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*Research Article*

# On Some Convergence Properties of the Modified Ishikawa Scheme for Asymptotic Demicontractive Self-Mappings with Matricial Parameterizing Sequences

**M. De la Sen**

*Institute of Research and Development of Processes (IIDP), Facultad de Ciencia y Tecnologia, Universidad del País Vasco, Leioa (Bizkaia), P.O. Box 644, 48080 Bilbao, Spain*

Correspondence should be addressed to M. De la Sen; manuel.delasen@ehu.eus

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This paper is focused on the modified Ishikawa iterative scheme by admitting that the parameterizing sequences might be vectors of distinct components. It is also assumed that the auxiliary self-mapping which supports the iterative scheme is asymptotically demicontractive.

## 1. Introduction

The study of iterative methods such as Krasnoselsky, Mann, and Ishikawa iterations as well as a large variety of extensions and their convergence properties have received special attention in the last decades. A detailed collection of existing results and new ones together with a very detailed discussion and comparative results between them is given in [1]. Ishikawa and Mann iterative schemes involve the use of an auxiliary self-mapping $T$ driven by one parameterizing sequence of scalars, say $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$ (Mann iterative scheme), or two ones, say $\{\alpha_n\}_{n=0}^{\infty}$, $\{\beta_n\}_{n=0}^{\infty} \subset [0,1]$ (Ishikawa iterative scheme). It has been proved in [2] that if any of the Ishikawa and Mann iterations converges to a fixed point of the auxiliary self-mapping $T$, then the other one converges to the same fixed point. The associated iterations are also compared to Picard's type iterations in [2]. See also [3] and some references therein. It turns out that fixed points of certain self-mappings are very relevant in stability studies since they are equilibrium points of differential systems, difference systems, or dynamic continuous-time and discrete-time systems. Therefore, their characterization and the study of their properties as attractors are of a major importance in the modelling issues of biological systems, epidemic models, and mechanical, electrical, and control systems, in general. In the context of sequences generated from iterative schemes involving an auxiliary self-mapping $T$, the usual assumptions being invoked are basically that $T$ is nonexpansive and defined on a nonempty closed convex subset of a normed space and that the sequences of parameters which generate the iterations are in $[0,1]$ and converge to zero and that the one being common to both schemes is a summable sequence. Further studies are developed in [4] for the case that two multivalued mappings are used as auxiliary mappings in the multivalued version of the Ishikawa iterative scheme. It is discussed in [1] that those iterations can be more robust against certain numerical errors than the Picard iterations. However, it is proved in [5] that, if the auxiliary self-mapping is a Zamfirescu operator [1], then Picard iteration through such an operator converges faster than those iterative schemes and than Mann's iteration which converges faster than Ishikawa iteration in the framework of nonempty convex closed subsets of Banach spaces. The Mann and Ishikawa iterations under a class of errors when the auxiliary mapping is strongly accretive are studied in [6]. On the other hand, the Ishikawa iterative scheme is investigated in [7] when the auxiliary mapping is quasi-contractive and the convergence properties are linked to Kannan contractions, Chatterjea-type contractions, and Zamfirescu operators since all of them are always quasi-contractive. The convergence

properties of the scheme are investigated in [8] under generalized nonexpansive mappings, while it was pointed out that although nonexpansive mappings are quasi-nonexpansive, if they have a fixed point, the converse is not true in general. Some close further formal results leading to several fixed point theorems concerning Ishikawa's schemes are also given in [9]. On the other hand, the properties of convergence of the Ishikawa iteration are discussed in [10, 11] where the auxiliary self-maps include two auxiliary multivalued self-maps with common fixed points which are strongly pseudocontractive in the first case and satisfy a concrete general contractive condition in the second one. Also, it is found in [12] that continuous monotone and generalized quasi-nonexpansive self-mappings on nonempty compact and convex subsets of Hilbert spaces converge strongly to one of their fixed points under Ishikawa's iterative scheme under certain standard conditions of its parameterizing sequences. On the other hand, it is proved in [13] that Mann's iteration scheme converges strongly to the unique fixed point of the auxiliary mapping provided such a mapping is a Lipschitzian strong pseudocontraction defined on a compact convex subset of a Hilbert space under certain conditions of its single parameterizing sequence. The parameterizing sequences of those iterative schemes take real values in $[0,1]$, even if the solution sequences of the Banach/Hilbert spaces are vector real or complex sequences, i.e., of dimension exceeding unity. In this paper, the modified Ishikawa scheme is revisited by admitting that the parameterizing sequences are real or complex sequences of matrices of, in general, distinct entries being of the same orders as the sequences generated from the iterative schemes. It is assumed that the auxiliary self-mapping is completely continuous, uniformly Lipschitzian, and asymptotically demicontractive. Such a self-map is defined, in general, on a nonempty compact subset of a Hilbert space but it is not assumed, in general, that it is convex since the uniqueness of the fixed point is not essential for the convergence purposes.

## 2. Some Preliminary Results

Some definitions and auxiliary results are given to be then invoked in the next section.

*Definition 1* (see [1]). Let $S$ be a nonempty subset of a normed linear space $E$ and let $T : S \longrightarrow S$ be a mapping. Then, $T$ is said to be *$k$-strict asymptotically pseudocontractive* if there exist a sequence $\{k_n\}_{n=1}^{\infty} \subset [1,\infty)$, such that $\{k_n\}_{n=1}^{\infty} \longrightarrow 1$, and a constant $k \in [0,1)$ such that for all $x, y \in S$ and $n \geq 1$

$$\begin{aligned}\left\|T^n x - T^n y\right\|^2 \leq k_n^2 \left\|x - y\right\|^2 \\ + k \left\|(x - T^n x) - (y - T^n y)\right\|^2 .\end{aligned} \tag{1}$$

*Definition 2* (see [1]). Let $T : S \longrightarrow S$ be a self-mapping with $S$ being a nonempty subset of a normed linear space $E$ and a fixed point set $F_T \neq \varnothing$. Then, $T$ is said to be *asymptotically demicontractive* if there exist a sequence $\{k_n\}_{n=1}^{\infty} \subset [1,\infty)$ such that $\{k_n\}_{n=1}^{\infty} \longrightarrow 1$ and a constant $k \in [0,1)$ such that for all $x, y \in S$, $p \in F_T$, and $n \geq 0$, one has

$$\left\|T^n x - p\right\|^2 \leq k_n^2 \left\|x - p\right\|^2 + k \left\|x - T^n x\right\|^2 . \tag{2}$$

Note that if $T$ is $k$-strict asymptotically pseudocontractive with $F_T \neq \varnothing$, then $T$ is asymptotically demicontractive since, by taking $y = p \in F_T$ in (1), one gets (2). Note also that the extended sequence $\{k_n\}_{n=0}^{\infty} = \{k_0\} \cup \{k_n\}_{n=1}^{\infty} \subset [1,\infty)$ can be considered in (1)–(2), with $k_0 \in [1,\infty)$, instead of $\{k_n\}_{n=1}^{\infty}$, since $x - T^0 x = 0$, for any $x \in S$ so that (1)–(2) still hold trivially for $n = 0$ since $\|x - y\|^2 = \|T^0 x - y\|^2 \leq k_0^2 \|x - y\|^2 \leq \|x - y\|^2$.

Let $S$ be a nonempty bounded closed subset of a normed linear space $E$ of dimension $r$ and let $T : S \longrightarrow S$ be an asymptotically demicontractive mapping. Consider the sequence $\{x_n\}_{n=0}^{\infty}$, with $x_n \in C^r$, defined by

$$\begin{aligned} x_0 &\in S \\ y_n &= (I_r - B_n) x_n + B_n T^n x_n \\ x_{n+1} &= (I_r - A_n) x_n + A_n T^n y_n \end{aligned} \tag{3}$$

for $n \geq 0$, where $A_n \in \mathbf{C}^{r\times r}$ and $B_n \in \mathbf{C}^{r\times r}$ for $n \geq 0$ and $I_r$ is the $r$th identity matrix. Note that (3) is a generalized modified Ishikawa iterative process in the sense that it is applicable to real or complex scalar ($r = 1$) and vector ($r \geq 2$) sequences and $\{A_n\}$ and $\{B_n\}$ are, in general, nondiagonal nonsparse matrices with, in general, distinct diagonal entries if $r \geq 2$. Equaion (3) may be equivalently rewritten as follows:

$$\begin{aligned} x_0 &\in S \\ y_n &= (1 - \beta_n) x_n + \beta_n T^n x_n + v_n \\ x_{n+1} &= (1 - \alpha_n) x_n + \alpha_n T^n y_n + u_n \\ v_n &= v_n(x_n) = (\beta_n I_r - B_n)(x_n - T^n x_n) \\ u_n &= u_n(x_n, y_n(x_n)) = (\alpha_n I_r - A_n)(x_n - T^n y_n) \end{aligned} \tag{4}$$

for $n \geq 0$, where $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ are real parameterizing sequences in $[0,1]$ which drive the iterative scheme together with the auxiliary self-mapping $T$. Note that the sequences $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ take into account the contributions of the disturbances related to the standard modified Ishikawa sequence caused by the presence of couplings between the components of $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ and the errors of the diagonal parts of the matrix sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ related to the case of being diagonal with identical diagonal entries defined by the sequences $\{\alpha_n I_r\}_{n=0}^{\infty}$ and $\{\beta_n I_r\}_{n=0}^{\infty}$, respectively.

The following result is concerned with the convergence of the solution of the iterated scheme when the sequences $\{x_n - T^n x_n\}_{n=0}^{\infty}$ and $\{x_n - T^n y_n\}_{n=0}^{\infty}$ converge to the limit kernels of the parameterizing matrix sequences errors related to a diagonal matrix with identical entries.

**Theorem 3.** *Let $S$ be a nonempty subset of a normed linear space $E$ of dimension $r$ and let the auxiliary self-mapping $T : S \longrightarrow S$ be an asymptotically demicontractive mapping. Then, the following properties hold:*

(i) $\limsup_{n\longrightarrow\infty}(\|T^n x_n - p + v_n\|^2 - (1+\theta_n(v_n))k_n^2\|x_n - p\|^2 - (1+\theta_n(v_n))[k + \|\beta_n I_r - B_n\|^2\theta_n(v_n)]\|x_n - T^n x_n\|^2) \le 0$

$$\limsup_{n\longrightarrow\infty}\left(\|T^n x_n - p + u_n\|^2 - (1+\vartheta_n(u_n)) \cdot k_n^2\|x_n - p\|^2 - (1+\vartheta_n(u_n)) \cdot \left[k + \|\alpha_n I_r - A_n\|^2\vartheta_n(u_n)\right]\|x_n - T^n x_n\|^2\right) \le 0 \tag{5}$$

*for any sequence $\{x_n\}_{n=0}^{\infty} \subset S$ and some real sequence $\{k_n\}_{n=0}^{\infty} \subset [1,\infty)$, such that $\{k_n\}_{n=0}^{\infty} \longrightarrow 1$, some real constant $k \in [0,1)$, and any $p \in F_T$, where*

$$\theta_n(v_n) = \begin{cases} 0 & \text{if } v_n = 0 \\ 1 & \text{if } v_n \neq 0; \end{cases}$$
$$\vartheta_n(u_n) = \begin{cases} 0 & \text{if } u_n = 0 \\ 1 & \text{if } u_n \neq 0 \end{cases} \tag{6}$$
$$\text{for } n \ge 0.$$

(ii) *Assume that $\{x_n - T^n x_n\}_{n=0}^{\infty} \longrightarrow p_B$ with $p_B \in K_B$ where $K_B$ exists and it is defined as a theoretic limit set by*

$$K_B = \liminf_{n\longrightarrow\infty}\bigcup_{n\ge1}\bigcap_{j\ge n}\left\{Ker\left(\beta_j I_r - B_j\right)\right\} = \limsup_{n\longrightarrow\infty}\bigcap_{n\ge1}\bigcup_{j\ge n}\left\{Ker\left(\beta_j I_r - B_j\right)\right\}(\neq\varnothing). \tag{7}$$

*Then, $\{v_n\}_{n=0}^{\infty} \longrightarrow 0$. This holds, in particular, if $k \in [0,1/2)$ leading to $K_B = \{0\}$, so that $p_B = 0$, and for some $p \in F_T$, $\{x_n\}_{n=0}^{\infty} \longrightarrow p$ which leads to $\{T^n x_n\}_{n=0}^{\infty} \longrightarrow p$.*

