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Research Article

Rectangular Metric-Like Type Spaces and Related Fixed Points

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In this paper, we introduce the concept of a rectangular metric-like space, along with its topology, and we prove some fixed point theorems for different contraction types. We also introduce the concept of modified metric-like spaces and we prove some topological and convergence properties under the symmetric convergence. Some examples are given to illustrate the new introduced metric type spaces.

1. Introduction

The generalization of Banach contraction principle, which has many applications in different branches of science and engineering, depends on either generalizing the metric type space or the contractive type mapping (see [1] and the references therein). The generalization of a metric space is based on reducing or modifying the metric axioms; for example, we cite quasi-metrics, partial metrics, $m-$metrics, S_p metrics, rectangular metrics, $b-$metrics. For more details, see [2–15]. Note that losing or weakening some of the metric axioms causes the loss of some topological properties, hence bringing obstacles in proving some fixed point theorems. These obstacles force researchers to develop new techniques in the development of fixed point theory in order to resolve more real concrete applications. In this article, we restrict ourselves on developing metric-like spaces by introducing modified metric-like spaces, rectangular metric-like spaces, and rectangular modified metric-like spaces. We shall prove some fixed point theorems in rectangular metric-like spaces. Examples will be given to support the given concepts. The notion of symmetric convergence will be also studied in the setting of modified metric-like spaces.

Definition 1 ([13] (partial metric space)). Let X be a nonempty set. A mapping $p : X \times X \longrightarrow \mathbb{R}^+$ is said to be a partial metric on X, if, for any $x, y, z \in X$, it satisfies the following conditions:

(p_1) $x = y$ if and only if $p(x, y) = p(x, x) = p(y, y)$;
(p_2) $p(x, x) \leq p(x, y)$;
(p_3) $p(x, y) = p(y, x)$;
(p_4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

In this case, the pair (X, p) is called a partial metric (PM) space.

Definition 2 ([9] (rectangular (or Branciari) metric space)). Let X be a nonempty set. A mapping $d : X \times X \longrightarrow \mathbb{R}^+$ is said to be a rectangular metric on X if, for any $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$, it satisfies the following conditions:

(R_1) $x = y$ if and only if $d(x, y) = 0$;
(R_2) $d(x, y) = d(y, x)$;
(R_3) $d(x, y) \leq d(x, u) + d(u, v) + d(v, y)$.

In this case, the pair (X, d) is called a rectangular metric (RM) space.

In [15], the notion of a rectangular metric space was extended to rectangular partial metric spaces as follows.

Definition 3 ([15] (rectangular partial metric space)). Let X be a nonempty set. A mapping $\rho : X \times X \longrightarrow \mathbb{R}^+$ is said to be a rectangular metric on X if, for any $x, y \in X$ and all distinct points $u, v \in X \setminus \{x, y\}$, it satisfies the following conditions:

(RP_1) $x = y$ if and only if $\rho(x, y) = \rho(x, x) = \rho(y, y)$;

(RP_2) $\rho(x, x) \le \rho(x, y)$;

(RP_3) $\rho(x, y) = \rho(y, x)$;

(RP_4) $\rho(x, y) \le \rho(x, u) + \rho(u, v) + \rho(v, y) - \rho(u, u) - \rho(v, v)$.

In this case, the pair (X, ρ) is called a rectangular partial metric (RPM) space.

It is clear that every rectangular metric space is a rectangular partial metric space, but the converse is not true.

Example 4 (see [15]). Let $X = [0, a]$ and $\alpha \ge a \ge 3$. Define the mapping $\rho : X \times X \longrightarrow \mathbb{R}^+$ by

$$\rho(x, y) = \begin{cases} x & \text{if } x = y \\ \dfrac{3\alpha + x + y}{2} & \text{if } x, y \in \{1, 2\},\ x \ne y \\ \dfrac{\alpha + x + y}{2} & \text{otherwise}. \end{cases} \quad (1)$$

Then (X, ρ) is a rectangular partial metric space, but it is not a rectangular metric space, because, for any $x > 0$, we have $\rho(x, x) = x \ne 0$.

For convergence, completeness, and examples of RM, PM, and RPM spaces, we refer to [9, 13, 15]. See also the papers [16, 17].

Definition 5 (see [11]). Let X be a nonempty set. A mapping $\sigma : X \times X \longrightarrow \mathbb{R}^+$ is said to be a metric-like on X if, for any $x, y, z \in X$, it satisfies the following conditions:

(σ_1) $\sigma(x, y) = 0$ implies $x = y$;

(σ_2) $\sigma(x, y) = \sigma(y, x)$;

(σ_3) $\sigma(x, y) \le \sigma(x, z) + \sigma(z, y)$.

In this case, the pair (X, σ) is called a metric-like space (ML-space).

