \dots, s_N) ds_1 \cdots ds_N, \end{aligned} \quad (44)$$

where $(t_1, \dots, t_N) \in \mathbb{R}_+^N$. The operator K is defined in such a way which is called the linear Volterra integral operator. The following lemma gives us a fundamental property of the Volterra integral operator (see [16]).

Lemma 16. *If the linear Volterra integral operator K maps $W_{loc}^{1,1}$ into itself, then it is continuous in the Fréchet metric topology $\tau_{loc}^{\mathcal{F}}$.*

Now, let us assume that $K_T := K|_{W^{n,1}([0, T]^N)}$; that is, K_T is the restriction of the operator K to the Banach space $W^{n,1}([0, T]^N)$. The operator $K_T : W^{n,1}([0, T]^N) \rightarrow W^{n,1}([0, T]^N)$ is continuous (cf. [16, 17]) and the norm of the linear operator K_T we denote by $\|K_T\|$ for $T \geq 0$.

Equation (42) will be discussed later under the following assumptions:

- (1) $N > 1$ is a natural number and $\xi : \mathbb{R}_+^N \rightarrow \mathbb{R}_+^N$ is a continuous function such that, for each $A \subset \mathbb{R}_+^N$, $\xi(A) \subset A$.
- (2) $f : \mathbb{R}_+^N \times \mathbb{R} \rightarrow \mathbb{R}$ is a locally absolutely continuous Carathéodory function and moreover, for each $x \in W_{loc}^{1,1}$, $(t_1, \dots, t_N) \in \mathbb{R}_+^N$, and $i = 0, 1, \dots, N$, we have

$$\left| \frac{\partial}{\partial t_i} f((t_1, \dots, t_N), x(t_1, \dots, t_N)) \right| \leq a(t_1, \dots, t_N) + b |x(t_1, \dots, t_N)|^{N/(N-1)}, \quad (45)$$

$$\left| \frac{\partial f}{\partial x}((t_1, \dots, t_N), x(t_1, \dots, t_N)) \right| \leq c(t_1, \dots, t_N) + d, \quad (46)$$

where $a \in L_{loc}^1(\mathbb{R}_+^N)$, $c \in L^\infty(\mathbb{R}_+^N)$ with $\bar{c} = \text{esssup}|c|$ and $b, d \geq 0$.

- (3) $T_0 : W_{loc}^{1,1} \rightarrow W_{loc}^{1,1}$ is a continuous operator and for each $x \in W_{loc}^{1,1}$, and $i = 0, 1, \dots, N$

$$\left| \frac{\partial}{\partial s_i} T_0 x(s_1, \dots, s_N) \right| \leq \left| \frac{\partial}{\partial s_i} x(s_1, \dots, s_N) \right|. \quad (47)$$

- (4) $\vartheta : \mathbb{R}_+^N \times \mathbb{R}_+^N \times \mathbb{R}^{N+2} \rightarrow \mathbb{R}_+$ satisfies Carathéodory conditions; that is, the function $\vartheta(\cdot, (t_1, \dots, t_N), (s_1, \dots, s_{N+2}))$ is measurable on $\mathbb{R}_+^N$ and the function $\vartheta((t_1, \dots, t_N), \cdot, \cdot)$ is continuous a.e. on $\mathbb{R}_+^N \times \mathbb{R}^{N+2}$.

- (5) $V : W_{loc}^{1,1} \rightarrow W_{loc}^{1,1}$ is an operator given by the formula

$$\begin{aligned} Vx(t_1, \dots, t_N) &:= \int_0^{t_1} \cdots \int_0^{t_N} \vartheta((t_1, \dots, t_N), \\ &\quad (s_1, \dots, s_N), x(\xi(s_1, \dots, s_N)), \frac{\partial x}{\partial s_1}(\xi(s_1, \dots, s_N)), \\ &\quad \dots, \frac{\partial x}{\partial s_N}(\xi(s_1, \dots, s_N)), \\ &\quad T_0 x(s_1, \dots, s_N)) ds_1 \cdots ds_N, \end{aligned} \quad (48)$$

where $x \in W_{loc}^{1,1}$ and it satisfies in the following conditions:

$$\begin{aligned} \text{(i)} \quad &(Vx(t_1, \dots, t_N))^{N/(N-1)} \\ &\leq \int_0^{t_1} \cdots \int_0^{t_N} \kappa((t_1, \dots, t_N), (s_1, \dots, s_N)) [a_1(\xi(s_1, \dots, s_N)) + M(x, s_1, \dots, s_N)] ds_1 \cdots ds_N, \end{aligned} \quad (49)$$

$$(ii) \quad \left| \frac{\partial Vx(t_1, \dots, t_N)}{\partial t_i} \right| \leq \int_0^{t_1} \cdots \int_0^{t_N} \kappa((t_1, \dots, t_N), (s_1, \dots, s_N)) [a_1(\xi(s_1, \dots, s_N)) + M(x, s_1, \dots, s_N)] ds_1 \cdots ds_N, \quad (50)$$

where $i = 1, \dots, N$, $a_1 : \mathbb{R}_+^N \rightarrow [1, \infty)$ is in $W_{loc}^{1,1}$, $\kappa : \mathbb{R}_+^N \rightarrow \mathbb{R}_+$ satisfies Carathéodory conditions, and

$$M(x, s_1, \dots, s_N) = \max \left\{ |x\xi(s_1, \dots, s_N)|, \left| \frac{\partial x}{\partial s_1} \xi(s_1, \dots, s_N) \right|, \dots, \left| \frac{\partial x}{\partial s_N} (\xi(s_1, \dots, s_N)) \right|, \right. \\ \left. |T_0 x(s_1, \dots, s_N)| \right\}. \quad (51)$$

(6) For all $T \geq 0$, $\|K_T\|(b + \bar{c} + d) < 1$.

We study the existence result in the following theorem.

Theorem 17. Under assumptions (1)–(6), (42) has at least one solution $x = x(t_1, \dots, t_N)$ which belongs to the space $W_{loc}^{1,1}$.

Proof. Let us consider the operator $H : W_{loc}^{1,1} \rightarrow W_{loc}^{1,1}$ defined by the right side of (42). Then we can assume that $H = FV$, where F is the superposition operator generated by f . Fix arbitrary $x \in W_{loc}^{1,1}$. For $i = 1, \dots, N$, we have

$$\begin{aligned} \frac{\partial}{\partial s_i} Hx(s_1, \dots, s_N) &= \frac{\partial}{\partial s_i} f((s_1, \dots, s_N), \\ Vx(s_1, \dots, s_N)) &= \frac{\partial f}{\partial s_i}((s_1, \dots, s_N), Vx(s_1, \dots, s_N)) \\ &+ \frac{\partial f}{\partial Vx}((s_1, \dots, s_N), \\ Vx(s_1, \dots, s_N)) &\frac{\partial Vx}{\partial s_i}(s_1, \dots, s_N). \end{aligned} \quad (52)$$

Hence,

$$\begin{aligned} &\max \left\{ \int_{[0,T]^N} \left| \frac{\partial}{\partial s_i} Hx(s_1, \dots, s_N) \right| ds_1 \cdots ds_N : i = 1, \dots, N \right\} \\ &\leq \max \left\{ \int_{[0,T]^N} \left| \frac{\partial f}{\partial s_i}((s_1, \dots, s_N), Vx(s_1, \dots, s_N)) \right| ds_1 \cdots ds_N : i = 1, \dots, N \right\} \\ &+ \max \left\{ \int_{[0,T]^N} \left| \frac{\partial f}{\partial Vx}((s_1, \dots, s_N), Vx(s_1, \dots, s_N)) \right| \left| \frac{\partial Vx}{\partial s_i}(s_1, \dots, s_N) \right| ds_1 \cdots ds_N : i = 1, \dots, N \right\}. \end{aligned} \quad (53)$$