(iii) *Assume that $\{x_n\}_{n=0}^{\infty} \longrightarrow p$ for some $p \in F_T$ and $\{A_n\}_{n=0}^{\infty}$ is pointwise convergent to the nonempty limit nonsingular $r$-matrix $A_\infty = \lim_{n\longrightarrow\infty}\bigcup_{n\ge1}\bigcap_{j\ge n}\{A_j\} = \liminf_{n\longrightarrow\infty}\bigcup_{n\ge1}\bigcap_{j\ge n}\{A_j\} = \limsup_{n\longrightarrow\infty}\bigcap_{n\ge1}\bigcup_{j\ge n}\{A_j\}$. Then,*

$$\begin{gathered}\{T^n y_n\}_{n=0}^{\infty} \longrightarrow p, \\ \{T^n y_n - x_n\}_{n=0}^{\infty} \longrightarrow 0, \\ \{T^n y_n - x_{n+1}\}_{n=0}^{\infty} \longrightarrow 0, \\ \{u_n\}_{n=0}^{\infty} \longrightarrow 0.\end{gathered} \tag{8}$$

*If, in addition, $\{v_n\}_{n=0}^{\infty} \longrightarrow 0$, then $\{y_n\}_{n=0}^{\infty} \longrightarrow p$.*

(iv) *Assume that $\{x_n - T^n y_n\}_{n=0}^{\infty} \longrightarrow p_A (\in K_A)$ such that the set $K_A$ exists and it is defined by*

$$K_A = \liminf_{n\longrightarrow\infty}\bigcup_{n\ge1}\bigcap_{j\ge n}\left\{Ker\left(\alpha_j I_r - A_j\right)\right\} = \limsup_{n\longrightarrow\infty}\bigcap_{n\ge1}\bigcup_{j\ge n}\left\{Ker\left(\alpha_j I_r - A_j\right)\right\}(\neq\varnothing). \tag{9}$$

*Then, $\{u_n\}_{n=0}^{\infty} \longrightarrow 0$ and $\{x_n - T^n y_n\}_{n=0}^{\infty} \longrightarrow 0$. This holds, in particular, if for some $p \in F_T$, $\{x_n\}_{n=0}^{\infty} \longrightarrow p$, $k \in [0,1/2)$; then $K_A = \{0\}$, so that $p_A = 0$, which leads to $\{T^n y_n\}_{n=0}^{\infty} \longrightarrow p$ and $\{y_n\}_{n=0}^{\infty} \longrightarrow p$.*

*Proof.* Since $T : S \longrightarrow S$ is asymptotically demicontractive, $F_T \neq \varnothing$. Then, there exists a sequence $\{k_n\}_{n=0}^{\infty} \subset [1,\infty)$ such that $\{k_n\}_{n=0}^{\infty} \longrightarrow 1$ and a real constant $k \in [0,1)$ such that, for any $x \in S$, $p \in F_T$, and one has:

$$\|T^n x - p\|^2 \le k_n^2\|x - p\|^2 + k\|x - T^n x\|^2; \quad n \ge 0 \tag{10}$$

Then, one gets from (4) and (10) that, for any $\{x_n\}_{n=0}^{\infty} \subset S$,

$$\begin{aligned}\|T^n x_n - p + v_n\|^2 &\le (1+\theta_n(v_n))\|T^n x_n - p\|^2 + 2\theta_n(v_n)\|v_n\|^2 \le (1+\theta_n(v_n))\left(k_n^2\|x_n - p\|^2 + k\|x_n - T^n x_n\|^2\right) + 2\theta_n(v_n)\|v_n\|^2 \le (1+\theta_n(v_n)) \\ &\cdot k_n^2\|x_n - p\|^2 + (1+\theta_n(v_n))\left[k + \theta_n(v_n)\|\beta_n I_r - B_n\|^2\right]\|x_n - T^n x_n\|^2 \\ &\le 2\left(k_n^2\|x_n - p\|^2 + \left(k + \|\beta_n I_r - B_n\|^2\right)\|x_n - T^n x_n\|^2\right); \quad n \ge 0\end{aligned} \tag{11}$$

$$\begin{aligned}\|T^n x_n - p + u_n\|^2 &\le (1+\vartheta_n(u_n))\|T^n x_n - p\|^2 + 2\vartheta_n(u_n)\|u_n\|^2 \le (1+\vartheta_n(u_n))\left(k_n^2\|x_n - p\|^2 + k\|x_n - T^n x_n\|^2\right) + 2\vartheta_n(u_n)\|u_n\|^2 \le (1 \\ &+ \vartheta_n(u_n))k_n^2\|x_n - p\|^2 + (1+\vartheta_n(u_n))\left[k + \vartheta_n(u_n)\|\alpha_n I_r - A_n\|^2\right]\|x_n - T^n x_n\|^2 \\ &\le 2\left(k_n^2\|x_n - p\|^2 + \left(k + \|\alpha I_r - A_n\|^2\right)\|x_n - T^n x_n\|^2\right); \quad n \ge 0.\end{aligned} \tag{12}$$

Property (i) follows directly from (11)-(12). On the other hand, under the existence condition of the set-theoretic limit $K_B$, it follows that the theoretic limit set exists as the nonempty limit $K_B = \lim_{n\longrightarrow\infty}\bigcup_{n\ge1}\bigcap_{j\ge n}\{Ker(\beta_j I_r - B_j)\} \supset \{p_B\}$ and, since $v_n = (\beta_n I_r - B_n)(x_n - T^n x_n)$ for $n \ge 0$, $\{v_n\}_{n=0}^{\infty} \longrightarrow 0$ if $\{x_n - T^n x_n\}_{n=0}^{\infty} \longrightarrow p_B$ since (a) $K_{Bn} = Ker(\beta_n I_r - B_n)$ for $n \ge 0$ is closed since each linear operator $(\beta_n I_r - B_n)$ is continuous and of finite dimension, so its Kernel is closed for any $n \ge 0$ and, (b) $p_B (\in KerK_B) \subseteq \bigcap_{j>n_0(n)} KerK_{Bj}$ for all $n \ge 0$ some nonnegative integer $n_0 = n_0(n)$ and each integer $n \ge 0$. Thus, $p_B$ is in all except finitely often (and also in infinitely often) sets $KerK_{Bj}$ since it belongs to $K_B$ because of the identity of limit supremum and limit infimum which defines $K_B$. Then,

$\{x_n - T^n x_n\}_{n=0}^{\infty} \longrightarrow p_B$, $\lim_{n\longrightarrow\infty} v_n = (\beta_n I_r - B_n)(x_n - T^n x_n) = K_B p_B = 0$. Also, for any $p \in F_T$, one has

$$\begin{aligned}\|T^n x_n - x_n\|^2 &= \|T^n x_n - p + p - x_n\|^2 \\ &\le 2\left(\|T^n x_n - p\|^2 + \|p - x_n\|^2\right) \\ &\le 2\left(k_n^2 \|x_n - p\|^2 + k\|x_n - T^n x_n\|^2 + \|p - x_n\|^2\right); \\ &\qquad n \ge 0\end{aligned} \tag{13}$$

and since $\{k_n\}_{n=0}^{\infty} \longrightarrow 1$, $\limsup_{n\longrightarrow\infty}[(1-2k)\|x_n - T^n x_n\|^2 - 4\|p - x_n\|^2] \le 0$. Thus, $\{x_n - T^n x_n\}_{n=0}^{\infty} \longrightarrow 0$ if $k \in [0, 1/2)$ and $\{x_n\}_{n=0}^{\infty} \longrightarrow p$. Then, $\{T^n x_n\}_{n=0}^{\infty} \longrightarrow p$. This holds, in particular, if $K_B = \{0\}$ (then $p_B = 0$), and one concludes that $\{v_n\}_{n=0}^{\infty} \longrightarrow 0$ from the third identity of (4). Property (ii) has been proved.

Note that if $\{x_n\}_{n=0}^{\infty} \longrightarrow p$, then the replacements of $(x_{n+1} - T^n y_n)$ and $\{u_n\}_{n=0}^{\infty}$ in (4) yield

$$\begin{aligned}0 &= \lim_{n\longrightarrow\infty}\left(x_{n+1} - T^n y_n - (1-\alpha_n)(x_n - T^n y_n) - u_n\right) \\ &= \lim_{n\longrightarrow\infty}\left(p - T^n y_n - (1-\alpha_n)(p - T^n y_n) - u_n\right) \\ &= \lim_{n\longrightarrow\infty}\left(\alpha_n (p - T^n y_n) - u_n\right) \\ &= \lim_{n\longrightarrow\infty}\left(\alpha_n (p - T^n y_n) + (A_n - \alpha_n I_r)(p - T^n y_n)\right) \\ &= \lim_{n\longrightarrow\infty}\left(A_n (p - T^n y_n)\right) = \lim_{n\longrightarrow\infty}\left(A_\infty (p - T^n y_n)\right) \\ &= A_\infty \lim_{n\longrightarrow\infty}(p - T^n y_n) = 0\end{aligned} \tag{14}$$

and $\lim_{n\longrightarrow\infty}(p - T^n y_n) \in KerA_\infty = \{0\}$ since $A_\infty$ exists and it is nonsingular. Therefore, $\{T^n y_n\}_{n=0}^{\infty} \longrightarrow p$, $\{T^n y_n - x_n\}_{n=0}^{\infty} \longrightarrow 0$, $\{T^n y_n - x_{n+1}\}_{n=0}^{\infty} \longrightarrow 0$, and $\{u_n\}_{n=0}^{\infty} \longrightarrow 0$. If, in addition, $\{v_n\}_{n=0}^{\infty} \longrightarrow 0$, then $\{y_n\}_{n=0}^{\infty} \longrightarrow p$ from (4). Property (iii) has been proved. On the other hand, if $\{x_n - T^n y_n\}_{n=0}^{\infty} \longrightarrow p_A (\in K_A)$, then $\{u_n\}_{n=0}^{\infty} \longrightarrow 0$ and $\{x_n - T^n y_n\}_{n=0}^{\infty} \longrightarrow 0$ from (4). If, for some $p \in F_T$, $\{x_n\}_{n=0}^{\infty} \longrightarrow p$ and $k \in [0, 1/2)$, then $p_A = 0$, $K_A = \{0\}$, $\{T^n y_n\}_{n=0}^{\infty} \longrightarrow p$, and then $\{y_n\}_{n=0}^{\infty} \longrightarrow p$ from (4); thus Property (iv) is proved. □

Note that no Property of Theorem 3 assumes any of the constraints $\liminf_{n\longrightarrow\infty}\alpha_n > 0$ or $\liminf_{n\longrightarrow\infty}\beta_n > 0$. Note also that, if $\{x_n\}_{n=0}^{\infty} \longrightarrow p (\in F_T)$ in Theorem 3, then $\{y_n\}_{n=0}^{\infty} \longrightarrow p$ is achievable if $\{A_n\}_{n=0}^{\infty} \longrightarrow A_\infty$ (nonsingular) pointwise or if $K_A = \{0\}$ and $k < 1/2$, each case under certain supplementary conditions. Note that

$$\begin{aligned}\|x_n - T^n y_n\| &= \|x_n - T^n x_n + T^n x_n - p + p - T^n y_n\| \\ &\le \|T^n x_n - p\| + \|p - T^n y_n\| + \|x_n - T^n x_n\| \\ &\le \sqrt{k_n^2 \|x_n - p\|^2 + k\|x_n - T^n x_n\|} \\ &\quad + \sqrt{k_n^2 \|y_n - p\|^2 + k\|y_n - T^n y_n\|} + \|x_n - T^n x_n\| \\ &\le k_n\left(\|x_n - p\| + \|y_n - p\|\right) + \sqrt{k}\left(\|x_n - T^n x_n\| \right. \\ &\quad \left. + \|y_n - T^n y_n\|\right) + \|x_n - T^n x_n\|; \quad n \ge 0\end{aligned} \tag{15}$$

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\left(\|x_n - T^n y_n\| - \|x_n - p\| - \|y_n - p\| \right. \\ &\quad \left. - \left(1 + \sqrt{k}\right)\|x_n - T^n x_n\| - \sqrt{k}\|y_n - T^n y_n\|\right) \le 0\end{aligned} \tag{16}$$

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\left(\|y_n - T^n x_n\| - \|y_n - p\| - \|x_n - p\| \right. \\ &\quad \left. - \left(1 + \sqrt{k}\right)\|y_n - T^n y_n\| - \sqrt{k}\|x_n - T^n x_n\|\right) \le 0.\end{aligned} \tag{17}$$

If $\{x_n\}_{n=0}^{\infty} \longrightarrow p$ and $\{x_n - T^n x_n\}_{n=0}^{\infty} \longrightarrow 0$, then

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\left(\|x_n - T^n y_n\| - \|y_n - p\| - \sqrt{k}\|y_n - T^n y_n\|\right) \\ &\quad \le 0\end{aligned} \tag{18}$$

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\left(\|y_n - T^n x_n\| - \|y_n - p\| \right. \\ &\quad \left. - \left(1 + \sqrt{k}\right)\|y_n - T^n y_n\|\right) \le 0.\end{aligned} \tag{19}$$

The subsequent result establishes that, under a similar condition as (2) of asymptotic demicontractiveness of an uniformly $L$-Lipschitzian operator [1] on a Hilbert space for given alternative specific constraints on $\{k_n\}$ and $k$, the boundedness of $\{x_n\}_{n=0}^{\infty}$ implies that of $\{x_n - Tx_n\}_{n=0}^{\infty}$ and $\{x_n - T^n x_n\}_{n=0}^{\infty}$. Also, the strong convergence of $\{x_n\}_{n=0}^{\infty}$ to a fixed point of $T$ implies the convergence to zero of $\{x_n - Tx_n\}_{n=0}^{\infty}$ and $\{x_n - T^n x_n\}_{n=0}^{\infty}$.

