Research Article

Eigenvalue Bounds for a Class of Schrödinger Operators in a Strip

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This paper is concerned with the estimation of the number of negative eigenvalues (bound states) of Schrödinger operators in a strip subject to Neumann boundary conditions. The estimates involve weighted L^1 norms and $L \ln L$ norms of the potential. Estimates involving the norms of the potential supported by a curve embedded in a strip are also presented.

1. Introduction

The celebrated Cwikel-Lieb-Rozenblum (CLR) inequality [1] gives an upper bound for the number of negative eigenvalues of the Schrödinger operator $-\Delta - V$, $V \geq 0$ on $L^2(\mathbb{R}^d)$, $d > 2$. It is known that the CLR inequality does not hold for $d = 2$ and one of the reasons is that the Sobolev space $H^1(\mathbb{R}^2)$ is not continuously embedded in $L^\infty(\mathbb{R}^2)$. However, $H^1(\mathbb{R}^2) \hookrightarrow L^q(\mathbb{R}^2)$ for all $q \in [2, +\infty)$ and there are estimates involving $\int_{\mathbb{R}^2} |V(x)|^r dx$, $\forall r > 1$ (see, e.g., [2–4]). More precisely, $H^1(\mathbb{R}^2)$ is embedded in a space of exponentially integrable functions which lies between $L^1(\mathbb{R}^2)$ and $L^p(\mathbb{R}^2)$, $p > 1$ (see, e.g., [5]). This gives rise to estimates involving a norm of V weaker than $\|V\|_{L^r}$, $r > 1$, namely, the Orlicz $L \ln L$ norm (see, e.g., [6, 7]). The strongest known estimates have been obtained in [6]. For more information regarding upper estimates for the number of negative eigenvalues of two-dimensional Schrödinger operators refer to [2–4, 6–9] and the references therein. This paper provides estimates for the number of negative eigenvalues of the Schrödinger operator $-\Delta - V$ on $L^2(S)$ whose domain is characterized by the Neumann boundary conditions, where S is an infinite straight strip. We use the results of Shargorodsky [6] to obtain improved versions of the estimates by Grigor'yan and Nadirashvili [3]. This improvement is achieved by replacing $\|V\|_{L^p}$, $p > 1$ in the estimates of [3, Section 7] by the $L \ln L$ norms of V . In addition, these estimates are extended to the case of strongly singular potentials (see Section 4).

The precise description of the operator here studied is as follows.

Let $S := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \in \mathbb{R}, 0 < x_2 < a\}$, $a > 0$ and $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function integrable on bounded subsets of S . Consider the following self-adjoint operator on $L^2(S)$:

$$H_V = -\Delta - V, \quad V \geq 0, \quad (1)$$

with homogeneous Neumann boundary conditions both at $x_2 = 0$ and $x_2 = a$. The main objective of this paper is to obtain estimates for the number of negative eigenvalues of (1) in terms of the norms of V .

The strategy used here is as follows: The problem is split into two problems. The first one is defined by the restriction of the quadratic form associated with operator (1) to the subspace of functions of the form $w(x_1)u_1(x_2)$, where u_1 is the first eigenfunction of the one-dimensional differential operator on $L^2((0, a))$ and, hence, is reduced to a well studied one-dimensional Schrödinger operator with the potential equal to a weighted mean value $\tilde{V}$ of V over $(0, a)$. The second problem is defined by a class of functions orthogonal to constant functions in the $L^2((0, a))$ inner product.

2. Preliminaries

Let (Ω, Σ, μ) be a measure space and let $\Psi : [0, +\infty) \rightarrow [0, +\infty)$ be a nondecreasing function. The Orlicz class $K_\Psi(\Omega)$ is the set of all of measurable functions $f : \Omega \rightarrow \mathbb{C}$ (or $\mathbb{R}$) such that

$$\int_{\Omega} \Psi(|f(x)|) d\mu(x) < \infty. \quad (2)$$

If $\Psi(t) = t^p, 1 \leq p < \infty$, this is just the $L^p(\Omega)$ space. The Orlicz space $L_{\Psi}(\Omega)$ is the linear span of the Orlicz class $K_{\Psi}(\Omega)$, that is, the smallest vector space containing $K_{\Psi}(\Omega)$.