Every metric-like space is a topological space whose topology is generated by the base consisting of the open σ−balls

$$B_\sigma(x, \delta) = \{y \in X : |\sigma(x, y) - \sigma(x, x)| < \delta\}, \quad x \in X,\ \delta > 0. \quad (2)$$

Note the difference between the balls $B_\sigma(x, \delta)$ and the balls $B_p(x, \delta)$, which is due to the absence of the smallness of the self distance condition $(p2)$ from the metric-like. Also, since the self distance need not be zero in metric-like spaces, then convergence and completeness in metric-like spaces still resemble the case of partial metric spaces. Indeed, a sequence $\{x_n\}$ in a metric-like space converges to a point $x \in X$ if and only if $\lim_{n\longrightarrow\infty}\sigma(x_n, x) = \sigma(x, x)$ and the sequence $\{x_n\}$ is called σ−Cauchy if $\lim_{m,n\longrightarrow\infty}\sigma(x_n, x_m)$ exists and is finite. The metric-like space (X, σ) is called complete if for each σ−Cauchy sequence $\{x_n\}$ there exists $x \in X$ such that

$$\lim_{n\longrightarrow\infty}\sigma(x_n, x) = \sigma(x, x) = \lim_{m,n\longrightarrow\infty}\sigma(x_n, x_m). \quad (3)$$

Remark 6. Metric-like spaces lose some topological and convergence properties as known for metric spaces. We state the following. For example, limits are not unique in ML-spaces. Take $X = \{a, b\}$ and let $\sigma(x, y) = 1$ for any $x, y \in X$. Then, clearly the constant sequence $\{x_n = 1\}$ converges to both a and b. Notice that $\sigma(a, a) = \sigma(b, b) = 1 \ne 0$. However, if $x_n \longrightarrow x$ and $x_n \longrightarrow y$ such that $x, y \in \Lambda = \{z \in X : \sigma(z, z) = 0\}$, then $\sigma(x, y) \le \sigma(x_n, x) + \sigma(x_n, y)$. Letting $n \longrightarrow \infty$, we conclude that $\sigma(x, y) = 0$, and hence $x = y$.

2. Main Results

Upon Remark 6 above, we define the following modified metric-like (mML) space.

Definition 7 (modified metric-like space). Let X be a nonempty set. A mapping $\sigma_m : X \times X \longrightarrow \mathbb{R}^+$ is said to be a modified metric-like on X, if for any $x, y, z \in X$, it satisfies the following conditions:

$(m\sigma_1)$ $\sigma_m(x, y) = 0$ implies $x = y$;

$(m\sigma_2)$ $\sigma_m(x, y) = \sigma_m(y, x)$;

$(m\sigma_3)$ $\sigma_m(x, y) \le \sigma_m(x, z) + \sigma_m(z, y) - \sigma_m(z, z)$.

The pair (X, σ_m) is called a modified metric-like space (mML-space).

It is clear that every partial metric space is an mML−space and every mML−space is an ML−space.

Example 8. Let $X = \{1, 2, 3, 4, 5\}$ and define the mapping $d : X \times X \longrightarrow \mathbb{R}^+$ such that $d(x, y) = 2$ for all $x \ne y$, $d(x, x) = 0$ for all $x \ne 1$, and $d(1, 1) = 3$. It is clear that the conditions (σ_1) and (σ_2) in Definition 5 are satisfied. We need to verify (σ_3). If $x \ne y$, then we have $d(x, u) + d(u, y) = 2 + 2 \ge d(x, y)$. Also, if $x = y = 1$, then $d(1, u) + d(u, 1) = 2 + 2 \ge d(1, 1)$. Finally, if $x = y \ne 1$, $d(x, u) + d(u, x) \ge 0 = d(x, x)$. Therefore, (X, d) is a ML-space, but it is not a mML−space, because $d(2, 1) + d(1, 3) - d(1, 1) = 2 + 2 - 3 = 1 \le d(2, 3)$.

As in the case of metric-like spaces, the open σ_m−balls in modified metric-like spaces are given as

$$B_{\sigma_m}(x, \delta) = \{y \in X : |\sigma_m(x, y) - \sigma_m(x, x)| < \delta\}, \quad x \in X,\ \delta > 0. \quad (4)$$

Also, note that a sequence $\{x_n\}$ in a modified metric-like space (X, σ_m) converges to a point $x \in X$ if and only if $\lim_{n\longrightarrow\infty}\sigma_m(x_n, x) = \sigma_m(x, x)$ and the sequence $\{x_n\}$ is called σ_m−Cauchy if $\lim_{n,l\longrightarrow\infty}\sigma_m(x_n, x_l)$ exists and is finite.