**Theorem 4.** *Let $H$ be a Hilbert space and let $T : H \longrightarrow H$ be an operator on a Hilbert space $H$ with a fixed point $p \in F_T$ which is uniformly $L$-Lipschitzian and satisfies (2) with $\{k_n\}_{n=0}^{\infty} \subset [1, \infty)$ and $k \in [0, 1/2)$. Then, the sequences $\{x_n - T^n x_n\}_{n=0}^{\infty}$ and $\{x_n - Tx_n\}_{n=0}^{\infty}$ are bounded if $\{x_n\}_{n=0}^{\infty}$ is bounded. If, furthermore, $\{x_n\}_{n=0}^{\infty} \longrightarrow p$, then $\{\|x_n - T^n x_n\|\}_{n=0}^{\infty} \longrightarrow 0$ and $\{\|x_n - Tx_n\|\}_{n=0}^{\infty} \longrightarrow 0$. If, in addition, $T$ is continuous at $p$, then $\{Tx_n\}_{n=0}^{\infty} \longrightarrow p$.*

*Proof.* Note that

$$\begin{aligned}&\|T^n x_n - p\| \le \sqrt{2}\left(k_n\|x_n - p\| + \sqrt{k}\|x_n - T^n x_n\|\right); \\ &\qquad n \ge 0, \\ &\|x_n - T^n x_n\| \le \|x_n - p\| + \|p - T^n x_n\| \\ &\qquad \le \left(1 + \sqrt{2}k_n\right)\|x_n - p\| \\ &\qquad + \sqrt{2k}\|x_n - T^n x_n\|; \quad n \ge 0.\end{aligned} \tag{20}$$

Thus,

$$\|x_n - T^n x_n\| \le \frac{1 + \sqrt{2}k_n}{1 - \sqrt{2k}}\|x_n - p\|; \quad n \ge 0 \tag{21}$$

$$\begin{aligned}\|x_n - Tx_n\| &\le \|T^n x_n - Tx_n\| + \|x_n - T^n x_n\| \\ &\le \frac{1 + \sqrt{2}k_n}{1 - \sqrt{2k}}\|x_n - p\| \\ &\quad + L\frac{1 + \sqrt{2}k_{n-1}}{1 - \sqrt{2k}}\|x_{n-1} - p\|; \quad n \ge 0\end{aligned} \tag{22}$$

so that if $\{x_n\}_{n=0}^{\infty}$ is bounded, then $\{\|x_n - T^n x_n\|\}_{n=0}^{\infty}$ and $\{\|x_n - Tx_n\|\}_{n=0}^{\infty}$ are bounded since $\{k_n\}_{n=0}^{\infty} \subset [1, \infty)$ and $k \in [0, 1/2)$. Furthermore, if $\{x_n\}_{n=0}^{\infty} \longrightarrow p$, then $\{\|x_n - T^n x_n\|\}_{n=0}^{\infty} \longrightarrow 0$ and $\{\|x_n - Tx_n\|\}_{n=0}^{\infty} \longrightarrow 0$. If, in addition, $T$ is continuous at $p$, then $\{Tx_n\}_{n=0}^{\infty} \longrightarrow p$ since $\{x_n\}_{n=0}^{\infty} \longrightarrow p$ and $\{\|x_n - Tx_n\|\}_{n=0}^{\infty} \longrightarrow 0$.

Now, an illustrative example with two "ad hoc" specific results (Lemmas 6 and 7) is given and discussed for the modified Ishikawa iterative scheme under structured errors for the case of a linear bounded operator $T$ on $\boldsymbol{R}^r$.

*Example 5.* Let the iterative scheme

$$x_{n+1} = \left[I_r - A_n + A_n T^n \left(I_r - B_n + B_n T^n\right)\right] x_n; \qquad x_0 \in \boldsymbol{R}^r \tag{23}$$

for $n \geq 0$. Consider the simplest case that $T$ defined on $\boldsymbol{R}^r$ is linear and bounded with norm upper-bound $K$. Note that the assumption that $T$ is a self-mapping on a bounded set to which, furthermore, the initial condition of the iteration belongs is removed. Now, assume that

$$A_n = \alpha_n I_r + \widetilde{A}_n; \qquad B_n = \beta_n I_r + \widetilde{B}_n \tag{24}$$

for $n \geq 0$ with $\alpha_n, \beta_n \in [0, 1]$ for $n \geq 0$. Note that $\widetilde{A}_n = (A_{dn} - \alpha_n I_r) + A_{odn}$ and $\widetilde{B}_n = (B_{dn} - \beta_n I_r) + B_{odn}$, where $A_{dn}$ and $B_{dn}$ are the diagonal parts of $A_n$ and $B_n$, which are zero if all their nonzero entries equalize $\alpha_n$ and, respectively, $\beta_n$, and $A_{odn}$ and $B_{odn}$ are their off-diagonal parts. Calculations via (4) and (24) yield

$$x_{n+1} = \left[1 - \alpha_n + \left(\alpha_n \left(1 - \beta_n\right) + \alpha_n \beta_n T^n\right) T^n\right] x_n + \Delta_n \left(x_n\right); \quad n \geq 0 \tag{25}$$

where

$$\begin{aligned} \Delta_n &= \Delta_n \left(x_n, y_n \left(x_n\right)\right) = \alpha_n T^n \widetilde{B}_n \left(T^n x_n - x_n\right) \\ &\quad + \widetilde{A}_n \left(T^n y_n \left(x_n\right) - x_n\right) = \Delta_n \left(x_n\right) \\ &= \left(\alpha_n T^n \widetilde{B}_n + \widetilde{A}_n T^n \widetilde{B}_n + \beta_n \widetilde{A}_n T^n + \widetilde{A}_n\right) \\ &\quad \cdot \left(T^n x_n - x_n\right); \quad n \geq 0 \end{aligned} \tag{26}$$

which is zero if $\widetilde{A}_n = \widetilde{B}_n = 0$ for any $n \geq 0$. Assume that $\varepsilon_n = \max\left(\varepsilon_{an}, \varepsilon_{bn}\right)$ where $\varepsilon_{an} \geq \|\widetilde{A}_n\|$ and $\varepsilon_{bn} \geq \|\widetilde{B}_n\|$ for $n \geq 0$. Then, since $\|T\| \leq K$, note that $\|\Delta_n\| \leq h_n \|x_n\|$; $n \geq 0$ with

$$h_n = \left(1 + K^n\right) \varepsilon_n \left[1 + \left(\alpha_n + \beta_n + \varepsilon_n\right) K^n\right]; \quad n \geq 0. \tag{27}$$

Let the *big Landau's* "$O$" and *small Landau's* "$o$" such that for, in general, complex vector sequences $\{f_n\}$ and $\{g_n\}$, $f_n = O(g_n)$ if $\|f_n\| \leq K_1 \|g_n\| + K_2$; $n \geq 0$ for some nonnegative real constants $K_1$ and $K_2$, and $f_n = o(g_n)$ if $f_n = O(g_n)$ and $\{\|f_n\| / \|g_n\|\} \longrightarrow 0$. Note the following:

(1) $\{h_n\} \longrightarrow 0$ if $(1 + K^n)\varepsilon_n[1 + (\alpha_n + \beta_n + \varepsilon_n)K^n] = o(1)$. In particular, this constraint holds if

(a) $K < 1$ and $\varepsilon_n \leq \overline{\varepsilon} < \infty$ for $n \geq 0$,

(b) $K = 1$ and $\{\varepsilon_n\}_{n=0}^{\infty} \longrightarrow 0$,

(c) $K > 1$ and $\{\varepsilon_n\}_{n=0}^{\infty} \longrightarrow 0$ according to $\varepsilon_n = o(K^{-2n})$,

(d) $\{\varepsilon_n\}_{n=0}^{\infty} \longrightarrow \infty$ and $K < 1$ satisfying $K^n = o(\varepsilon_n^{-2})$; that is, $\{\varepsilon_n\}_{n=0}^{\infty}$, equivalently, is unbounded according to $\varepsilon_n = o(K^{-n/2})$.

(2) $\{h_n x_n\}_{n=0}^{\infty} \longrightarrow 0$ if $(1 + K^n)\varepsilon_n[1 + (\alpha_n + \beta_n + \varepsilon_n)K^n] = o(\|x_n\|^{-1})$. In particular, this holds if $K < 1$ and $\varepsilon_n = o(\|x_n\|^{-1})$, if $K > 1$ and $\varepsilon_n = o(K^{-2n}((1 + \alpha_n + \beta_n)\|x_n\|)^{-1})$, and if $K = 1$ and $\varepsilon_n = o((1 + \alpha_n + \beta_n)^{-1}\|x_n\|^{-1})$.

(3) $\{h_n\}_{n=0}^{\infty}$ is bounded if there exists a nonnegative finite real constant $C$ such that $h_n = a_n b_n \leq C$ for $n \geq 0$, where $a_n = (1 + K^n)\varepsilon_n$ and $b_n = [1 + (\alpha_n + \beta_n + \varepsilon_n)K^n]$ for $n \geq 0$. This holds, in particular, if $a_n = O(b_n^{-1})$ and $\{b_n\}$ is bounded or if $b_n = O(a_n^{-1})$ and $\{a_n\}$ is bounded. Note from (25) that

$$\begin{aligned} &\|x_{n+1}\| \\ &\quad \leq \left(\left[1 - \alpha_n + \left(\alpha_n \left(1 - \beta_n\right) + \alpha_n \beta_n K^n\right) K^n\right] + h_n\right) \\ &\quad \cdot \|x_n\|; \quad n \geq 0. \end{aligned} \tag{28}$$

The following two results hold directly from (28) and the given conditions in the case when $\{h_n\}_{n=0}^{\infty} \longrightarrow 0$, $\{h_n x_n\}_{n=0}^{\infty} \longrightarrow 0$ and boundedness of $\{h_n\}_{n=0}^{\infty}$ in Example 5.