Definition 1. A continuous nondecreasing convex function $\Psi : [0, +\infty) \rightarrow [0, +\infty)$ is called an N -function if

$$\begin{aligned} \lim_{t \rightarrow 0+} \frac{\Psi(t)}{t} &= 0 \\ \text{and } \lim_{t \rightarrow \infty} \frac{\Psi(t)}{t} &= \infty. \end{aligned} \quad (3)$$

The function $\Phi : [0, +\infty) \rightarrow [0, +\infty)$ defined by

$$\Phi(t) := \sup_{s \geq 0} (st - \Psi(s)) \quad (4)$$

is called complementary to Ψ .

Definition 2. An N -function Ψ is said to satisfy a global Δ_2 -condition if there exists a positive constant k such that, for every $t \geq 0$,

$$\Psi(2t) \leq k\Psi(t). \quad (5)$$

Similarly Ψ is said to satisfy a Δ_2 -condition near infinity if there exists $t_0 > 0$ such that (5) holds for all $t \geq t_0$.

Let Φ and Ψ be mutually complementary N -functions, and let $L_{\Phi}(\Omega)$ and $L_{\Psi}(\Omega)$ be the corresponding Orlicz spaces. We will use the following norms on $L_{\Psi}(\Omega)$:

$$\|f\|_{\Psi} = \|f\|_{\Psi, \Omega} = \sup \left\{ \left| \int_{\Omega} fg d\mu \right| : \int_{\Omega} \Phi(|g|) d\mu \leq 1 \right\} \quad (6)$$

and

$$\begin{aligned} \|f\|_{(\Psi)} &= \|f\|_{(\Psi, \Omega)} \\ &= \inf \left\{ \kappa > 0 : \int_{\Omega} \Psi \left(\frac{|f|}{\kappa} \right) d\mu \leq 1 \right\}. \end{aligned} \quad (7)$$

These two norms are equivalent:

$$\|f\|_{(\Psi)} \leq \|f\|_{\Psi} \leq 2 \|f\|_{(\Psi)}, \quad \forall f \in L_{\Psi}(\Omega) \quad (8)$$

(see [5]).

Note that

$$\int_{\Omega} \Psi \left(\frac{|f|}{\kappa_0} \right) d\mu \leq C_0, \quad C_0 \geq 1 \implies \|f\|_{(\Psi)} \leq C_0 \kappa_0. \quad (9)$$

It follows from (9) with $\kappa_0 = 1$ that

$$\|f\|_{(\Psi)} \leq \max \left\{ 1, \int_{\Omega} \Psi(|f|) d\mu \right\}. \quad (10)$$

We will need the following equivalent norm on $L_{\Psi}(\Omega)$ with $\mu(\Omega) < \infty$ which was introduced in [7]:

$$\begin{aligned} \|f\|_{\Psi}^{(\text{av})} &= \|f\|_{\Psi, \Omega}^{(\text{av})} \\ &= \sup \left\{ \left| \int_{\Omega} fg d\mu \right| : \int_{\Omega} \Phi(|g|) d\mu \leq \mu(\Omega) \right\}. \end{aligned} \quad (11)$$

We will use the following pair of pairwise complementary N -functions:

$$\begin{aligned} \mathcal{A}(s) &= e^{|s|} - 1 - |s|, \\ \mathcal{B}(s) &= (1 + |s|) \ln(1 + |s|) - |s|, \end{aligned} \quad (12)$$

$$s \in \mathbb{R}.$$

Let $I_1, I_2 \subseteq \mathbb{R}$ be nonempty open intervals. We denote by $L_1(I_1, L_{\mathcal{B}}(I_2))$ the space of measurable functions $f : I_1 \times I_2 \rightarrow \mathbb{C}$ such that

$$\|f\|_{L_1(I_1, L_{\mathcal{B}}(I_2))} := \int_{I_1} \|f(x_1, \cdot)\|_{\mathcal{B}, I_2} dx_1 < +\infty. \quad (13)$$

Let us recall that a sequence $\{a_n\}$ belongs to the “weak l_1 -space” (Lorentz space) $l_{1,w}$ if the quasinorm

$$\|\{a_n\}\|_{1,w} = \sup_{s > 0} (s \text{ card } \{n : |a_n| > s\}) \quad (14)$$

is finite. It is a quasinorm in the sense that it satisfies the weak version of the triangle inequality:

$$\|\{a_n\} + \{b_n\}\|_{1,w} \leq 2 (\|\{a_n\}\|_{1,w} + \|\{b_n\}\|_{1,w}). \quad (15)$$

Quasinorm (14) induces a topology on $l_{1,w}$ in which this space is nonseparable. The closure of the set of elements a_n with only finite number of nonzero terms is a separable subspace in $l_{1,w}$. It is well known that $l_1 \subset l_{1,w}$ and

$$\|\{a_n\}\|_{1,w} \leq \|\{a_n\}\|_{l_1} \quad (16)$$

(see, e.g., [10] for more details).