Definition 9 (symmetrical convergence in modified metric–like spaces). We shall say that a sequence $\{x_n\}$ in a modified metric-like space (X,σ_m) is symmetrically convergent to $x \in X$ if, for every $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that, for each $n \geq n_0$, we have

$$x_n \in B_{\sigma_m}(x,\epsilon) \text{ and } x \in B_{\sigma_m}(x_n,\epsilon). \tag{5}$$

Equivalently,

$$\lim_{n\longrightarrow\infty}\sigma_m(x_n,x) = \lim_{n\longrightarrow\infty}\sigma_m(x_n,x_n) = \sigma_m(x,x). \tag{6}$$

We shall denote $x_n \longrightarrow^s x$ for symmetrical convergence. It is clear that symmetrical convergence implies σ_m-convergence. Our first result is as follows.

Theorem 10. *Let (X,σ_m) be an* mML*-space. Then one has the following:*

(1) *If $x_n \longrightarrow^s x$, then $\{x_n\}$ is σ_m-Cauchy.*

(2) *If $\{x_n\}$ is σ_m-Cauchy and has a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \longrightarrow^s x$, then $x_n \longrightarrow^s x$.*

(3) *If $\{x_n\}$ and $\{y_n\}$ are σ_m-Cauchy sequences, then $\lim_{n\longrightarrow\infty}\sigma_m(x_n,y_n)$ exists.*

(4) *If $x_n \longrightarrow^s x$ and $y_n \longrightarrow^s y$, then $\lim_{n\longrightarrow\infty}\sigma_m(x_n,y_n) = \sigma(x,y)$.*

Proof.

(1) Assume $x_n \longrightarrow^s x$. Then $\lim_{n\longrightarrow\infty}\sigma_m(x_n.y_n) = \sigma(x,y)$. By $(m\sigma_3)$, for each $l,n \in \mathbb{N}$ we have

$$\sigma_m(x_n,x_l) \leq \sigma_m(x_n,x) + \sigma_m(x,x_l) - \sigma_m(x,x) \tag{7}$$

and

$$\begin{aligned}\sigma_m(x,x) &\leq \sigma_m(x,x_n) + \sigma_m(x_n,x_l) + \sigma_m(x_l,x)\\ &\quad - \sigma_m(x_n,x_n) - \sigma_m(x_l,x_l).\end{aligned} \tag{8}$$

Let $l,n \longrightarrow \infty$. Then $\sigma_m(x,x) \leq \lim_{n,l\longrightarrow\infty}\sigma_m(x_l,x_n) \leq \sigma_m(x,x)$. Hence $\lim_{n,l\longrightarrow\infty}\sigma_m(x_l,x_n) = \sigma_m(x,x)$, and so $\{x_n\}$ is σ_m-Cauchy.

(2) Let $\{x_n\}$ be a σ_m-Cauchy and have a subsequence $\{x_{n_i}\}$ such that $x_{n_i} \longrightarrow^s x$. Then

$$\lim_{i\longrightarrow\infty}\sigma_m(x_{n_i},x) = \lim_{i\longrightarrow\infty}\sigma_m(x_{n_i},x_{n_i}) = \sigma_m(x,x). \tag{9}$$

Since $\{x_n\}$ is σ_m–Cauchy, there exists $r > 0$ such that $\lim_{l,n\longrightarrow\infty}\sigma_m(x_n,x_l) = r$. It is clear that $\sigma_m(x,x) = r$ as well. On the other hand, by $(m\sigma_3)$, we have

$$\begin{aligned}\sigma_m(x_n,x) &\leq \sigma_m(x_n,x_{n_i}) + \sigma_m(x_{n_i},x)\\ &\quad - \sigma_m(x_{n_i},x_{n_i}),\end{aligned} \tag{10}$$

and

$$\sigma_m(x_{n_i},x) \leq \sigma_m(x_{n_i},x_n) + \sigma_m(x_n,x) - \sigma_m(x_n,x_n). \tag{11}$$

Therefore,

$$\begin{aligned}&\sigma_m(x_{n_i},x) - \sigma_m(x_{n_i},x_n) + \sigma_m(x_n,x_n) \leq \sigma_m(x_n,x)\\ &\quad\leq \sigma_m(x_n,x_{n_i}) + \sigma_m(x_{n_i},x) - \sigma_m(x_{n_i},x_{n_i}).\end{aligned} \tag{12}$$

If we take $n,i \longrightarrow \infty$, then $\sigma_m(x,x) - r + r \leq \lim_{n,l\longrightarrow\infty}\sigma_m(x_n,x_l) \leq r + \sigma_m(x,x) - \sigma_m(x,x)$. It follows that $\lim_{n\longrightarrow\infty}\sigma_m(x_n,x) = \lim_{n\longrightarrow\infty}\sigma_m(x_n,x_n) = \sigma_m(x,x)$, and hence $x_n \longrightarrow^s x$.