On the other hand, by (45), (49), and condition (3) we have

$$\begin{aligned} &\max \left\{ \int_{[0,T]^N} \left| \frac{\partial f}{\partial s_i}((s_1, \dots, s_N), Vx(s_1, \dots, s_N)) \right| ds_1 \cdots ds_N : i = 1, \dots, N \right\} \\ &\leq \int_{[0,T]^N} |a(s_1, \dots, s_N)| + b |Vx(s_1, \dots, s_N)|^{N/(N-1)} ds_1 \cdots ds_N \\ &\leq \|a\|_1^T \\ &+ b \int_{[0,T]^N} \left(\int_0^{s_1} \cdots \int_0^{s_N} \kappa((s_1, \dots, s_N), (\tau_1, \dots, \tau_N)) [a_1(\xi(\tau_1, \dots, \tau_N)) + M(x, \tau_1, \dots, \tau_N)] d\tau_1 \cdots d\tau_N \right) ds_1 \cdots ds_N \\ &\leq \|a\|_1^T + b \int_{[0,T]^N} Ka_1(s_1, \dots, s_N) + b \int_{[0,T]^N} KM(x, s_1, \dots, s_N) ds_1 \cdots ds_N \leq \|a\|_1^T + b \|K_T\| (\|a\|_{1,1}^T + \|x\|_{1,1}^T), \end{aligned} \quad (54)$$

where $\|a\|_1^T = \int_{[0,T]^N} |a(s_1, \dots, s_N)| ds_1 \cdots ds_N$.

Applying (46), (50), and condition (3) we have

$$\max \left\{ \int_{[0,T]^N} \left| \frac{\partial f}{\partial Vx}((s_1, \dots, s_N), Vx(s_1, \dots, s_N)) \right| \left| \frac{\partial Vx}{\partial s_i}(s_1, \dots, s_N) \right| ds_1 \cdots ds_N : i = 1, \dots, N \right\}$$

$$\begin{aligned}
&\leq \max \left\{ \int_{[0,T]^N} \left| \frac{\partial Vx}{\partial s_i} (s_1, \dots, s_N) \right| (c(s_1, \dots, s_N) + d) ds_1 \cdots ds_N : i = 1, \dots, N \right\} \leq (\bar{c} + d) \\
&\cdot \int_{[0,T]^N} \left(\int_0^{s_1} \cdots \int_0^{s_N} \kappa((s_1, \dots, s_N), (\tau_1, \dots, \tau_N)) [a_1(\xi(\tau_1, \dots, \tau_N)) + M(x, \tau_1, \dots, \tau_N)] d\tau_1 \cdots d\tau_N \right) ds_1 \cdots ds_N \\
&\leq (\bar{c} + d) \int_{[0,T]^N} K[a_1(s_1, \dots, s_N) + M(x, s_1, \dots, s_N)] ds_1 \cdots ds_N \leq (\bar{c} + d) \|K_T\| (\|a_1\|_{1,1}^T + \|x\|_{1,1}^T).
\end{aligned} \tag{55}$$

Thus, combining (53), (54), and (55) it can be deduced that

$$\begin{aligned}
&\max \left\{ \int_{[0,T]^N} \left| \frac{\partial}{\partial s_i} Hx(s_1, \dots, s_N) \right| ds_1 \cdots ds_N : i = 1, \dots, N \right\} \leq \|a\|_1^T + (b + \bar{c} + d) \|K_T\| (\|a_1\|_{1,1}^T \\
&+ \|x\|_{1,1}^T).
\end{aligned} \tag{56}$$

In addition, from (45) and using the same techniques as above, one can show that

$$\begin{aligned}
&\int_{[0,T]^N} |Hx(s_1, \dots, s_N)| ds_1 \cdots ds_N \\
&\leq \|a\|_1^T + b \|K_T\| (\|a_1\|_{1,1}^T + \|x\|_{1,1}^T).
\end{aligned} \tag{57}$$

Now (56) and (57) imply that

$$\|Hx\|_{1,1}^T \leq A(T) + B(T) \|x\|_{1,1}^T, \tag{58}$$

where $A(T) = \|a\|_1^T + (b + \bar{c} + d) \|K_T\| \|a_1\|_{1,1}^T$ and $B(T) = (b + \bar{c} + d) \|K_T\|$.

Next, we define the nonnegative, nondecreasing function $r : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ and the subset B of $W_{\text{loc}}^{1,1}$ as follows:

$$r(t) := \frac{A(t)}{1 - B(t)}, \tag{59}$$

$$B := \{x \in W_{\text{loc}}^{1,1} : \|x\|_{1,1}^t \leq r(t) \text{ for } t \geq 0\}.$$