3. Estimating the Number of Negative Eigenvalues in a Strip

Define (1) via its quadratic form

$$q_{V,S}[u] := \int_S |\nabla u(x)|^2 dx - \int_S V(x) |u(x)|^2 dx, \quad (17)$$

$$\text{dom}(q_{V,S}) = W_2^1(S) \cap L^2(S, V dx),$$

where $W_2^1(S)$ denotes the standard Sobolev space $H^1(S)$. Let $N_-(q_{V,S})$ denote the number of negative eigenvalues of (1) repeated according to their multiplicities. Then $N_-(q_{V,S})$ is given by

$$N_-(q_{V,S}) = \sup_L \{\dim L : q_{V,S}[u] < 0, \forall u \in L \setminus \{0\}\}, \quad (18)$$

where L denotes a linear subspace of $\text{dom}(q_{V,S})$ (see, e.g., [10, Theorem 10.2.3]).

Let

$$\begin{aligned} I_n &:= [2^{n-1}, 2^n], \quad n > 0, \\ I_0 &:= [-1, 1], \\ I_n &:= [-2^{|n|}, -2^{|n|-1}], \quad n < 0, \end{aligned} \quad (19)$$

and

$$\begin{aligned}\mathcal{G}_n &:= \int_{I_n} \int_0^a |x_1| V(x) dx, \quad n \neq 0, \\ \mathcal{G}_0 &:= \int_{I_0} \int_0^a V(x) dx.\end{aligned}\quad (20)$$

Furthermore, let

$$\begin{aligned}J_n &:= (n, n+1), \quad n \in \mathbb{Z}, \\ I &:= (0, a) \\ \text{and } \mathcal{D}_n &:= \|V\|_{L_1(J_n, L_{\mathcal{B}}(I))}.\end{aligned}\quad (21)$$

Then we have the following result.

Theorem 3. *There exist constants $c, C > 0$ such that*

$$N_-(q_{V,S}) \leq 1 + C \left(\sum_{\{n \in \mathbb{Z}, \mathcal{G}_n > c\}} \sqrt{\mathcal{G}_n} + \sum_{\{n \in \mathbb{Z}, \mathcal{D}_n > c\}} \mathcal{D}_n \right), \quad (22)$$

$\forall V \geq 0.$

Let

$$L_1 := \{u \in L^2(S) : u(x) = u(x_1)\} \quad (23)$$

and $P : L^2(S) \rightarrow L_1$ be a projection defined by

$$Pv(x) := \frac{1}{a} \int_0^a v(x) dx_2 = Pv(x_1). \quad (24)$$

Indeed, P is a projection since $P^2 = P$. Let $L_2 := (I - P)L^2(S)$, and then one can show that $L^2(S) = L_1 \oplus L_2$. Here and below $\oplus$ denotes a direct orthogonal sum.

Indeed for all $v \in L_2$ we have

$$\begin{aligned}\int_0^a v(x) dx_2 &= \int_0^a (I - P)v(x) dx_2 \\ &= \int_0^a v(x) dx_2 - \int_0^a Pv(x) dx_2 \\ &= \int_0^a v(x) dx_2 - \int_0^a Pv(x_1) dx_2 \\ &= \int_0^a v(x) dx_2 - \int_0^a v(x) dx_2 = 0.\end{aligned}\quad (25)$$

Now pick $w \in L_1$ and $v \in L_2$, then,

$$\begin{aligned}\langle v, w \rangle_{L^2(S)} &= \int_S v(x) \overline{w(x_1)} dx \\ &= \int_{\mathbb{R}} \left(\int_0^a v(x) dx_2 \right) \overline{w(x_1)} dx_1 = 0.\end{aligned}\quad (26)$$

Similarly, let

$$\begin{aligned}\mathcal{H}_1 &:= \{u \in W_2^1(S) : u(x) = u(x_1)\} \\ \text{and } \mathcal{H}_2 &:= (I - P)W_2^1(S),\end{aligned}\quad (27)$$

and then

$$W_2^1(S) = \mathcal{H}_1 \oplus \mathcal{H}_2. \quad (28)$$

Indeed, for all $v \in \mathcal{H}_1$ and all $w \in \mathcal{H}_2$ we have

$$\begin{aligned}\langle v, w \rangle_{W_2^1(S)} &= \int_S \left(v(x_1) \overline{w(x)} + v_{x_1}(x_1) \overline{w_{x_1}(x)} \right. \\ &\quad \left. + v_{x_2}(x_1) \overline{w_{x_2}(x)} \right) dx = 0.\end{aligned}\quad (29)$$