(3) Assume that $\{x_n\}$ and $\{y_n\}$ are σ_m-Cauchy sequences in X. Then there exist $r_1,r_2 > 0$ such that $\lim_{n,l\longrightarrow\infty}\sigma_m(x_n,x_l) = r$ and $\lim_{n,l\longrightarrow\infty}\sigma_m(y_n,y_l) = r_2$. It is sufficient to prove that the sequence $\{\sigma_m(x_n,y_n)\}$ is Cauchy in $\mathbb{R}$. By $(m\sigma_3)$, for each $n,l \in \mathbb{N}$, we have

$$\begin{aligned}\sigma_m(x_n,y_n) &\leq \sigma_m(x_n,x_l) + \sigma_m(x_l,y_l) + \sigma_m(y_l,x_n)\\ &\quad - \sigma_m(x_l,x_l) - \sigma_m(y_l,y_l),\end{aligned} \tag{13}$$

and

$$\begin{aligned}\sigma_m(x_l,y_l) &\leq \sigma_m(x_l,x_n) + \sigma_m(x_n,y_n) + \sigma_m(y_n,y_l)\\ &\quad - \sigma_m(x_n,x_n) - \sigma_m(y_n,y_n).\end{aligned} \tag{14}$$

From what is mentioned, it follows that

$$\begin{aligned}&\sigma_m(x_n,x_n) + \sigma_m(y_n,y_n) - \sigma_m(x_l,x_n) - \sigma_m(y_n,y_l)\\ &\quad\leq \sigma_m(x_n,y_n) - \sigma_m(x_l,y_l)\\ &\quad\leq \sigma_m(x_n,x_l) + \sigma_m(y_l,y_n) - \sigma_m(x_l,x_l)\\ &\qquad - \sigma_m(y_l,y_l).\end{aligned} \tag{15}$$

Let $n,l \longrightarrow \infty$, then

$$\begin{aligned}r_1 + r_2 - r_1 - r_2 &\leq \lim_{n,l\longrightarrow\infty}(\sigma_m(x_n,y_n) - \sigma_m(x_l,y_l)\\ &\leq r_1 + r_2 - r_1 - r_2.\end{aligned} \tag{16}$$

Hence $|\sigma_m(x_n,y_n) - \sigma_m(x_l,y_l)| = 0$, and so $\{\sigma_m(x_n,y_n)\}$ is Cauchy in $\mathbb{R}$.

(4) Assume that $x_n \longrightarrow^s x$ and $y_n \longrightarrow^s y$. Then

$$\lim_{n\longrightarrow\infty}\sigma_m(x_n,x) = \lim_{n\longrightarrow\infty}\sigma_m(x_n,x_n) = \sigma_m(x,x), \tag{17}$$

and

$$\lim_{n\longrightarrow\infty}\sigma_m(y_n,y) = \lim_{n\longrightarrow\infty}\sigma_m(y_n,y_n) = \sigma_m(y,y). \tag{18}$$

Now, for each $n \in \mathbb{N}$, we have

$$\begin{aligned}\sigma_m(x_n,y_n) &\leq \sigma_m(x_n,x) + \sigma_m(x,y) + \sigma_m(y,y_n)\\ &\quad - \sigma_m(x,x) - \sigma_m(y,y),\end{aligned} \tag{19}$$

and

$$\sigma_m(x,y) \le \sigma_m(x,x_n) + \sigma_m(x_n,y_n) + \sigma_m(y_n,y) - \sigma_m(x_n,x_n) - \sigma_m(y_n,y_n). \tag{20}$$

Finally, letting $n \longrightarrow \infty$ yields that

$$\underline{\lim}_{n\longrightarrow\infty}\sigma_m(x_n,y_n) \le \sigma_m(x,y) \le \underline{\lim}_{n\longrightarrow\infty}\sigma_m(x_n,y_n), \tag{21}$$

and thus $\lim_{n\longrightarrow\infty}\sigma(x_n,y_n) = \sigma(x,y)$.

□

Now, we introduce the concepts of rectangular metric-like and rectangular modified metric-like spaces.

Definition 11. Let X be a nonempty set and $\rho_r : X^2 \longrightarrow [0,\infty)$ be a function. If the following conditions are satisfied for all x, y in X,

(1) $\rho_r(x,y) = 0 \Longrightarrow x = y$;

(2) $\rho_r(x,y) = \rho_r(y,x)$;

(3) $\rho_r(x,y) \le \rho_r(x,u) + \rho_r(u,v) + \rho_r(v,y)$, for all distinct $u, v \in X \setminus \{x,y\}$,

then the pair (X,ρ_r) is called a rectangular metric-like (RML) space.

Definition 12. Let X be a nonempty set and $\rho_{mr} : X^2 \longrightarrow [0,\infty)$ be a function. If the following conditions are satisfied for all x, y in X,

(1) $\rho_{rm}(x,y) = 0 \Longrightarrow x = y$;

(2) $\rho_{rm}(x,y) = \rho_{rm}(y,x)$;

(3) $\rho_{rm}(x,y) \le \rho_{rm}(x,u) + \rho_{rm}(u,v) + \rho_{rm}(v,y) - \rho_{rm}(u,u) - \rho_{rm}(v,v)$, for all distinct $u, v \in X \setminus \{x,y\}$,

then the pair (X,ρ_{rm}) is called a rectangular modified metric-like (RMML) space.