The set B is bounded and convex in $W_{\text{loc}}^{1,1}$. Inequality (58) ensures that H maps B into itself; that is,

$$H : B \longrightarrow B. \tag{60}$$

Furthermore, the set

$$\begin{aligned}
&\pi_T(B) \\
&= \{x \in W^{1,1}([0, T]^N) : \|x\|_{1,1}^t \leq r(t) \text{ for } t \in [0, T]\}
\end{aligned} \tag{61}$$

is convex and closed in the Banach space $W^{1,1}([0, T]^N)$, so it is weakly closed in $W^{1,1}([0, T]^N)$ for $T \geq 0$. Since

$$B = \bigcap_{T \geq 0} (\pi_T^{-1}(\pi_T(B))), \tag{62}$$

then B is closed in the topology τ_{loc}^ω ; that is,

$$\overline{B}^\omega = B. \tag{63}$$

Regarding condition (5) and applying majorant principle (cf. [16, 17]) we infer that the operator $V|_{W^{1,1}([0, T]^N)} : W^{1,1}([0, T]^N) \rightarrow W^{1,1}([0, T]^N)$ is continuous. In aid to this fact, together with Proposition 2, we conclude that

$$V : W_{\text{loc}}^{1,1} \longrightarrow W_{\text{loc}}^{1,1} \tag{64}$$

is continuous in the topology τ_{loc}^ω .

Further, take a nonempty subset X of the set B and arbitrary number $T \geq 0$. Fix a number $\varepsilon > 0$ and a nonempty subset $\mathcal{D} \subset [0, T]^N$ such that $\mathcal{D}$ is measurable and $m(\mathcal{D}) \leq \varepsilon$. Then, for arbitrarily fixed $x \in X$ we have

$$\begin{aligned}
&\max \left\{ \int_{\mathcal{D}} |D^i Hx(s_1, \dots, s_N)| ds_1 \cdots ds_N : i = 0, 1, \dots, N \right\} \leq \int_{\mathcal{D}} |a(s_1, \dots, s_N)| ds_1 \cdots ds_N \\
&+ (b + \bar{c} + d) \|K_T\| \left(\int_{\mathcal{D}} |a_1(s_1, \dots, s_N)| ds_1 \cdots ds_N \right. \\
&\left. + \int_{\mathcal{D}} M(x, s_1, \dots, s_N) ds_1 \cdots ds_N \right).
\end{aligned} \tag{65}$$

Taking into account absolute continuity of the integrals of the functions a and a_1 and letting $\varepsilon \rightarrow 0$, we get

$$\mu_T(HX) \leq \|K_T\| (b + \bar{c} + d) \mu_T(X). \tag{66}$$

Put $B_0 = B$, $B_n = \overline{\text{conv}H(B_{n-1})}^\omega$, $n = 1, 2, \dots$. In view of (60) and (63), we have $B_n \subset B_{n-1}$ for all $n \in \mathbb{N}$, by which together with (66) we get

$$\mu_T(B_n) \leq (\|K_T\| (b + \bar{c} + b))^n \mu_T(B), \quad T \geq 0. \tag{67}$$

It follows then from condition (6) that $\lim_{n \rightarrow \infty} \mu_T(B_n) = 0$. Due to condition (6°) in Theorem 8 we deduce that the set $B_\infty = \bigcap_{n=1}^\infty B_n$ is nonempty, convex, and compact in τ_{loc}^ω . This means

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{x \in B_{\infty}} \left\{ \sup \left\{ \max_{i=0, \dots, N} \int_{\mathcal{D}} |D^i x(t_1, \dots, t_N)| dt_1 \cdots dt_N : \mathcal{D} \subset [0, T]^N, m(\mathcal{D}) \leq \varepsilon \right\} \right\} \right\} = 0 \quad (68)$$

for each $T \geq 0$.

In the sequel, let us consider the sequence of the operators $P_n : B \rightarrow B$, $n = 1, 2, \dots$, defined by formula

$$(P_n x)(t_1, \dots, t_N) := \begin{cases} x(t_1, \dots, t_N), & |x(t_1, \dots, t_N)| \leq n, \\ n \operatorname{sgn}(x(t_1, \dots, t_N)), & |x(t_1, \dots, t_N)| > n. \end{cases} \quad (69)$$

It is readily verified that

$$\lim_{\varepsilon \rightarrow 0} \left\{ \sup_{x \in B_{\infty}} \left\{ \sup \left\{ \max_{|\alpha| \leq 1} \int_{\mathcal{D}} |D^\alpha VP_n x(s_1, \dots, s_N)| ds_1 \cdots ds_N : \mathcal{D} \subset [0, T]^N, \mu(\mathcal{D}) < \varepsilon \right\} \right\} \right\} = 0. \quad (71)$$