**Lemma 6**. *The solution sequence $\{x_n\}_{n=0}^{\infty}$ is bounded for any finite initial condition $x_0$ if the following constraints hold*:

$$\liminf_{n \longrightarrow \infty} \left(h_n + \left(\alpha_n \left(1 - \beta_n\right) + \alpha_n \beta_n K^n\right) K^n - \alpha_n\right) > -1 \tag{29}$$

$$\limsup_{n \longrightarrow \infty} \left(h_n + \left(\alpha_n \left(1 - \beta_n\right) + \alpha_n \beta_n K^n\right) K^n - \alpha_n\right) < 0. \tag{30}$$

*The second constraint holds, in particular, if*

$$\begin{aligned} &\left(1 + K^n\right) \varepsilon_n \left[1 + \left(\alpha_n + \beta_n + \varepsilon_n\right) K^n\right] \\ &\quad = \max\left(o(1), o\left(\left(1 + \alpha_n + \beta_n\right)^{-1} \|x_n\|^{-1}\right)\right), \end{aligned} \tag{31}$$

$$\limsup_{n \longrightarrow \infty} \left(\left(\alpha_n \left(1 - \beta_n\right) + \alpha_n \beta_n K^n\right) K^n - \alpha_n\right) < 0 \tag{32}$$

*since then either $\{h_n\}_{n=0}^{\infty} \longrightarrow 0$ or $\{h_n x_n\}_{n=0}^{\infty} \longrightarrow 0$, or $h_n \leq C$ for $n \geq 0$, and*

$$\liminf_{n \longrightarrow \infty} \left(\left(\alpha_n \left(1 - \beta_n\right) + \alpha_n \beta_n K^n\right) K^n - \alpha_n\right) > -1 \tag{33}$$

$$\limsup_{n \longrightarrow \infty} \left(\left(\alpha_n \left(1 - \beta_n\right) + \alpha_n \beta_n K^n\right) K^n - \alpha_n\right) < -C. \tag{34}$$

□

**Lemma 7.** *Assume that $T : \boldsymbol{R}^r \longrightarrow \boldsymbol{R}^r$ and that (29)-(30) hold for the sequence $\{h_n\}_{n=0}^{\infty}$, such that $\|\Delta_n\| \leq h_n \|x_n\|$, and for $h_n \equiv 0$ for $n \geq 0$ and $x_0 \in S$ for some given bounded set $S \subset \boldsymbol{R}^r$.*

*Then, the following properties hold:*

$$\text{(i)}\quad \left\| \sum_{i=0}^{n} \left( \prod_{j=i+1}^{n} \left[1-\alpha_j\right] \right) \Delta_i\left(x_i\right) x_0 \right\| < \infty \tag{35}$$

(a) *if the constraints (29)-(30) also hold for the perturbation-free case, i.e., if* $\{h_n\}_{n=0}^{\infty} \equiv 0$, *or*

(b) *if* $\sum_{i=0}^{\infty}(\prod_{j=i+1}^{\infty}[1-\alpha_j]) < \infty$ *(it suffices that* $\sum_{i=0}^{\infty}(1-\alpha_i) < \infty$*) and* $(1+K^n)\varepsilon_n[1+(\alpha_n+\beta_n+\varepsilon_n)K^n] = o(1)$.

(ii) $\{\Delta_n(x_n)\}_{n=0}^{\infty} \longrightarrow 0$ if the set-theoretic limit $Ker\Delta_{\infty}^0 = \lim_{n\longrightarrow\infty} Ker(\Delta_n^0)$ exists, $Ker\Delta_{\infty}^0 \cap S \neq \varnothing$, and $\{T^n x_n - x_n\}_{n=0}^{\infty} \longrightarrow p_{\Delta^0} \in Ker\Delta_{\infty}^0 \cap S$, a particular case arises if $0 \in S$, $Ker\Delta_{\infty}^0 = \{0\}$, and $\{T^n x_n - x_n\}_{n=0}^{\infty} \longrightarrow 0$.

(iii) $\{\Delta_n(x_n)\}_{n=0}^{\infty} \longrightarrow 0$ if $\Delta_n^0 \longrightarrow \Delta_{\infty}^0$ as $n \longrightarrow \infty$ pointwise and $\{T^n x_n - x_n\}_{n=0}^{\infty} \longrightarrow p_{\Delta^0} \in Ker\Delta_{\infty}^0 \cap S$.

(iv) Assume that the theoretic limit set $T_{S\infty} = \lim_{n\longrightarrow\infty} T^n(S)$ exists; that is,

$$\begin{aligned} T_{S\infty} &= \lim_{n\longrightarrow\infty} \bigcup_{n\geq 1} \bigcap_{j\geq n} \left\{T^j(S)\right\} = \liminf_{n\longrightarrow\infty} \bigcup_{n\geq 1} \bigcap_{j\geq n} \left\{T^j(S)\right\} \\ &= \limsup_{n\longrightarrow\infty} \bigcap_{n\geq 1} \bigcup_{j\geq n} \left\{T^j(S)\right\} \end{aligned} \tag{36}$$

with $T_{S\infty} \subseteq S$, either $Ker\Delta_{\infty}^0 = \lim_{n\longrightarrow\infty} Ker(\Delta_n^0)$ exists or $\Delta_n^0 \longrightarrow \Delta_{\infty}^0$ pointwise as $n \longrightarrow \infty$, $0 \in Ker\Delta_{\infty}^0 \cap S$, and $\{x_n\} \longrightarrow p \in F_T$, Then, $\{\Delta_n(x_n)\} \longrightarrow 0$.

*Proof.* Assume that $\{x_n^0\}$ is then sequence generated from the iteration scheme with $\{\Delta_n\} \equiv 0$ for $x_0^0 = x_0$. One gets from (25) that the perturbed and disturbance-free solutions are

$$x_{n+1} = x_{n+1}^0 + \sum_{i=0}^{n} \left( \prod_{j=i+1}^{n} \left[1-\alpha_j\right] \right) \Delta_i\left(x_i\right); \quad n \geq 0 \tag{37}$$

$$\begin{aligned} x_{n+1}^0 = \Bigg[ &\prod_{j=0}^{n} \left[1-\alpha_j\right] \\ &+ \sum_{i=0}^{n} \left( \prod_{j=i+1}^{n} \left[1-\alpha_j\right] \right) \left(\alpha_i\left(1-\beta_i\right) + \alpha_i\beta_i T^i\right) T^i \Bigg] \\ &\cdot x_0; \quad n \geq 0. \end{aligned} \tag{38}$$

Since (29)-(30) hold for the sequence $\{h_n\}_{n=0}^{\infty}$ and for $h_n \equiv 0$, one has from Lemma 6 that, for any $x_0 \in S$: $\{x_n\}_{n=0}^{\infty}$, $\{x_n^0\}_{n=0}^{\infty}$ are bounded and then $\|x_{n+1} - x_{n+1}^0\| = \|\sum_{i=0}^{n}(\prod_{j=i+1}^{n}[1-\alpha_j])\Delta_i(x_i)x_0\| < \infty$ for $n \geq 0$. Thus, Property (i) holds under conditions (a). On the other hand, if $(1+K^n)\varepsilon_n[1+(\alpha_n+\beta_n+\varepsilon_n)K^n] = o(1)$, then $\{h_n x_n\}_{n=0}^{\infty} \longrightarrow 0$ since $\{\|x_n\|\}_{n=0}^{\infty}$ is bounded from Lemma 6 and $\{h_n\} \longrightarrow 0$, and then $\{\|\Delta_n\|\} \longrightarrow 0$. Thus, $\sup_{n\geq 0}\|\Delta_n\| < \infty$ since convergent sequences are bounded. Since $\sum_{i=0}^{\infty}(\prod_{j=i+1}^{\infty}[1-\alpha_j]) < \infty$, it follows that (35) holds, and then Property (i) holds under conditions (b). Property (ii) follows directly since $\Delta(x_n) = \Delta_n^0 x_n \longrightarrow 0$ as $n \longrightarrow \infty$ from (26) since the set- theoretic limit $Ker\Delta_{\infty}^0 = \lim_{n\longrightarrow\infty} Ker(\Delta_n^0)$ exists, and $\{T^n x_n - x_n\}_{n=0}^{\infty} \longrightarrow p_{\Delta^0} \in Ker\Delta_{\infty}^0 \cap S(\neq \varnothing)$. To prove Property (iii), note that if $\{\Delta_n^0\}_{n=0}^{\infty} \longrightarrow \Delta_{\infty}^0$, then $\{\Delta_n^0\chi_n - \Delta_{\infty}^0\chi_n\}_{n=0}^{\infty} \longrightarrow 0$ for $\chi_n \in Dom(\Delta_n^0)$. Since $\{T^n x_n - x_n\}_{n=0}^{\infty} \longrightarrow p_{\Delta^0} \in Ker\Delta_{\infty}^0 \cap S$, then

$$\begin{aligned} &\left\{\Delta_n^0 \cdot \left(p_{\Delta^0} + T^n x_n - x_n - p_{\Delta^0}\right) - \Delta_{\infty}^0 \right. \\ &\quad \left. \cdot \left(p_{\Delta^0} + T^n x_n - x_n - p_{\Delta^0}\right)\right\} \longrightarrow \left(\left\{\Delta_n^0 \cdot p_{\Delta^0}\right\} - \Delta_{\infty}^0 \right. \\ &\quad \left. \cdot p_{\Delta^0}\right) = 0. \end{aligned} \tag{39}$$

Since $p_{\Delta^0} \in Ker\Delta_{\infty}^0 \cap S$, then

$$\begin{aligned} &\Delta_{\infty}^0 \cdot p_{\Delta^0} = 0, \\ &\left\{\Delta_n^0 \cdot p_{\Delta^0}\right\}_{n=0}^{\infty} \longrightarrow 0, \\ &\left\{\Delta_n^0 \cdot \left(T^n x_n - x_n\right)\right\}_{n=0}^{\infty} = \left\{\Delta_n\left(\left(T^n x_n - x_n\right)\right)\right\}_{n=0}^{\infty} \\ &\quad \longrightarrow 0. \end{aligned} \tag{40}$$

Property (iv) follows if either Property (ii) or Property (iii) holds and, furthermore, there exists the set-theoretic limit $T_{S\infty} \subseteq S$, $0 \in Ker\Delta_{\infty}^0 \cap S$, and $\{x_n\}_{n=0}^{\infty} \longrightarrow p \in F_T \subset S$ since $\{T^n x_n - x_n\}_{n=0}^{\infty} \longrightarrow 0$. □

Inspired by Example 5, we get the following result, to be then used, on sufficient conditions for asymptotic vanishing of the perturbation sequence $\{\Delta_n(x_n)\}_{n=0}^{\infty}$ by removing the assumption of linearity of $T$.

**Theorem 8.** *Assume that E is a normed linear space and that* $\{T^n\}_{n=0}^{\infty}$ *is a sequence of bounded operators on* $S \subset E$. *Define*

$$\begin{aligned} \Delta_n &= \Delta_n\left(x_n\right) = x_{n+1} \\ &\quad - \left[\left(1-\alpha_n\right)x_n + \alpha_n T^n\left(\left(1-\beta_n\right)x_n + \alpha_n\beta_n T^n x_n\right)\right] \end{aligned} \tag{41a}$$

$$\begin{aligned} h_n &= (1+L) \\ &\quad \cdot \left[\left(1+\beta_n\right)\left\|\widetilde{A}_n\right\| + \alpha_n L\left\|\widetilde{B}_n\right\| + L\left\|\widetilde{A}_n\right\|\left\|\widetilde{B}_n\right\|\right]; \end{aligned} \tag{41b}$$

$$\varepsilon_n = \max\left(\left\|\widetilde{A}_n\right\|, \left\|\widetilde{B}_n\right\|\right)$$

*for* $n \geq 0$. *Then, the following Items hold:*

(i) $\{\Delta_n\}_{n=0}^{\infty} \longrightarrow 0$ *if* $\varepsilon_n = o(\|x_n\|^{-1})$, *and* $\{h_n\} \longrightarrow 0$ *if either* (a) $\|\widetilde{A}_n\| \longrightarrow 0$ *and* $\|\widetilde{B}_n\| \longrightarrow 0$ *as* $n \longrightarrow \infty$, *or* (b) $\|\widetilde{A}_n\| \longrightarrow 0$ *and* $\alpha_n \longrightarrow 0$ *as* $n \longrightarrow \infty$.

(ii) *Define the sequences* $\widetilde{x}_n = T^n x_n - x_n$ *and* $\widetilde{G}_n = \alpha_n T^n \widetilde{B}_n + \widetilde{A}_n$ *for* $n \geq 0$ *and assume that* $\{\widetilde{x}_n\}_{n=0}^{\infty} \longrightarrow \widetilde{x}$, $\{\widetilde{G}_n\}_{n=0}^{\infty} \longrightarrow \widetilde{G}$ *pointwise, such that* $\widetilde{G}$ *is closed, and* $\{\widetilde{G}_n\widetilde{x}_n\}_{n=0}^{\infty} \longrightarrow v_{\widetilde{G}}$. *Then,* $\widetilde{G}\widetilde{x} = v_{\widetilde{G}}$ *and*

(ii. a) $\Delta_n - u_n = \widetilde{G}_n x_n$ *for* $n \geq 0$

(ii. b) *If* $\{\widetilde{G}_n x_n\}_{n=0}^{\infty} \longrightarrow 0$, *equivalently if* $\{\Delta_n - u_n\}_{n=0}^{\infty} \longrightarrow 0$, *then* $\widetilde{x} = 0$ *if* $\widetilde{G}$ *is nonsingular.*

(ii. c) *If* $\widetilde{x} = 0$ *and* $\widetilde{G}.0 = 0$, *in particular if* $\widetilde{G} = 0$, *then* $\{\Delta_n - u_n\}_{n=0}^{\infty} \longrightarrow 0$.