Example 13. Let $X = \{1,2,3,4,5\}$ and define the mapping $\rho_r : X^2 \longrightarrow [0,\infty)$ by

$$\rho_r(x,y) = \begin{cases} 2.5 & \textit{for } x \ne y \\ 5 & \textit{if } x = y = 1 \\ 0 & \textit{otherwise.} \end{cases} \tag{22}$$

It is clear that conditions (1) and (2) in Definition 11 are satisfied. We need to verify the last condition. For all distinct $u, v \in X \setminus \{x,y\}$, we have $\rho_r(x,u) + \rho_r(u,v) + \rho_r(v,y) = 2.5 + \rho_r(u,v) + 2.5 = 5 + \rho_r(u,v) \ge \rho_r(x,y)$, for all $x, y \in X$. Therefore, (X,ρ_r) is a RML-space, but it is not a RMML-space. Indeed, $\rho_r(2,1) + \rho_r(1,1) + \rho_r(1,3) - \rho_r(1,1) - \rho_r(1,1) = 2.5 + 5 + 2.5 - 5 - 5 = 0 \le \rho_r(2,3) = 2.5$. Moreover, the space (X,ρ_r) is not a rectangular partial metric, because the condition (RP_4) of Definition 2 does not hold.

Example 14. Let (X,ρ_r) be a RML-space. Consider $\rho_{rm}(x,y) = \rho_r(x,y) + a$ for $a > 0$. Then the space (X,ρ_{rm}) is a RMML-space.

Example 15. Let $X = (0,1)$ and define the mapping $\rho_{mr} : X^2 \longrightarrow [0,\infty)$ by $\rho_{mr}(x,y) = |x-y| + 2$. Then (X,ρ_{rm}) is a RMML-space. Indeed, for any $x, y \in X$ and distinct $u, v \in X\setminus\{x,y\}$, we have $\rho_{mr}(x,u)+\rho_{mr}(u,v)+\rho_{mr}(v,y)-\rho_{mr}(u,u)-\rho_{mr}(v,v) = \rho_r(x,u)+\rho_r(u,v)+\rho_r(v,y)-\rho_r(u,u)-\rho_r(v,v)+\alpha \le \rho_r(x,y) - \rho_r(u,u) - \rho_r(v,v) + \alpha \le \rho_r(x,y) + \alpha = \rho_{rm}(x,y)$.

Definition 16.

(1) A sequence $\{x_n\}$ is called ρ_r−convergent (resp., ρ_{rm}−convergent) in a rectangular metric-like space (X,ρ_r) (resp., a rectangular modified metric-like space (X,ρ_{rm})), if there exists $x \in X$ such that $\lim_{n\longrightarrow\infty}\rho_r(x_n,x) = \rho_r(x,x)$
(resp. $\lim_{n\longrightarrow\infty}\rho_{rm}(x_n,x) = \rho_{rm}(x,x)$).

(2) A sequence $\{x_n\}$ is called ρ_r−Cauchy if and only if $\lim_{n,m\longrightarrow\infty}\rho_r(x_n,x_m)$
(resp., $\lim_{n,p\longrightarrow\infty}\rho_{rm}(x_n,x_p)$) exists and is finite.

(3) A rectangular metric-like space (X,ρ_r) is called ρ_r−complete (resp., ρ_{rm}−complete) if, for every ρ_r−Cauchy (resp., ρ_{rm}−Cauchy) sequence $\{x_n\}$ in X, there exists $x \in X$, such that $\lim_{n\longrightarrow\infty}\rho_r(x_n,x) = \lim_{n,m\longrightarrow\infty}\rho_r(x_n,x_m) = \rho_r(x,x)$
(resp., $\lim_{n\longrightarrow\infty}\rho_{rm}(x_n,x) = \lim_{n,p\longrightarrow\infty}\rho_{rm}(x_n,x_p) = \rho_{rm}(x,x)$).

Remark 17. The convergence defined in Definition 16 is the convergence obtained in the sense of the topology generated by the open balls $B_\varrho(x,\delta) = \{y \in X : |\varrho(x,y) - \varrho(x,x)| < \delta\}$, $x \in X$, $\varrho \in \{\rho_r, \rho_{rm}\}$. This convergence is weaker than the symmetric convergence discussed before.

Definition 18 (continuity of maps). Let (X,ϱ_1) and (Y,ϱ_2) be two metric type (such as ML, MML, PM, RPM, RML, RMML) spaces. The mapping $f : X \longrightarrow Y$ is said (sequentially) continuous at $x \in X$ if and only if $f(x_n) \overset{\varrho_2}{\longrightarrow} f(x)$, whenever $\{x_n\}$ is a sequence in X such that $x_n \overset{\varrho_1}{\longrightarrow} x$.