Let us fix a number $T > 0$ and let us put

$$A_{x,n} := \{(t_1, \dots, t_N) \in [0, T]^N : x(t_1, \dots, t_N) \neq (P_n x)(t_1, \dots, t_N)\} \quad \text{for } x \in B_{\infty}, n \in \mathbb{N}. \quad (72)$$

In view of (70) and condition (5) we derive that

$$\begin{aligned} & \int_{\mathcal{D}} |VP_n x(s_1, \dots, s_N)| ds_1 \cdots ds_N \\ &= \int_{A_{x,n} \cap \mathcal{D}} |VP_n x(s_1, \dots, s_N)| ds_1 \cdots ds_N \\ &+ \int_{\mathcal{D} - A_{x,n}} |Vx(s_1, \dots, s_N)| ds_1 \cdots ds_N \\ &\leq \int_{A_{x,n} \cap \mathcal{D}} [K(a_1(\xi(s_1, \dots, s_N))) \\ &+ M(P_n x, s_1, \dots, s_N)]^{(N-1)/N} ds_1 \cdots ds_N \\ &+ \int_{\mathcal{D} - A_{x,n}} [K(a_1(\xi(s_1, \dots, s_N))) \\ &+ M(x, s_1, \dots, s_N)]^{(N-1)/N} ds_1 \cdots ds_N \\ &\leq \int_{\mathcal{D}} [K(a_1(s_1, \dots, s_N)) \\ &+ M(x, s_1, \dots, s_N)]^{(N-1)/N} ds_1 \cdots ds_N \\ &\leq \|K_T\|^{(N-1)/N} \left[\int_{\mathcal{D}} a_1(s_1, \dots, s_N) ds_1 \cdots ds_N \right. \\ &\left. + \int_{\mathcal{D}} |D^\alpha x(s_1, \dots, s_N)| ds_1 \cdots ds_N \right]. \end{aligned} \quad (73)$$

$$|D^i P_n x(t_1, \dots, t_N)| \leq |D^i x(t_1, \dots, t_N)| \quad \text{for } i = 0, \dots, N. \quad (70)$$

We now apply Lemma 12 to prove VB_{∞} is relatively compact in $W_{\text{loc}}^{1,1}$ with the topology $\tau_{\text{loc}}^{\mathcal{F}}$. For the sake of convenience, we fulfill it in 2 steps.

Step 1. We are going to show that $VP_n B_{\infty}$ is relatively compact in the topology $\tau_{\text{loc}}^{\mathcal{F}}$. To do this, we apply Lemma 11. First we show that $\mu_T(VP_n B_{\infty}) = 0$; that is,

Taking into account absolute continuity of the integral of the function a_1 , it yields that

$$\begin{aligned} & \int_{\mathcal{D}} |VP_n x(s_1, \dots, s_N)| ds_1 \cdots ds_N \\ &\leq \|K_T\|^{(N-1)/N} \int_{\mathcal{D}} |D^\alpha x(s_1, \dots, s_N)| ds_1 \cdots ds_N. \end{aligned} \quad (74)$$

On the other hand, using again (70) and condition (5) we have that

$$\begin{aligned} & \int_{\mathcal{D}} \left| \frac{\partial}{\partial s_i} VP_n x(s_1, \dots, s_N) \right| ds_1 \cdots ds_N \\ &= \int_{A_{x,n} \cap \mathcal{D}} \left| \frac{\partial}{\partial s_i} VP_n x(s_1, \dots, s_N) \right| ds_1 \cdots ds_N \\ &+ \int_{\mathcal{D} - A_{x,n}} \left| \frac{\partial}{\partial s_i} Vx(s_1, \dots, s_N) \right| ds_1 \cdots ds_N \\ &\leq \|K_T\| \max_{|\alpha| \leq 1} \left\{ \int_{\mathcal{D}} |D^\alpha x(s_1, \dots, s_N)| ds_1 \cdots ds_N \right\} \\ &\quad \text{for } i = 1, \dots, n. \end{aligned} \quad (75)$$