(iii) *If* $\{\widetilde{B}_n\}_{n=0}^{\infty} \longrightarrow \widetilde{B}$ *pointwise, such that* $\widetilde{B}$ *is closed, and* $\{\widetilde{B}_n\widetilde{x}_n\}_{n=0}^{\infty} \longrightarrow v_{\widetilde{B}}$, *then* $\widetilde{B}\widetilde{x} = v_{\widetilde{B}}$, *and if* $\widetilde{x} = 0$ *and* $\widetilde{B}.0 = 0$, *in particular if* $\widetilde{B} = 0$, *then* $\{v_n\}_{n=0}^{\infty} \longrightarrow 0$.

(iv) *If* $\{\widetilde{G}_n\}_{n=0}^{\infty} \longrightarrow \widetilde{G}$ *pointwise,* $\{\widetilde{B}_n\}_{n=0}^{\infty} \longrightarrow \widetilde{B}$ *pointwise,* $\widetilde{G}$ *and* $\widetilde{B}$ *are closed,* $\{\widetilde{x}_n\}_{n=0}^{\infty} \longrightarrow \widetilde{x}$, $\{\widetilde{G}_n\widetilde{x}_n\}_{n=0}^{\infty} \longrightarrow v_{\widetilde{G}}$, *and* $\{\widetilde{B}_n\widetilde{x}_n\}_{n=0}^{\infty} \longrightarrow v_{\widetilde{B}}$, *then* $x \in Dom\widetilde{G} \cap Dom\widetilde{B}$ *and rank* $\begin{bmatrix} \widetilde{G} \\ \widetilde{B} \end{bmatrix} = rank \begin{bmatrix} \widetilde{G} & v_{\widetilde{G}} \\ \widetilde{B} & v_{\widetilde{B}} \end{bmatrix}$. *If* $v_{\widetilde{G}} = v_{\widetilde{B}} = 0$, *then* $\widetilde{x} \in Ker\widetilde{G} \cap Ker\widetilde{B}$, *and* $\widetilde{x} = 0$ *if* $\widetilde{G}$ *or* $\widetilde{B}$ *is nonsingular.*

*Proof.* From the last two equations of (4), we have

$$\begin{aligned} v_n &= \widetilde{B}_n\left(T^n x_n - x_n\right); \quad n \geq 0 \\ u_n &= \widetilde{A}_n\left(T^n y_n - x_n\right) = \widetilde{A}_n\Big(T^n\big[(1-\beta_n)x_n \\ &\quad + \beta_n T^n x_n - \widetilde{B}_n x_n + \widetilde{B}_n T^n x_n\big] - x_n\Big); \quad n \geq 0 \end{aligned} \tag{42}$$

$$\begin{aligned} \Delta_n &= \Delta_n(x_n) = \Big[\widetilde{A}_n\left(I_r + T^n\left(\beta_n I_r + \widetilde{B}_n\right)\right) \\ &\quad + \alpha_n T^n \widetilde{B}_n\Big]\left(T^n x_n - x_n\right) = \left(\alpha_n T^n \widetilde{B}_n + \widetilde{A}_n\right)\left(T^n x_n \right. \\ &\quad \left. - x_n\right) + u_n; \quad n \geq 0 \end{aligned} \tag{43}$$

without requiring the linearity of $T$ for $n \geq 0$. Since $\|T^n\| = \sup_{x\in S, \|x\|\leq 1}(\|T^n x\|/\|x\|) \leq L$ for $n \geq 0$, one gets that $\|\Delta_n\| \leq h_n\|x_n\|$ for $n \geq 0$, hence Property (i) of the lemma. Property (ii. a) follows by direct calculations, Property (ii. b) follows since if $\{\widetilde{x}_n\}_{n=0}^{\infty} \longrightarrow \widetilde{x}$ and $\{\widetilde{G}_n x_n\}_{n=0}^{\infty} \longrightarrow 0$, then $\widetilde{x} \in Dom\widetilde{G}$ and $\widetilde{G}\widetilde{x} = v_{\widetilde{G}}$ since $\widetilde{G}$ is closed. As a result, if $v_{\widetilde{G}} = 0$, then $\widetilde{x} \in Ker\widetilde{G}$ and $\widetilde{x} = 0$ if $\widetilde{G}$ is nonsingular since then $Ker\widetilde{G} = \{0\}$. Property (ii. c) follows since $\widetilde{G}_n \longrightarrow \widetilde{G}$ pointwise as $n \longrightarrow \infty$, $\widetilde{x} = 0$, and $\widetilde{G}.0 = 0$, in particular if $\widetilde{G} = 0$, then

$$\begin{aligned} \widetilde{G}_n\widetilde{x}_n &= \widetilde{G}\widetilde{x} + \left(\widetilde{G}_n - \widetilde{G}\right)\widetilde{x} + \widetilde{G}_n\left(\widetilde{x}_n - \widetilde{x}\right) \\ &= \left(\widetilde{G}_n - \widetilde{G}\right)\widetilde{x} + \widetilde{G}_n\left(\widetilde{x}_n - \widetilde{x}\right), \quad n \geq 0 \end{aligned} \tag{44}$$

and then $\widetilde{G}_n\widetilde{x}_n = \Delta_n - u_n \longrightarrow 0$ as $n \longrightarrow \infty$. Property (iii) follows under a similar reasoning to that used to prove Property (ii. c). Property (iv) follows from Properties [(ii)-(iii)] and the Rouché-Frobenius theorem from Linear Algebra. □

Assume that $\{\Delta_n\}_{n=0}^{\infty} \longrightarrow 0$. Then $\{u_n\}_{n=0}^{\infty} \longrightarrow 0$ via (43) if $T^n (n \geq 1)$ and $T$ are operators on $E$, $\{T^n\}_{n=0}^{\infty} \longrightarrow T$, such that $T$ is closed, pointwise and either

(a) $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$, or

(b) $(y_n - x_n) \longrightarrow 0$ and $(y_n - T^n x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ and $T.0 = 0$ (then $T^n(y_n - x_n) \longrightarrow T(y_n - x_n) \longrightarrow T.0 = 0$, $T^n(y_n - T^n x_n) \longrightarrow T(y_n - T^n x_n) \longrightarrow T.0 = 0$ and $0 \in DomT$), or

(c) $(y_n - x_n) \longrightarrow 0$, $\beta_n \longrightarrow 0$ as $n \longrightarrow \infty$.

Note that $(y_n - x_n) \longrightarrow 0$ and $(y_n - T^n x_n) \longrightarrow 0$ as $n \longrightarrow \infty$ if $T : S \longrightarrow S$, $p \in F_T \cap S$, and $x_n, y_n \longrightarrow p$ as $n \longrightarrow \infty$.

*Definition 9* (see [1]). Let $S$ be a nonempty subset of a normed linear space $E$. $T : S \longrightarrow S$ is said to be *uniformly L-Lipschitzian with constant* $L > 0$ if $\|T^n x - T^n y\| \leq L\|x - y\|$ for $n \geq 1$.

Note that if $T$ is asymptotically nonexpansive, then it is 0-strict pseudocontractive and uniformly $\sup_{n\geq 1}(k_n)$-Lipschitzian, [1]. Note also that if $T$ is uniformly $L$- Lipschitzian and $0 \in S$, then the sequence of operators $\{T^n\}_{n=0}^{\infty}$ is uniformly bounded since by taking $y = 0$, one has $\|T^n x\| \leq L\|x\|$ for all $x \in S$ and $n \geq 1$.

**Lemma 10.** *Let* $T : S \longrightarrow S$ *be asymptotically demicontractive. The following properties hold:*

(i) *If* $T$ *Lipschitzian with constant* $L_T$ *(i.e.,* $T$ *is* $L_T$*-Lipschitzian), then for any* $p \in F_T$

$$\begin{aligned} &\limsup_{n\longrightarrow\infty}\Big(\|T^n x - T^n y\| \\ &\quad - \sqrt{k}\left(\|x - p\| + \|y - p\|\right)\left(1 + L_T^n\right)\Big) \leq 0. \end{aligned} \tag{45}$$

(ii) *If* $T$ *is k-strict asymptotically pseudocontractive and uniformly L-Lipschitzian, then for any* $p \in F_T$

$$\begin{aligned} \|T^n x - T^n y\| &\leq \sqrt{k}(1 + L)\left(\|x - p\| + \|y - p\|\right) \\ &\qquad \text{for all } x, y \in S,\ n \geq 1. \end{aligned} \tag{46}$$

*Let* $T : S \longrightarrow S$ *be k-strict asymptotically pseudocontractive. The following properties hold:*

(iii) *If* $T$*is k-strict asymptotically pseudocontractive and* $L_T$*-Lipschitzian with constant, then*

$$\begin{aligned} &\limsup_{n\longrightarrow\infty}\Big(\|T^n x - T^n y\| - \left(1 + \sqrt{k}\left(1 + L_T^n\right)\right)\|x - y\|\Big) \\ &\quad \leq 0 \quad \text{for all } x, y \in S. \end{aligned} \tag{47}$$

(iv) *If* $T$ *is k-strict asymptotically pseudocontractive and uniformly L-Lipschitzian then*

$$\begin{aligned} \|T^n x - T^n y\| &\leq \left(1 + \sqrt{k}(1 + L)\right)\|x - y\| \\ &\qquad \text{for all } x, y \in S,\ n \geq 0. \end{aligned} \tag{48}$$

*Proof.* One gets from (2) that

$$\begin{aligned} \|T^n x - T^n y\| &\leq \|T^n x - p\| + \|T^n y - p\| \\ &\leq k_n\left(\|x - p\| + \|y - p\|\right) \\ &\quad + \sqrt{k}\left(\|x - T^n x\| + \|y - T^n y\|\right) \\ &\leq \left(k_n + \sqrt{k}\right)\left(\|x - p\| + \|y - p\|\right) \\ &\quad + \sqrt{k}\left(\|T^n p - T^n x\| + \|T^n p - T^n y\|\right) \\ &\leq \left(k_n + \sqrt{k}\left(1 + L_T^n\right)\right)\left(\|x - p\| + \|y - p\|\right), \end{aligned} \tag{49}$$

for all $x, y \in S$ and $n \geq 0$. Property (i) follows directly since $\{k_n\}_{n=0}^{\infty} \longrightarrow 1$. Property (ii) follows by replacing $L_T^n \longrightarrow L$ in (49). On the other hand, if $T$ is $k$-strict asymptotically pseudocontractive and $L_T$-Lipschitzian, then for a constant $k \in [0,1)$ and some real sequence $\{k_n\}_{n=0}^{\infty}(\subset [1,\infty)) \longrightarrow 1$, one gets Property (iii) since for any $x, y \in S$

$$\begin{aligned}\|T^n x - T^n y\| &\leq k_n \|x - y\| \\ &\quad + \sqrt{k}\,\|x - y + T^n y - T^n x\| \\ &\leq \left(k_n + \sqrt{k}\left(1 + L_T^n\right)\right)\|x - y\|; \quad n \geq 0.\end{aligned} \tag{50}$$

Property (iv) follows by replacing $L_T^n \longrightarrow L$ in (50). □

**Lemma 11** (see [1]). *Let $x$, $y$, and $z$ be points in Hilbert space and $\lambda \in [0,1]$. Then*

$$\begin{aligned}\|\lambda x + (1-\lambda) y - z\|^2 &= \lambda \|x - z\|^2 + (1-\lambda)\|y - z\|^2 \\ &\quad - \lambda(1-\lambda)\|x - y\|^2.\end{aligned} \tag{51}$$

## 3. Main Results

The main result follows.