Our second main result concerns an existence and uniqueness theorem on rectangular metric-like spaces.

Theorem 19. *Let (X,ρ_r) be a ρ_r−complete rectangular metric-like space and T be a self mapping on X. If there exists $0 < k < 1$ such that*

$$\rho_r(Tx,Ty) \le k\rho_r(x,y) \quad \textit{for all } x, y \in X, \tag{23}$$

then T has a unique fixed point u in X such that $\rho_r(u,u) = 0$.

Proof. Let x in X be arbitrary. Using (23), we have

$$\rho_r\left(T^n x, T^{n+1}x\right) \le k\rho_r\left(T^{n-1}x, T^n x\right) \le \cdots \le k^n\rho_r(x,Tx), \tag{24}$$

for all $n \ge 1$. We distinguish two cases.

Case 1. Let $T^n x = T^m x$ for some integers $n \neq m$. For example, take $m > n$. We have $T^{m-n}(T^n x) = T^n x$. Choose $y = T^n x$ and $p = m - n$. Then

$$T^p y = y, \tag{25}$$

that is, y is a periodic point of T. By (23) and (24), we have

$$\rho_r(y, Ty) = \rho_r\left(T^p y, T^{p+1} y\right) \leq k^p \rho_r(y, Ty). \tag{26}$$

Since $k \in (0,1)$, we get $\rho_r(y, Ty) = 0$, so $y = Ty$; that is, y is a fixed point of T.

Case 2. Suppose that $T^n x \neq T^m x$ for all integers $n \neq m$.

We rewrite (24) as

$$\rho_r\left(T^n x, T^{n+1} x\right) \leq k^n \rho_r(x, Tx) \leq \frac{k^n}{1-k} \rho_r(x, Tx). \tag{27}$$

Similarly, by (23), we have

$$\begin{aligned} \rho_r\left(T^n x, T^{n+2} x\right) &\leq k \rho_r\left(T^{n-1} x, T^{n+1} x\right) \leq \cdots \\ &\leq k^n \rho_r\left(x, T^2 x\right) \leq \frac{k^n}{1-k} \rho_r\left(x, T^2 x\right). \end{aligned} \tag{28}$$

Now, if $m > 2$ is odd, then consider $m = 2p + 1$ with $p \geq 1$. By (23) and (27), we have

$$\begin{aligned} &\rho_r\left(T^n x, T^{n+m} x\right) \\ &\quad \leq \rho_r\left(T^n x, T^{n+1} x\right) + \rho_r\left(T^{n+1} x, T^{n+2} x\right) + \cdots \\ &\qquad + \rho_r\left(T^{n+2p} x, T^{n+2p+1} x\right) \\ &\quad \leq k^n \rho_r(x, Tx) + k^{n+1} \rho_r(x, Tx) + \cdots \\ &\qquad + k^{n+2p} \rho_r(x, Tx) \\ &\quad = k^n \rho_r(x, Tx)\left[1 + k + k^2 + \cdots + k^{2p}\right] \\ &\quad \leq \frac{k^n}{1-k} \rho_r(x, Tx). \end{aligned} \tag{29}$$

On the other hand, if $m > 2$ is even, then consider $m = 2p$ with $p \geq 2$. Again, by (23), (27), and (28),

$$\begin{aligned} \rho_r\left(T^n x, T^{n+m} x\right) &\leq \rho_r\left(T^n x, T^{n+2} x\right) \\ &\quad + \rho_r\left(T^{n+2} x, T^{n+3} x\right) + \cdots \\ &\quad + \rho_r\left(T^{n+2p-1} x, T^{n+2p} x\right) \\ &\leq k^n \rho_r\left(x, T^2 x\right) + k^{n+2} \rho_r(x, Tx) \\ &\quad + k^{n+3} \rho_r(x, Tx) + \cdots \\ &\quad + k^{n+2p-1} \rho_r(x, Tx) \\ &\leq k^n \rho_r\left(x, T^2 x\right) + \frac{k^{n+2}}{1-k} \rho_r(x, Tx) \\ &\leq k^n \rho_r\left(x, T^2 x\right) + \frac{k^n}{1-k} \rho_r(x, Tx). \end{aligned} \tag{30}$$

We deduce from all cases that

$$\rho_r\left(T^n x, T^{n+m} x\right) \leq k^n \rho_r\left(x, T^2 x\right) + \frac{k^n}{1-k} \rho_r(x, Tx) \quad \text{for all } n, m \geq 0. \tag{31}$$

The right-hand side tends to 0 as $n \longrightarrow \infty$, so the sequence $\{T^n x\}$ is ρ_r−Cauchy in the ρ_r−complete rectangular metric-like space (X, ρ_r). Due to Definition 16, there exists some $u \in X$ such that

$$\lim_{n \longrightarrow \infty} \rho_r\left(T^n x, u\right) = \lim_{n,m \longrightarrow \infty} \rho_r\left(T^n x, T^m x\right) = \rho_r(u, u). \tag{32}$$

In view of (31), we get

$$\begin{aligned} \rho_r(u, u) &= \lim_{n \longrightarrow \infty} \rho_r\left(T^n x, u\right) = \lim_{n,m \longrightarrow \infty} \rho_r\left(T^n x, T^m x\right) \\ &= 0. \end{aligned} \tag{33}$$

We shall prove that $Tu = u$. Mention that we are still in case 2, that is, $T^n x \neq T^m x$ for all integers $n \neq m$. Now, we distinguish three subcases.