Regarding (74) and (75) we obtain

$$\begin{aligned} \mu_T(VP_n B_{\infty}) &\leq \max \{ \|K_T\|, \|K_T\|^{(N-1)/N} \} \mu_T(B_{\infty}) \\ &= 0, \quad \text{for } T \geq 0. \end{aligned} \quad (76)$$

Hence $\mu_T(VP_n B_{\infty}) = 0$, for $T \geq 0$. It enforces that the bounded set $VP_n B_{\infty}$ is relatively compact in the topology $\tau_{\text{loc}}^{\omega}$. It remains to prove that $VP_n B_{\infty}$ is countably a.e. equicontinuous on N -cells. Let us fix $T > 0$ and $\varepsilon > 0$. According to Luzin's Theorem, there is a closed subset $\mathcal{D}_{\varepsilon} \subset [0, T]^N$ such that $m(\mathcal{D}_{\varepsilon}') \leq \varepsilon$, and the functions $\vartheta|_{\mathcal{D}_{\varepsilon} \times \mathbb{R}_+^N \times \mathbb{R}^{N+2}}$ and $\kappa|_{\mathcal{D}_{\varepsilon} \times \mathbb{R}_+^N}$ are continuous.

Let us take arbitrary n -tuples $(t_1, \dots, t_N), (s_1, \dots, s_N) \in \mathbb{R}_+^N$ such that, for each $i = 1, \dots, N$, $s_i > t_i$. Then, for $x \in B_\infty$, by condition (5) we have

$$|VP_n x(s_1, \dots, s_N) - VP_n x(t_1, \dots, t_N)| \leq T^N \omega^T(\vartheta,$$

$$\|(s_1, \dots, s_N) - (t_1, \dots, t_N)\|) \times [\|a_1\|_{1,1}^T + \|x\|_{1,1}^T]$$

$$+ \int_{t_1}^{s_1} \dots \int_{t_N}^{s_N} \vartheta((s_1, \dots, s_N), (\tau_1, \dots, \tau_N),$$

$$P_n x(\xi(\tau_1, \dots, \tau_N)), \frac{\partial P_n x}{\partial \tau_1}(\xi(\tau_1, \dots, \tau_N)), \dots,$$

$$\frac{\partial P_n x}{\partial \tau_N}(\xi(\tau_1, \dots, \tau_N)), T_0 P_n x(\tau_1, \dots,$$

$$\tau_N)) d\tau_1 \dots d\tau_N \leq T^N \omega^T(\vartheta,$$

$$\|(s_1, \dots, s_N) - (t_1, \dots, t_N)\|) \times [\|a_1\|_{1,1}^T + \|x\|_{1,1}^T]$$

$$+ \left(\int_0^{s_1} \dots \int_0^{s_N} \kappa((s_1, \dots, s_N), (\tau_1, \dots, \tau_N)) \right.$$

$$\cdot [a_1(\xi(\tau_1, \dots, \tau_N))$$

$$+ M(P_n x, \tau_1, \dots, \tau_N)] d\tau_1 \dots d\tau_N \Big)^{(N-1)/N}$$

$$\leq T^N \omega^T(\vartheta, \|(s_1, \dots, s_N) - (t_1, \dots, t_N)\|)$$

$$\times [\|a_1\|_{1,1}^T + \|x\|_{1,1}^T] + \left(\bar{\kappa} \int_0^{s_1} \dots \int_0^{s_N} a_1(\tau_1, \dots,$$

$$\tau_N) d\tau_1 \dots d\tau_N + \bar{\kappa}$$

$$\begin{aligned} & \cdot \max \left\{ \int_0^{s_1} \dots \int_0^{s_N} |D^j x(\tau_1, \dots, \tau_N)| d\tau_1 \dots d\tau_N : i \right. \\ & \left. = 0, \dots, N \right\} \Big)^{(N-1)/N}, \end{aligned} \quad (77)$$

where $\omega^T(\vartheta, \cdot)$ denotes the modulus of continuity of the function ϑ on the set $\mathcal{D}_\varepsilon \times [0, T]^N \times [-n, n]^{N+2}$ and

$$\begin{aligned} \bar{\kappa} = \max \{ & \kappa((t_1, \dots, t_N), (s_1, \dots, s_N)) : \\ & ((t_1, \dots, t_N), (s_1, \dots, s_N)) \in \mathcal{D}_\varepsilon \times [0, T]^N \}. \end{aligned} \quad (78)$$

From uniform continuity of the function ϑ on the compact set $\mathcal{D}_\varepsilon \times [0, T]^N \times [-n, n]^{N+2}$, absolute continuity of the integral of the function a_1 , and $\mu_T(B_\infty) = 0$, we yield

$$|VP_n x(s_1, \dots, s_N) - VP_n x(t_1, \dots, t_N)|, \quad (79)$$

is arbitrarily small provided the number $\|(s_1, \dots, s_N) - (t_1, \dots, t_N)\|$ is small enough. This proves that $VP_n B_\infty$ is countably a.e. equicontinuous on N -cells.