**Theorem 12.** *Let $S$ be a nonempty bounded closed subset of a Hilbert space $H$ and let $T : S \longrightarrow S$ be a completely continuous uniformly $L$-Lipschitzian asymptotically demicontractive mapping. Suppose that the following conditions hold:*

(a) $\{k_n - 1\}_{n=0}^{\infty}$ *is summable,*

(b) $0 < a \leq \alpha_n \leq 1 - k(1 - \beta_n)$, $n \geq 0$,

(c) $\{\beta_n I_r - B_n\}_{n=0}^{\infty}$ *and* $\{\alpha_n I_r - A_n\}_{n=0}^{\infty}$ *are summable, and either*

(d1) $0 < b \leq \beta_n < \min(\beta_{0n}, 1)$, $n \geq 0$, *where*

$$\begin{aligned}\beta_{0n} = \frac{1}{2kL^2}\Bigg[&\left(\left(k_n^2 + kL^2\lambda_n(1+\lambda_n) + k\right)^2\right. \\ &\left. + 4\left(k + k_n^2 - kk_n^2\right)\right)^{1/2} - \left(k_n^2 + kL^2\lambda_n(1+\lambda_n)\right. \\ &\left. + k\right)\Bigg]\end{aligned} \tag{52}$$

*for any $n \geq 0$, where $\lambda_n = 1$ if $v_n \neq 0$ and $\lambda_n = 0$ if $v_n = 0$, or*

(d2) $0 < b \leq \beta_n < 1 - k$, $n \geq 0$ *(implying that $k < 1 - b$), and $\{\beta_n(\beta_n + \lambda_n(1+\lambda_n)) - (1-\beta_n)/L^2\}_{n=0}^{\infty}$ is summable with positive sum, or*

(d3) $0 < b \leq \beta_n < (\sqrt{(L^2\lambda_n(1+\lambda_n)+1)^2 + 4L^2} - L^2\lambda_n - 1)/2L^2$, $n \geq 0$ *(implying that $k < 1 - b$), and $\{\beta_n + k - 1\}_{n=0}^{\infty}$ is summable with positive sum. Then, $\{x_n\}_{n=0}^{\infty}$ converges strongly to some fixed point of $T$ in $S$.*

*Proof.* One gets from the third equation of (4) and ((41a) and (41b)) that

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T^n y_n + (\Delta_n - \alpha_n T^n v_n); \quad n \geq 0, \tag{53}$$

$$x_{n+1} - \widehat{p}_n = (1-\alpha_n)(x_n - p) + \alpha_n(T^n y_n - p); \quad n \geq 0 \tag{54}$$

where $\widehat{p}_n = p + u_n$ and $u_n = \Delta_n - \alpha_n T^n v_n$, $n \geq 0$. Equations (53)-(54) agree with ((41a) and (41b)) and (4). Since $T$ is asymptotically demicontractive, $F_T \neq \varnothing$. Let $p \in F_T$. One gets from Lemma 11, with $z = 0$, that

$$\begin{aligned}\|x_{n+1} - \widehat{p}_n\|^2 &= \|x_{n+1} - p - u_n\|^2 \\ &= (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\|T^n y_n - p\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|x_n - T^n y_n\|^2; \quad n \geq 0.\end{aligned} \tag{55}$$

The following relations will be used:

$$\begin{aligned}\|x_{n+1} - \widehat{p}_n\|^2 &= \|x_{n+1} - p + (\alpha_n T^n v_n - \Delta_n)\|^2 \\ &\geq \left(\|x_{n+1} - p\| - \|\alpha_n T^n v_n - \Delta_n\|\right)^2 = \|x_{n+1} - p\|^2 \\ &\quad + \left(\|\alpha_n T^n v_n - \Delta_n\| - \sigma_n\|x_{n+1} - p\|\right) \\ &\quad \cdot \|\alpha_n T^n v_n - \Delta_n\|; \quad n \geq 0\end{aligned} \tag{56}$$

where $\sigma_n = 2$ if $\alpha_n T^n v_n - \Delta_n \neq 0$ and $\sigma_n = 0$ if $\alpha_n T^n v_n - \Delta_n = 0$ for any $n \geq 0$. Then, one has from (55) and (56), again since $T$ is asymptotically demicontractive, that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq (1-\alpha_n)\|x_n - p\|^2 + \alpha_n\left[k_n^2\|y_n - p\|^2\right. \\ &\quad \left. + k\|T^n y_n - y_n\|^2 - (1-\alpha_n)\|x_n - T^n y_n\|^2\right] \\ &\quad + \left(\sigma_n\|x_{n+1} - p\| - \|\alpha_n T^n v_n - \Delta_n\|\right)\|\alpha_n T^n v_n - \Delta_n\|; \\ &\quad n \geq 0.\end{aligned} \tag{57}$$

On the other hand, one gets from the second equation of (4) that

$$y_n - \widehat{p}_{yn} = (1-\beta_n)(x_n - p) + \beta_n(T^n x_n - p); \quad n \geq 0 \tag{58}$$

where

$$\widehat{p}_{yn} = p + v_n = \widehat{p}_n + v_n + \alpha_n T^n v_n - \Delta_n; \quad n \geq .0 \tag{59}$$

The following relations are afterward used:

$$\begin{aligned}\|y_n - \widehat{p}_{yn}\|^2 &= \|y_n - p - v_n\|^2 \geq \left(\|y_n - p\| - \|v_n\|\right)^2 \\ &= \|y_n - p\|^2 \\ &\quad + \left(\|v_n\| - (1+\lambda_n)\|y_n - p\|\right)\|v_n\|; \\ &\quad n \geq 0\end{aligned} \tag{60}$$

where $\lambda_n = 1$ if $v_n \neq 0$ and $\lambda_n = 0$ if $v_n = 0$ for any $n \geq 0$. Again, using Lemma 11, (58), (59), and (60), one gets that

$$\begin{aligned}\|y_n - p\|^2 + (\|v_n\| - \lambda_n \|y_n - p\|)\|v_n\| &= \|y_n - \hat{p}_{yn}\|^2 \\ &= (1-\beta_n)\|x_n - p\|^2 + \beta_n \|T^n x_n - p\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\leq (1-\beta_n)\|x_n - p\|^2 \\ &\quad + \beta_n\left(k_n^2\|x_n - p\|^2 + k\|T^n x_n - x_n\|^2\right) \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2; \quad n \geq 0.\end{aligned} \tag{61}$$

Then,

$$\begin{aligned}\|y_n - p\|^2 \leq \left[1 + \beta_n\left(k_n^2 - 1\right)\right]\|x_n - p\|^2 \\ + \beta_n(\beta_n + k - 1)\|x_n - T^n x_n\|^2 \\ + (\lambda_n\|y_n - p\| - \|v_n\|)\|v_n\|; \quad n \geq 0.\end{aligned} \tag{62}$$

Since $T$ is asymptotically demicontractive and $L$-uniformly Lipschitzian, one gets from (4), Lemma 11, and the second relation of (4) that

$$\begin{aligned}&\|y_n - T^n y_n\|^2 + \lambda_n\|v_n\|^2 \\ &\quad - \lambda_n(1+\lambda_n)\|y_n - T^n y_n\|\|v_n\| \\ &\quad = \|y_n - T^n y_n - v_n\|^2 \\ &\quad = \|(1-\beta_n)(x_n - T^n y_n) + \beta_n(T^n x_n - T^n y_n)\|^2 \\ &\quad = (1-\beta_n)\|x_n - T^n y_n\|^2 + \beta_n\|T^n x_n - T^n y_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\quad \leq (1-\beta_n)\|x_n - T^n y_n\|^2 + L^2\beta_n\|x_n - y_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\quad \leq (1-\beta_n)\|x_n - T^n y_n\|^2 \\ &\quad + L^2\beta_n\|\beta_n(x_n - T^n x_n) - v_n\|^2 \\ &\quad - \beta_n(1-\beta_n)\|x_n - T^n x_n\|^2 \\ &\quad \leq (1-\beta_n)\|x_n - T^n y_n\|^2 \\ &\quad + \beta_n\left[L^2\beta_n^2(1+\lambda_n) + \beta_n - 1\right]\|x_n - T^n x_n\|^2 \\ &\quad + L^2\beta_n\lambda_n(1+\lambda_n)\|v_n\|^2; \quad n \geq 0\end{aligned} \tag{63}$$

which leads to

$$\begin{aligned}\|y_n - T^n y_n\|^2 &\leq (1-\beta_n)\|x_n - T^n y_n\|^2 \\ &\quad + \beta_n\left[L^2\beta_n^2(1+\lambda_n) + \beta_n - 1\right]\|x_n - T^n x_n\|^2 \\ &\quad + \lambda_n\left[\left(L^2\beta_n(1+\lambda_n) - 1\right)\|v_n\|\right. \\ &\quad \left. + (1+\lambda_n)\|y_n - T^n y_n\|\right]\|v_n\|; \quad n \geq 0.\end{aligned} \tag{64}$$

Now, one gets from (62) and (64) into (57) that

$$\begin{aligned}\|x_{n+1} - p\|^2 &\leq \|x_n - p\|^2 + \alpha_n\left(k_n^2\left(1 + \beta_n\left(k_n^2 - 1\right)\right)\right. \\ &\quad \left. - 1\right)\|x_n - p\|^2 + \alpha_n\beta_n\left(k_n^2(\beta_n + k - 1)\right. \\ &\quad \left. + k\left(L^2\beta_n(\beta_n + \lambda_n) + \beta_n - 1\right)\right)\|x_n - T^n x_n\|^2 \\ &\quad + \alpha_n(k(1-\beta_n) + \alpha_n - 1)\|x_n - T^n y_n\|^2 \\ &\quad + \alpha_n\left(\left[k\lambda_n\left(L^2\beta_n(1+\lambda_n) - 1\right) - k_n^2\right]\|v_n\|\right. \\ &\quad \left. + k_n^2(\lambda_n(1+\lambda_n)\|y_n - p\|)\right)\|v_n\| + (\sigma_n\|x_{n+1} - p\| \\ &\quad - \|\alpha_n T^n v_n - \Delta_n\|)\|\alpha_n T^n v_n - \Delta_n\|; \quad n \geq 0.\end{aligned} \tag{65}$$

Thus, the constraints on the sequences $\{k_n\}$, $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ and $\{A_n\}_{n=0}^{\infty}$ being implicit in the conditions (a) to (c) of the theorem statement lead to

$$\begin{aligned}\|x_{n+1} - p\|^2 - \|x_n - p\|^2 &\leq \alpha_n\left(k_n^2\left(1 + \beta_n\left(k_n^2 - 1\right)\right)\right. \\ &\quad \left. - 1\right)\|x_n - p\|^2 + \alpha_n\beta_n\left(k_n^2(\beta_n + k - 1)\right. \\ &\quad \left. + k\left(L^2\beta_n(\beta_n + \lambda_n) + \beta_n - 1\right)\right)\|x_n - T^n x_n\|^2 \\ &\quad - \alpha_n|k(1-\beta_n) + \alpha_n - 1|\|x_n - T^n y_n\|^2 \\ &\quad + \alpha_n\left(\left[k\lambda_n\left(L^2\beta_n(1+\lambda_n) - 1\right) - k_n^2\right]\|v_n\|\right. \\ &\quad \left. + k_n^2(\lambda_n(1+\lambda_n)\|y_n - p\|)\right)\|v_n\| + (\sigma_n\|x_{n+1} - p\| \\ &\quad - \|\alpha_n T^n v_n - \Delta_n\|)\|\alpha_n T^n v_n - \Delta_n\|; \quad n \geq 0\end{aligned} \tag{66}$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\|x_{n+1} - p\|^2 - \|x_n - p\|^2\right) &\leq M_1 M_2 + \left(\sup_{n\geq 0}\|x_n\right. \\ &\quad \left. - T^n x_n\|^2\right)\left(\sum_{n=0}^{\infty}\alpha_n\beta_n\left(k_n^2(\beta_n + k - 1)\right.\right. \\ &\quad \left.\left. + k\left(L^2\beta_n(1+\lambda_n)(\beta_n + \lambda_n) + \beta_n - 1\right)\right)\right) + M_3\end{aligned} \tag{67}$$

for some finite real constant $M_3$; since $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, and $\{\Delta_n\}_{n=0}^{\infty}$ are summable as a result following from condition (c), the definitions of $\{u_n\}_{n=0}^{\infty}$, $\{v_n\}_{n=0}^{\infty}$, and $\{\Delta_n\}_{n=0}^{\infty}$ from (4) and (43) with $M_1 = \sup_{n\geq 0}(\alpha_n(k_n + 1)(1 + k_n^2\beta_n)\|x_n - p\|^2) <$