Subcase 1. If, for all $n \geq 0$, $T^n x \notin \{u, Tu\}$, the rectangular inequality implies that

$$\begin{aligned} \rho_r(u, Tu) &\leq \rho_r\left(u, T^n x\right) + \rho_r\left(T^n x, T^{n+1} x\right) \\ &\quad + \rho_r\left(T^{n+1} x, Tu\right) \\ &\leq \rho_r\left(u, T^n x\right) + \rho_r\left(T^n x, T^{n+1} x\right) \\ &\quad + k \rho_r\left(T^n x, u\right). \end{aligned} \tag{34}$$

Taking limit as $n \longrightarrow \infty$ and using (27) and (33), we get $\rho_r(u, Tu) = 0$; that is, $Tu = u$.

Subcase 2. If there exists an integer N such that $T^N x = u$, due to case 2, $T^n x \neq u$ for all $n > N$. Similarly, $T^n x \neq Tu$ for all $n > N$. We reach subcase 1, so u is a fixed point of T.

Subcase 3. If there exists an integer N such that $T^N x = Tu$, again, necessarily $T^n x \neq u$ and $T^n x \neq Tu$ for all $n > N$. Similarly, we get $Tu = u$.

We deduce that u is a fixed point of T. To show the uniqueness of the fixed point u, assume that T has another fixed point v. By (23),

$$\rho_r(u, v) = \rho_r(Tu, Tv) \leq k \rho_r(u, v), \tag{35}$$

which holds unless $\rho_r(u, v) = 0$, so $u = v$. □

Next, we present the following example.

Example 20. Let $X = \{1, 2, 3, 4, 5\}$ and define the mapping $\rho_r : X^2 \longrightarrow [0, \infty)$ by

$$\rho_r(x, y) = \begin{cases} 2.5 & \text{for } x \neq y \neq 1 \\ 5 & \text{if } x = y = 1 \\ 3 & \text{if } x = 1 \text{ or } y = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{36}$$

It is not difficult to see that (X, ρ_r) is a complete rectangular metric-like space. Let T be a self mapping on X defined by $T2 = T3 = T4 = T5 = 2$ and $T1 = 3$. Note that T satisfies the contraction of Theorem 19 with $k = 29/30$, and 2 is the unique fixed point of T.

Our third main result is as follows.

Theorem 21. *Let (X, ρ_r) be a ρ_r−complete rectangular metric-like space and T be a self mapping on X. If there exists $0 < k < 1$ such that*

$$\rho_r(Tx, Ty) \le k \max\{\rho_r(x, Tx), \rho_r(y, Ty)\} \quad \text{for all } x, y \in X, \tag{37}$$

then T has a unique fixed point u in X such that $\rho_r(u, u) = 0$.

Proof. Fix x in X. If $\rho_r(T^n x, T^{n+1}x) = 0$ for some n, then x_n is a fixed point of T. The proof is completed. From now on, we assume that $\rho_r(T^n x, T^{n+1}x) > 0$, so $T^n x \ne T^{n+1}x$ for all n. By (37), we have

$$\begin{aligned} 0 &< \rho_r\left(T^n x, T^{n+1}x\right) \\ &\le k \max\left\{\rho_r\left(T^{n-1}x, T^n x\right), \rho_r\left(T^n x\ , T^{n+1}x\right\}, \right. \end{aligned} \tag{38}$$

for all $n \ge 1$. If, for some $n \ge 1$, $\max\{\rho_r(T^{n-1}x, T^n x), \rho_r(T^n x, T^{n+1}x\} = \rho_r(T^n x, T^{n+1}x\}$, then

$$0 < \rho_r\left(T^n x, T^{n+1}x\right) \le k\rho_r\left(T^n x, T^{n+1}x\right), \tag{39}$$

which is a contradiction. Thus,

$$\begin{aligned} 0 < \rho_r\left(T^n x, T^{n+1}x\right) &\le k\rho_r\left(T^{n-1}x, T^n x\right) \\ &\le k^n \rho_r(x, Tx), \quad \text{for all } n \ge 1. \end{aligned} \tag{40}$$

Here, the sequence $\{\rho_r(T^n x, T^{n+1}x)\}$ is nonincreasing. As the proof of Theorem 19, we distinguish two cases.