Note that if $(t_1, \dots, t_N), (s_1, \dots, s_N) \in \mathbb{R}_+^N$ are arbitrary, then it is enough to break down the integration intervals into suitable subintervals and applying the same reasoning as above, to see that $VP_n B_\infty$ is countably a.e. equicontinuous on N -cells. We then conclude by applying Lemma 11 that $VP_n B_\infty$ is relatively compact in $\tau_{\text{loc}}^\mathcal{F}$.

Step 2. We are going to prove that $\lim_{n \rightarrow \infty} \mathbb{D}(VB_\infty, VP_n B_\infty) = 0$, where $\mathbb{D}(\cdot, \cdot)$ denotes the Hausdorff distance in the Fréchet metric space $(W_{\text{loc}}^{1,1}, d)$. Since all functions of the set B_∞ are uniformly bounded in $\|\cdot\|_{1,1}^T$,

$$\lim_{n \rightarrow \infty} \sup \{m(A_{x,n}) : x \in B_\infty\} = 0. \quad (80)$$

Using our assumptions, for arbitrarily fixed $x \in B_\infty$, we have

$$\begin{aligned} \|Vx - VP_n x\|_{1,1}^T &= \max_{|i| \leq 1} \int_{[0,T]^N} |D^i (Vx(t_1, \dots, t_N) - VP_n x(t_1, \dots, t_N))| dt_1 \dots dt_N \\ &= \max_{|i| \leq 1} \int_{A_{x,n}} |D^i (Vx(t_1, \dots, t_N) - VP_n x(t_1, \dots, t_N))| dt_1 \dots dt_N \leq \max_{|i| \leq 1} \int_{A_{x,n}} |D^i Vx(t_1, \dots, t_N)| dt_1 \dots dt_N \\ &+ \max_{|i| \leq 1} \int_{A_{x,n}} |D^i VP_n x(t_1, \dots, t_N)| dt_1 \dots dt_N \\ &\leq \max \left\{ \int_{A_{x,n}} [K(a_1(\xi(t_1, \dots, t_N)) + M(x, t_1, \dots, t_N))]^{(N-1)/N} dt_1 \dots dt_N, \int_{A_{x,n}} [K(a_1(\xi(t_1, \dots, t_N)) + M(x, t_1, \dots, t_N))] dt_1 \dots dt_N \right\} \\ &+ \max \left\{ \int_{A_{x,n}} [K(a_1(\xi(t_1, \dots, t_N)) + M(P_n x, t_1, \dots, t_N))]^{(N-1)/N} dt_1 \dots dt_N, \int_{A_{x,n}} [K(a_1(\xi(t_1, \dots, t_N)) + M(P_n x, t_1, \dots, t_N))] dt_1 \dots dt_N \right\} \leq 2 \|K_T\| \int_{A_{x,n}} [a_1(t_1, \dots, t_N) \\ &+ M(x, t_1, \dots, t_N)] dt_1 \dots dt_N. \end{aligned} \quad (81)$$

By absolute continuity of integral of function a_1 and applying (68) and (80) we derive that

$$\lim_{n \rightarrow \infty} \sup_{x \in B_\infty} \|Vx - VP_n x\|_{1,1}^T = 0, \quad \text{for } T \geq 0. \quad (82)$$

This equality together with inclusion $VP_n B_\infty \subset VB_\infty$ implies that

$$\lim_{n \rightarrow \infty} \mathbb{D}(VB_\infty, VP_n B_\infty) = 0. \quad (83)$$