$+\infty$, $T : S \longrightarrow S$, and $S$ are bounded so $\{\|x_n - p\|\}$ and $\{\|x_n - T^n x_n\|\}_{n=0}^{\infty}$ are bounded and $\{k_n\}_{n=0}^{\infty}$ is bounded since $T : S \longrightarrow S$ is asymptotically demicontractive and $M_2 = \sum_{n=0}^{\infty}(k_n - 1) < \infty$. If condition (d1) holds then the convex parabola $p_n = p_n(\beta_n) = k_n^2(\beta_n + k - 1) + k(L^2\beta_n(1+\lambda_n)(\beta_n+\lambda_n) + \beta_n - 1) \le 0$ for any $n \ge 0$ such that $\beta_n \in [\max(0, \beta_{1n}), \beta_{0n}]$ where $\beta_{1n} \le 0$ and $\beta_{0n} > 0$ are the roots of $p_n(\beta_n) = 0$. Therefore, one gets from (67), since $\min_{n\ge 0}(\alpha_n\beta_n) \ge ab > 0$, that

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\|x_{n+1} - p\|^2 - \|x_n - p\|^2\right)\\&\quad + \sup_{n\ge 0}\left(\alpha_n\beta_n|p_n|\right)\left(\sum_{n=0}^{\infty}\|x_n - T^n x_n\|^2\right) \le M_4\\&\quad = M_1M_2 + M_3.\end{aligned} \tag{68}$$

As a result, $\{x_n - p\}_{n=0}^{\infty} \longrightarrow (x^* - p)$, $\{x_n\}_{n=0}^{\infty} \longrightarrow x^*(\in S)$, since $S$ is closed, and $\{\|x_n - T^n x_n\|\}_{n=0}^{\infty} \longrightarrow 0$. Conditions (d2) allow modifying (68) as

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\|x_{n+1} - p\|^2 - \|x_n - p\|^2\right)\\&\quad + \sup_{n\ge 0}\left[\alpha_n\beta_n k_n^2(1 - k - \beta_n)\right]\left(\sum_{n=0}^{\infty}\|x_n - T^n x_n\|^2\right)\\&\quad \le M_6 = M_4 + M_5\end{aligned} \tag{69}$$

where $M_5 = k\sum_{n=0}^{\infty}\alpha_n\beta_n(L^2\beta_n(\beta_n+\lambda_n) + \beta_n - 1)\sup_{n\ge 0}\|x_n - T^n x_n\|^2$ and the same conclusions on the convergence of the involved sequences follow. On the other hand, conditions (d3) allow modifying (68) as

$$\begin{aligned}&\sum_{n=0}^{\infty}\left(\|x_{n+1} - p\|^2 - \|x_n - p\|^2\right)\\&\quad + k\sup_{n\ge 0}\left[\alpha_n\beta_n\left(L^2\beta_n(1+\lambda_n)(\beta_n+\lambda_n) + \beta_n\right.\right.\\&\quad \left.\left. - 1\right)\right]\left(\sum_{n=0}^{\infty}\|x_n - T^n x_n\|^2\right) \le M_8 = M_4 + M_7\end{aligned} \tag{70}$$

where $M_7 = \sum_{n=0}^{\infty}\alpha_n\beta_n k_n^2(k + \beta_n - 1)\sup_{n\ge 0}\|x_n - T^n x_n\|^2$ and the same conclusions follow again on the convergence of the involved sequences.

Now, it is proven that $\{\|x_n - Tx_n\|\}_{n=0}^{\infty} \longrightarrow 0$ proceeding by contradiction. Assume that $\{\|x_n - Tx_n\|\}_{n=0}^{\infty}$ does not converge to 0 as $n \longrightarrow \infty$. Then, there is a subsequence $\{\|x_{n_i} - Tx_{n_i}\|\}_{i=0}^{\infty}(\subseteq \{\|x_n - T^n x_n\|\}_{n=0}^{\infty})$ which does not converge to zero. Since $T$ is uniformly $L$-Lipschitzian, one has

$$\begin{aligned}L\|x_n - T^n x_n\| &\ge \left\|T^{n+1}x_n - Tx_n\right\|\\&\ge \left|\|x_n - Tx_n\| - \left\|x_n - T^{n+1}x_n\right\|\right|\\&\ge \left|\|x_n - Tx_n\| - \|x_n - T^n x_n\| + \left\|T^n x_n - T^{n+1}x_n\right\|\right|\\&\ge \left|\|x_n - Tx_n\| - \|x_n - T^n x_n\| - L\|x_n - Tx_n\|\right|\\&= \left|(1 - L)\|x_n - Tx_n\| - \|x_n - T^n x_n\|\right|\\&\ge |1 - L|\,\|x_n - Tx_n\| - \|x_n - T^n x_n\|;\quad n \ge 0\end{aligned} \tag{71}$$

and then, since $\{\|x_n - T^n x_n\|\}_{n=0}^{\infty} \longrightarrow 0$, the subsequence $\{\|x_{n_i} - T^n x_{n_i}\|\}_{i=0}^{\infty}(\subseteq \{\|x_n - T^n x_n\|\}_{n=0}^{\infty}) \longrightarrow 0$ with $n_{i+1} > n_i$ for any $i \ge 0$ such that $\|x_{n_i} - Tx_{n_i}\| \le ((1+L)/|1-L|)\|x_{n_i} - T^n x_{n_i}\|$. Thus, $\{\|x_{n_i} - Tx_{n_i}\|\}_{i=0}^{\infty} \longrightarrow 0$, hence a contradiction. Then, $\{\|x_n - T^n x_n\|\}_{n=0}^{\infty} \longrightarrow 0$ implies that $\{\|x_n - Tx_n\|\}_{n=0}^{\infty} \longrightarrow 0$. Since the set $S$ is bounded and closed and $T$ is completely continuous, $\{Tx_n\}_{n=0}^{\infty}$ has a subsequence $\{Tx_{n_i}\}_{i=0}^{\infty}$ such that $\{Tx_{n_i}\}_{i=0}^{\infty} \longrightarrow q$ for some $q \in S$. Since $\{\|x_{n_i} - Tx_{ni}\|\}_{i=0}^{\infty} \longrightarrow 0$, $\{x_{n_i}\}_{i=0}^{\infty} \longrightarrow q$ and $q \in F_T$ since $T$ is continuous. Now, the constraints (68) to (70) imply in any case that $\|x_{n+1} - p\|^2 \le \|x_n - p\|^2 + |s_n|$ for $n \ge 0$ such that $\sum_{n=0}^{\infty}|s_n| < \infty$ with $\{x_n - p\}_{n=0}^{\infty} \longrightarrow (x^* - p)$ and $\{x_{n_i} - q\}_{i=0}^{\infty}(\subseteq \{x_n - q\}_{n=0}^{\infty}) \longrightarrow 0$. One concludes from Lemma 1.7 of [1] that $\{x_n - q\}_{n=0}^{\infty} \longrightarrow (x^* - q) = 0$, hence $\{x_n - p\}_{n=0}^{\infty} \longrightarrow x^* = q$. □

The following result is a parallel one to Theorem 12 under the assumption that $T$ satisfies an *asymptotically demicontractive-like condition* (2), in the sense that (2) holds with $k_n \in [0, 1]$ and $\{k_n\} \longrightarrow 1$ and it has a unique fixed point.

**Theorem 13.** *Let $S$ be a nonempty bounded closed subset of a of a Hilbert space $H$ and let $T : S \longrightarrow S$ be a uniformly $L$-Lipschitzian satisfying (2) and being continuous at $p \in F_T$ such that $S \cap F_T = \{p\}$ and $\{k_n\}_{n=0}^{\infty} \subseteq [0, 1]$, $\{k_n\}_{n=0}^{\infty} \longrightarrow 1$ and $k \in [0, 1)$. Suppose also that the following conditions hold:*

(a) $\{\alpha_n(k_n^2(1 - \beta_n(1 - k_n^2)) - 1)\}_{n=0}^{\infty} \longrightarrow 0$ *and* $\lim_{n\longrightarrow\infty}\sum_{j=0}^{n}\alpha_j(k_j^2(1 - \beta_j(1 - k_n^2)) - 1) = +\infty$,

(b) $0 < a \le \alpha_n \le \min(1 - k(1 - \beta_n), 1/(1 - k_n^2))$, $n \ge 0$,

(c) $\{\beta_n I_r - B_n\}_{n=0}^{\infty}$ *and* $\{\alpha_n I_r - A_n\}_{n=0}^{\infty}$ *are summable, and either (d1) or (d2) below holds:*

(d1) $\beta_n \in [\min(0, (a - (1 - k))/k), \min((1 - \alpha_n(1 - k_n^2))/\alpha_n k_n^2(1 - k_n^2), \beta_{1n}))$, $n \ge 0$, *where*

$$\begin{aligned}\beta_{1n} = \frac{1}{2kL^2}&\left[\left(kL^2\lambda_n + k + k_n^2\right)\right.\\&\left. + 4kL^2\left(k\left(1 - k_n^2\right) + k_n^2\right)\right]^{1/2}\end{aligned} \tag{72}$$

(d2) $\beta_n \in [\min((a - (1 - k))/k, \beta_{1n}), 1]n \ge 0$ *and* $\{\alpha_n\beta_n(k_n^2(\beta_n + k - 1) + k(L^2\beta_n(\beta_n + \lambda_n) + \beta_n - 1))\}_{n=0}^{\infty}$ *is summable.*

*Then,* $\{x_n\}_{n=0}^{\infty} \longrightarrow p$.

*Proof.* It is possible to obtain from (66), under the given conditions, that

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \left[1-\alpha_n\left(k_n^2\left(1-\beta_n\left(1-k_n^2\right)\right)-1\right)\right]\|x_n \\ &-p\|^2+\alpha_n\beta_n\left(k_n^2\left(\beta_n+k-1\right)\right. \\ &+k\left(L^2\beta_n\left(\beta_n+\lambda_n\right)+\beta_n-1\right)\Big)\|x_n-T^nx_n\|^2 \\ &-\alpha_n\left|k\left(1-\beta_n\right)+\alpha_n-1\right|\|x_n-T^ny_n\|^2 \\ &+\alpha_n\left(\left[k\lambda_n\left(L^2\beta_n-1\right)-k_n^2\right]\|v_n\|\right. \\ &+k_n^2\left(\lambda_n\|y_n-p\|\right)\Big)\|v_n\|+\left(\sigma_n\|x_{n+1}-p\|\right. \\ &-\|\alpha_nT^nv_n-\Delta_n\|\Big)\|\alpha_nT^nv_n-\Delta_n\|\le\Big[1 \\ &-\alpha_n\left(k_n^2\left(1-\beta_n\left(1-k_n^2\right)\right)-1\right)\Big]\|x_n-p\|^2+\gamma_n; \\ &\qquad n\ge 0\end{aligned} \tag{73}$$

where

$$\begin{aligned}\gamma_n &= \gamma_{1n}+\alpha_n\left(\left[k\lambda_n\left(L^2\beta_n-1\right)-k_n^2\right]\|v_n\|\right. \\ &+k_n^2\left(\lambda_n\|y_n-p\|\right)\Big)\|v_n\|+\left(\sigma_n\|x_{n+1}-p\|\right. \\ &-\|\alpha_nT^nv_n-\Delta_n\|\Big)\|\alpha_nT^nv_n-\Delta_n\|;\quad n\ge 0\end{aligned} \tag{74}$$

and $\gamma_{1n}=0$ under condition (d1) since the second right-hand-side term of (73) is nonpositive and can be omitted in the analysis and

$$\begin{aligned}\gamma_{1n} &= \alpha_n\beta_n\left(k_n^2\left(\beta_n+k-1\right)\right. \\ &+k\left(L^2\beta_n\left(\beta_n+\lambda_n\right)+\beta_n-1\right)\Big)\|x_n-T^nx_n\|^2\end{aligned} \tag{75}$$

under condition (d2) is nonnegative and $\{\gamma_{1n}\}_{n=0}^{\infty}$ is summable since $\{\|x_n-T^nx_n\|\}_{n=0}^{\infty}$ is bounded since $S$ is bounded. The third right-hand-side term of (73) is nonpositive under condition (b), and it can be omitted in the analysis, and the last two terms of (73) are summable from condition (c). Then, $\{\gamma_n\}_{n=0}^{\infty}$ is summable and has nonnegative elements while $\{\alpha_n(k_n^2(1-\beta_n(1-k_n^2))-1)\}_{n=0}^{\infty}\longrightarrow 0$ and $\lim_{n\longrightarrow\infty}\sum_{k=0}^{n}\alpha_k(k_j^2(1-\beta_j(1-k_n^2))-1)=+\infty$ from conditions (a). The result $\{x_n-p\}_{n=0}^{\infty}\longrightarrow 0$ follows from Venter's theorem, [14]. □

**Corollary 14.** *Theorem 13 holds if the second constraint of the condition (a) is modified as follows:*

$$\limsup_{n\longrightarrow\infty}\left(n-\sum_{j=0}^{n}\alpha_j\left(k_j^2\left(1-\beta_j\left(1-k_n^2\right)\right)-1\right)-g_n\right)\le 0 \tag{76a}$$

$$\liminf_{n\longrightarrow\infty}\left(n-\sum_{j=0}^{n}\alpha_j\left(k_j^2\left(1-\beta_j\left(1-k_n^2\right)\right)-1\right)-g_{0n}\right)\ge 0 \tag{76b}$$

*for some real sequences* $\{g_n\}_{n=0}^{\infty}\subseteq[0,K_g]\subset[0,\infty)$ *and* $\{g_{0n}\}_{n=0}^{\infty}\subseteq[0,K_g]$ *fulfilling* $g_{0n}\le g_n$ *for* $n\ge 0$.