Case 1. Let $T^n x = T^m x$ for some integers $n \ne m$ (take $m - n \ge 2$). Let $y = T^n x = T^p y$, where $p = m - n$. Again, using (40),

$$\rho_r(y, Ty) = \rho_r\left(T^p y, T^{p+1}y\right) \le k^p \rho_r(y, Ty). \tag{41}$$

Since $k \in (0, 1)$, we get $\rho_r(y, Ty) = 0$, so y is a fixed point of T.

Case 2. Suppose that $T^n x \ne T^m x$ for all integers $n \ne m$.

We rewrite (40) as

$$\rho_r\left(T^n x, T^{n+1}x\right) \le k^n \rho_r(x, Tx) \le \frac{k^n}{1-k}\rho_r(x, Tx). \tag{42}$$

Similarly, by (37) and (42), we have

$$\begin{aligned} &\rho_r\left(T^n x, T^{n+2}x\right) \\ &\quad \le k \max\left\{\rho_r\left(T^{n-1}x, T^n x\right), \rho_r\left(T^{n+1}x, T^{n+2}x\right)\right\} \\ &\quad \le k \max\left\{k^{n-1}\rho_r(x, Tx), k^{n+1}\rho_r(x, x)\right\} \\ &\quad = k^n \rho_r(x, Tx) \le \frac{k^n}{1-k}\rho_r(x, Tx). \end{aligned} \tag{43}$$

Now, if $m > 2$ is odd, then consider $m = 2p + 1$ with $p \ge 1$. By (37) and (42), we have

$$\begin{aligned} &\rho_r\left(T^n x, T^{n+m}x\right) \\ &\quad \le \rho_r\left(T^n x, T^{n+1}x\right) + \rho_r\left(T^{n+1}x, T^{n+2}x\right) + \cdots \\ &\qquad + \rho_r\left(T^{n+2p}x, T^{n+2p+1}x\right) \\ &\quad \le k^n \rho_r(x, Tx)\left[1 + k + k^2 + \cdots + k^{2p}\right] \\ &\quad \le \frac{k^n}{1-k}\rho_r(x, Tx). \end{aligned} \tag{44}$$

On the other hand, if $m > 2$ is even, then consider $m = 2p$ with $p \ge 2$. Again, by (37), (42),

$$\begin{aligned} \rho_r\left(T^n x, T^{n+m}x\right) &\le \rho_r\left(T^n x, T^{n+2}x\right) \\ &\quad + \rho_r\left(T^{n+2}x, T^{n+3}x\right) + \cdots \\ &\quad + \rho_r\left(T^{n+2p-1}x, T^{n+2p}x\right) \\ &\le k^n \rho_r(x, Tx) + k^{n+2}\rho_r(x, Tx) \\ &\quad + k^{n+3}\rho_r(x, Tx) + \cdots \\ &\quad + k^{n+2p-1}\rho_r(x, Tx) \\ &\le \frac{k^n}{1-k}\rho_r(x, Tx). \end{aligned} \tag{45}$$

We deduce from all cases that

$$\rho_r\left(T^n x, T^{n+m}x\right) \le \frac{k^n}{1-k}\rho_r(x, Tx) \quad \text{for all } n, m \ge 0. \tag{46}$$

The right-hand side tends to 0 as $n \longrightarrow \infty$, so the sequence $\{T^n x\}$ is ρ_r−Cauchy in the ρ_r−complete rectangular metric-like space (X, ρ_r). Due to Definition 16, there exists some $u \in X$ such that, in view of (46),

$$\begin{aligned} \rho_r(u, u) &= \lim_{n\longrightarrow\infty}\rho_r\left(T^n x, u\right) = \lim_{n,m\longrightarrow\infty}\rho_r\left(T^n x, T^m x\right) \\ &= 0. \end{aligned} \tag{47}$$

We shall prove that $Tu = u$. Without loss of generality, we may assume that, for all $n \ge 0$, $T^n x \notin \{u, Tu\}$. By the rectangular inequality, we have

$$\begin{aligned} \rho_r(u, Tu) &\le \rho_r\left(u, T^n x\right) + \rho_r\left(T^n x, T^{n+1}x\right) \\ &\quad + \rho_r\left(T^{n+1}x, Tu\right) \\ &\le \rho_r\left(u, T^n x\right) + \rho_r\left(T^n x, T^{n+1}x\right) \\ &\quad + k \max\left\{\rho_r\left(T^n x, T^{n+1}x\right), \rho(u, Tu)\right\}. \end{aligned} \tag{48}$$

Taking limit as $n \longrightarrow \infty$ and using (42) and (47), we get $\rho_r(u, Tu) = 0$, that is, $Tu = u$.

Now, let v be a fixed point of T. From (37),

$$\rho_r(u, v) = \rho_r(Tu, Tv) \le k\rho_r(u, v). \tag{49}$$

It is true unless $\rho_r(v, v) = 0$, so u is the unique fixed point of T. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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