Regarding (83), Step 1, and Lemma 12 we derive that VB_∞ is also relatively compact in $\tau_{loc}^{\mathcal{F}}$. Taking into account the continuity of $F : W_{loc}^{1,1} \rightarrow W_{loc}^{1,1}$ (see Theorem 15), we get that $HB_\infty = FVB_\infty$ is also relatively compact in $\tau_{loc}^{\mathcal{F}}$. Let us take $Y = \overline{\text{conv}HB_\infty}^{\mathcal{F}}$. We observe that Y is compact and convex subset of the space $W_{loc}^{1,1}$ with the topology $\tau_{loc}^{\mathcal{F}}$. From continuity of V and in view of Theorem 15 the operator $H = FV$ transforms continuously the set Y into itself. So, applying

$$x(t_1, t_2) = \frac{1}{(t_1^2 + 1)(t_2^2 + 1)} + \arctan \left(\frac{1}{100} t_1^3 t_2^3 + \frac{1}{100} t_1 t_2 \int_0^{t_1} \int_0^{t_2} \sin^2 \sqrt{(x(s_1, s_2))^{10} + e^{\partial x(s_1, s_2)/\partial s_1} + \cosh \left(\frac{\partial x(s_1, s_2)}{\partial s_2} \right) + \frac{1}{4} (x(s_1, s_2))^2 ds_1 ds_2} \right). \quad (84)$$

Equation (84) is a special case of (42) with

$$\xi(t_1, t_2) = (t_1, t_2),$$

$$f(t_1, t_2, x) = \frac{1}{(t_1^2 + 1)(t_2^2 + 1)} + \arctan x,$$

$$T_0(x) = \frac{1}{2}x,$$

$$\vartheta((t_1, t_2), (s_1, s_2), (x_1, x_2, x_3, x_4))$$

the classical Tychonoff fixed point principle we conclude that there is a solution $x \in B_\infty \subset W_{loc}^{1,1}$ of (42). This completes the proof of our theorem. $\square$

Now, we present an example which verifies the effectiveness and applicability of Theorem 17.

Example 18. Consider the following Volterra integrodifferential equation:

$$= \frac{1}{100} \left(t_1^2 t_2^2 + t_1 t_2 \sin^2 \sqrt{x_1^{10} + e^{x_2} + \cosh x_3 + x_4^2} \right),$$

$$\kappa((t_1, t_2), (s_1, s_2)) = \frac{1}{5}. \quad (85)$$

It is easy to see that ξ satisfies in condition (1) of Theorem 17. Also, f satisfies Carathéodory conditions and if we define $a(t_1, t_2) = 2(t_1 + t_2)/(t_1^2 + 1)(t_2^2 + 1)$, $b = d = 1/1000$, and $c(t_1, t_2) = 1$, then condition (2) of Theorem 17 holds. Moreover, let us take

$$Vx(t_1, t_2) = \frac{1}{100} t_1^3 t_2^3 + \frac{1}{100} t_1 t_2 \int_0^{t_1} \int_0^{t_2} \sin^2 \sqrt{(x(s_1, s_2))^{10} + e^{\partial x(s_1, s_2)/\partial s_1} + \cosh \left(\frac{\partial x(s_1, s_2)}{\partial s_2} \right) + \frac{1}{4} (x(s_1, s_2))^2 ds_1 ds_2}, \quad (86)$$

$$a_1(s_1, s_2) = 36(s_1 s_2)^5 + 50(s_1 s_2)^4 + 16(s_1 s_2)^3 + 18(s_1 s_2^2 + s_1^2 s_2) + 4(s_1 + s_2) + 1.$$

Then we have

$$(Vx)^2 \leq \frac{1}{10^4} ((t_1 t_2)^6 + 2(t_1 t_2)^5 + (t_1 t_2)^4),$$

$$\left| \frac{\partial Vx}{\partial t_1} \right| \leq \frac{1}{100} (3t_1^2 t_2^3 + 2t_1 t_2^2) \quad (87)$$

$$\left| \frac{\partial Vx}{\partial t_2} \right| \leq \frac{1}{100} (3t_1^3 t_2^2 + 2t_1^2 t_2).$$

It can be easily shown that condition (5) of Theorem 17 holds, too. On the other hand, $\|K_T\|(b + \bar{c} + d) = 1002/5000 < 1$.

Consequently, all the conditions of Theorem 17 are satisfied. It implies that the integrodifferential equation (84) has at least one solution in the space $W_{loc}^{1,1}$.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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