*Proof.* Note that ((76a) and (76b)) are guaranteed if

$$\begin{aligned}n-g_{0n} &\ge \sum_{j=0}^{n}\alpha_j\left(k_j^2\left(1-\beta_j\left(1-k_n^2\right)\right)-1\right)\ge n-g_n \\ &\ge n-K_g;\quad n\ge 0\end{aligned} \tag{77}$$

and there exists some finite integer $n_0\ge\max(1,K_g)$ such that

$$\sum_{j=0}^{n}\alpha_j\left(k_j^2\left(1-\beta_j\left(1-k_n^2\right)\right)-1\right)\ge 0\quad\text{for } n\ge n_0 \tag{78}$$

and, since $\{\alpha_n\}_{n=0}^{\infty}\subset(0,1)$ and $\{k_n\}_{n=0}^{\infty}\longrightarrow 1$, one has $\{\alpha_n(k_n^2(1-\beta_n(1-k_n^2))-1)\}_{n=0}^{\infty}\longrightarrow 0$ since

$$\lim_{n\longrightarrow\infty}\sum_{j=0}^{n}\alpha_j\left(k_j^2\left(1-\beta_j\left(1-k_n^2\right)\right)-1\right)=+\infty. \tag{79}$$

Therefore, condition (a) of Theorem 13 holds. □

Two examples are now given concerning the use of simple auxiliary asymptotically demicontractive self-mappings in the generalized modified Ishikawa's iterative schemes under matrix parameterizing sequences.

*Example 15.* The condition that the mapping $T$ is asymptotically demicontractive prior to its use as an auxiliary self-mapping in a generalized modified Ishikawa's scheme includes many other stronger properties for such a self-mapping as, for instance, contractive, asymptotically contractive, and asymptotically pseudocontractive self-mappings. Consider a self-mapping defined on the first closed orthant of $\boldsymbol{R}^n$ which generates sequences $\{x_n\}_{n=0}^{\infty}$ as $x_n=T^nx$ with $x=x_0$ for $x_0\in\boldsymbol{R}_{0+}^n$, such that $F(T)\supset\{0\}$, where $\boldsymbol{R}_{0+}=\{z\in\boldsymbol{R}:z_0\ge 0\}$, subject to

$$\begin{aligned}\|x_n\|^2 &= k_n^2\|x\|^2+k\|x_n-x\|^2-\nu_n\|x\|^2 \\ &\le k_n^2\|x\|^2+k\|x_n-x\|^2;\quad n\ge 0\end{aligned} \tag{80}$$

for some sequences $\{k_n\}_{n=0}^{\infty}\subset[1,\infty)$ such that $\{k_n\}_{n=0}^{\infty}\longrightarrow 1$and $\{\nu_n\}_{n=0}^{\infty}\subset[0,k_n^2\|x\|^2+k\|x_n-x\|^2]$, and some real constant $k\in[0,1)$. From the above constraint, it follows that

$$\|x_n\|^2=\frac{k_n^2+2k}{1-2k}\|x\|^2-\frac{\nu_n}{1-2k}\|x\|^2;\quad n\ge 0 \tag{81}$$

provided that $k \in [0, 1/2)$. Assume also that $k_n^2 = 1 - \gamma_n + \mu_n$ for $n \geq 0$ with $\mu_n \geq \gamma_n \geq 0$ for $n \geq 0$, $\sum_{n=0}^{\infty} \gamma_n = +\infty$, $\{\mu_n\}_{n=0}^{\infty}$, $\{\gamma_n\}_{n=0}^{\infty} \longrightarrow 0$, and

$$\{\nu_n\}_{n=0}^{\infty} \subseteq [0, \min(\mu_n + 2k(2-\gamma_n), 1 + \mu_n - \gamma_n + 2k)] \tag{82}$$

since $k_n^2 = 1 - \gamma_n + \mu_n$, with $\{\nu_n\}_{n=0}^{\infty} \longrightarrow 4k$. One has from (81) that such an equation can be rewritten as

$$\|x_{n+1}\|^2 = \frac{k_{n+1}^2 + 2k - \nu_{n+1}}{k_n^2 + 2k - \nu_n} \|x_n\|^2; \quad n \geq 0 \tag{83}$$

and also as

$$\begin{aligned} \|x_n\|^2 &= \frac{1 + \mu_n - \gamma_n + 2k}{1 - 2k} \|x\|^2 - \frac{\nu_n}{1 - 2k} \\ &= (1 - \gamma_n) \|x\|^2 + \frac{\mu_n - \nu_n + 2k(2 - \gamma_n)}{1 - 2k} \|x\|^2; \\ &\qquad n \geq 0. \end{aligned} \tag{84}$$

Thus, $\{x_n\}_{n=0}^{\infty}$ converges to some fixed point of the self-mapping $T$ on $\boldsymbol{R}_{0+}^n$, such that $\{\|x_n\|\}_{n=0}^{\infty} \longrightarrow \|x\|$; that is, the fixed point has the same norm as that of the initial condition of the generated sequence which is, furthermore, asymptotically demicontractive and asymptotically nonexpansive in view of (80) and (81). If a generalized modified Ishikawa's iterative process with matrix parameterizing sequences is implemented with such an auxiliary mapping $T$ on $\boldsymbol{R}_{0+}^n$, then any such obtained generated sequence converges asymptotically to zero under the given constraints of the results in the paper body. That is the case of Theorems 3 and 4 and Theorem 13 under the respective groups of invoked extra constraints on the matrix parameterizing sequences.

*Example 16.* Consider a self-mapping $T$ on $\boldsymbol{R}_{0+}^n$ with $F(T) \supset \{0\}$ such that any sequence $\{x_n\}_{n=0}^{\infty}$ generated as $x_{n+1} = Tx_n = T^{n+1}x_0$, $n \geq 0$, for a given $x_0 \in \boldsymbol{R}_{0+}^n$, such that for some norm $\|\cdot\|$ (for instance, the Euclidean norm, $z_n = \|x_n\|$ for $n \geq 0$) satisfies the following norm constraint:

$$z_{n+1} = (1 + \gamma_n - \delta_n) z_n + \omega_n; \quad n \geq 0 \tag{85}$$

where $\{\gamma_n\}_{n=0}^{\infty}$, $\{\delta_n\}_{n=0}^{\infty}$, and $\{\omega_n\}_{n=0}^{\infty}$ are parameterizing real sequences subject to the constraints:

(a) $\delta_n \geq \gamma_n \geq 0$, $n \geq 0$;

(b) $\delta_n z_n + \sqrt{k}|z_n - z_0| \geq \omega_n \geq 0$, $n \geq 0$, for some $k \in [0, 1)$;

(c) $\{\delta_n\}_{n=0}^{\infty} \longrightarrow 0$; $\{\gamma_n\}_{n=0}^{\infty} \longrightarrow 0$;

(d) $\sum_{n=0}^{\infty} \delta_n = +\infty$;

(e) $\sum_{n=0}^{\infty} \gamma_n = +\infty$;

(f) $\sum_{n=0}^{\infty} (\delta_n - \gamma_n) z_n < +\infty$;

(g) $\sum_{n=0}^{\infty} (\omega_n + \gamma_n z_n) < +\infty$.

Note that (81) is identical to

$$z_{n+1} = (1 - \delta_n) z_n + \gamma_n z_n + \omega_n; \quad n \geq 0. \tag{86}$$

Note from (a), (b), (c), (d), (g), and Venter's theorem, [14], that $\{z_n\}_{n=0}^{\infty} \longrightarrow 0$ which implies that $\{x_n\}_{n=0}^{\infty} \longrightarrow 0$. Note also that, since $z_n \geq 0$ for $n \geq 0$, (85), or (86), and the constraints (b) and (f) imply that

$$z_{n+1} = |z_{n+1}| \leq (1 + \gamma_n) z_n + \sqrt{k} |z_n - z_0|; \quad n \geq 0 \tag{87}$$

with $\{k_n\}_{n=0}^{\infty} \subset [1, \infty)$, where $k_n = \sqrt{1 + \gamma_n}$, $n \geq 0$, so that $\{k_n\}_{n=0}^{\infty} \longrightarrow 1$ from the constraint (c). Since (87) implies that the squared norm of the elements of $\{x_n\}_{n=0}^{\infty}$ satisfies the asymptotic demi-contractiveness condition (2), it follows that the self-mapping $T$ on $S \equiv \boldsymbol{R}_{0+}^n$ is asymptotically demicontractive and $\{x_n\}_{n=0}^{\infty} \longrightarrow 0$ which is a similar conclusion to that obtained from Venter's theorem.

Note that conditions (b) and (f) can be guaranteed if there exist either a finite set or sequence of nonnegative integer numbers $\{n_k\}_{k=0}^{c_{nS}}$ with $c_{nS} = card(\{n_k\}_{k=0}^{c_{nS}}) \leq \chi_0$ ($\chi_0$ denoting the infinity cardinal of denumerable sets) and an associated set or sequence of nonnegative real numbers $\{M_n\}_{n=0}^{c_{nS}}$ such that

$$\begin{aligned} &\sum_{k=0}^{c_{nS}} \sum_{n=n_k}^{n_{k+1}} \left[ (\omega_n - \gamma_n) z_n - \sqrt{k} |z_n - z_0| \right] \\ &\quad \leq \sum_{k=0}^{c_{nS}} \sum_{n=n_k}^{n_{k+1}} (\delta_n - \gamma_n) z_n = \sum_{n=0}^{\infty} (\delta_n - \gamma_n) z_n = M \\ &\quad = \sum_{k=0}^{c_{nS}} M_k < +\infty \end{aligned} \tag{88}$$

which holds if the parameterizing sequences fulfil the constraints:

$$\begin{aligned} &\delta_n - \gamma_n \leq \frac{M_k}{x_n}; \\ &\qquad n = n_k, n_k + 1, \ldots, n_{k+1} - 1, \ k = 0, 1, \ldots, c_{nS} \end{aligned} \tag{89}$$

with the convention that $n_{c_{nS}+1} - 1 = +\infty$. Note that the condition (g) with (b) is guaranteed, for instance, if

$$\sum_{n=0}^{\infty} \left(\delta_n + \gamma_n + \sqrt{k}\right) z_n = M_a = \sum_{k=0}^{c_{nSa}} M_{ak} < +\infty \tag{90}$$

for a finite set or for sequence of nonnegative real numbers $\{M_{an}\}_{n=0}^{c_{nSa}}$, which is guaranteed if

$$\begin{aligned} &\delta_n - \gamma_n + \sqrt{k} \leq \frac{M_k}{x_n}; \\ &\qquad n = n_k, n_k + 1, \ldots, n_{k+1} - 1, \ k = 0, 1, \ldots, c_{nSa}. \end{aligned} \tag{91}$$

If a generalized modified Ishikawa's iterative process with matrix parameterizing sequences is implemented with such an auxiliary mapping $T$ on $\boldsymbol{R}_{0+}^n$, then any such obtained generated sequence converges asymptotically to zero under the given constraints of the above results given in the paper body.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The author declares that he does not have any conflicts of interest.

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