This is so because $v, v_{x_1} \in L_1, \overline{w}, \overline{w_{x_1}} \in L_2$, and $v_{x_2} = 0$. To see this note that $v(x_1)$ and $v_{x_1}(x_1)$ do not depend on x_2 implying that $v_{x_1} \in L_1$. Also, $w \in L_2 \iff \int_0^a w(x) dx_2 = 0$. So, $(d/dx_1) \int_0^a w(x) dx_2 = 0 \implies \int_0^a w_{x_1}(x) dx_2 = 0 \implies w_{x_1} \in L_2$. Hence $\int_S v_{x_1}(x_1) \overline{w_{x_1}(x)} dx = 0$.

Now for all $u \in W_2^1(S)$, $u = v + w$, $v \in \mathcal{H}_1, w \in \mathcal{H}_2$ one has

$$\begin{aligned}\int_S |\nabla u(x)|^2 dx &= a \int_{\mathbb{R}} |v'(x_1)|^2 dx_1 + \int_S |\nabla w(x)|^2 dx \\ &\quad + \underbrace{\int_S \nabla v(x_1) \cdot \overline{\nabla w(x)} dx + \int_S \nabla w(x) \cdot \overline{\nabla v(x_1)} dx}_{=0} \\ &= a \int_{\mathbb{R}} |v'(x_1)|^2 dx_1 + \int_S |\nabla w(x)|^2 dx\end{aligned}\quad (30)$$

and

$$\begin{aligned}\int_S V(x) |u(x)|^2 dx &= \int_S V(x) |v(x_1)|^2 dx \\ &\quad + \int_S V(x) |w(x)|^2 dx \\ &\quad + \int_S 2V(x) \operatorname{Re}(v \cdot w) dx \\ &\leq 2 \int_{\mathbb{R}} \widetilde{V}(x_1) |v(x_1)|^2 dx_1 \\ &\quad + 2 \int_S V(x) |w(x)|^2 dx,\end{aligned}\quad (31)$$

where

$$\widetilde{V}(x_1) = \frac{1}{a} \int_0^a V(x) dx_2. \quad (32)$$

So

$$\begin{aligned}\int_S |\nabla u(x)|^2 dx - \int_S V(x) |u(x)|^2 dx &\geq \int_{\mathbb{R}} |v'(x_1)|^2 dx_1 - 2 \int_{\mathbb{R}} \widetilde{V}(x_1) |v(x_1)|^2 dx_1 \\ &\quad + \int_S |\nabla w(x)|^2 dx - 2 \int_S V(x) |w(x)|^2 dx.\end{aligned}\quad (33)$$

Hence

$$N_-(q_{V,S}) \leq N_-(q_{1,2\tilde{V},\mathbb{R}}) + N_-(q_{2,2V,S}), \quad (34)$$

where $q_{1,2\tilde{V},\mathbb{R}}$ and $q_{2,2V,S}$ denote the restrictions of the form $q_{2V,S}$ to the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Let

$$\begin{aligned} \mathcal{G}_n &:= \int_{I_n} |x_1| \tilde{V}(x_1) dx_1, \quad n \neq 0, \\ \mathcal{G}_0 &:= \int_{I_0} \tilde{V}(x_1) dx_1. \end{aligned} \quad (35)$$

Then similarly to the estimate before (39) in [6] one has

$$N_-(q_{1,2\tilde{V},\mathbb{R}}) \leq 1 + 7.61 \sum_{\{n \in \mathbb{Z}, \mathcal{G}_n > 0.046\}} \sqrt{\mathcal{G}_n}. \quad (36)$$

In terms of the original potential V

$$\begin{aligned} \mathcal{G}_n &= \int_{I_n} |x_1| \tilde{V}(x_1) dx_1 \\ &= \int_{I_n} |x_1| \left(\int_0^a V(x) dx_2 \right) dx_1 \\ &= \int_{I_n \times (0,a)} |x_1| V(x) dx \end{aligned} \quad (37)$$

and

$$\begin{aligned} \mathcal{G}_0 &= \int_{I_0} \tilde{V}(x_1) dx_1 = \int_{I_0} \left(\int_0^a V(x) dx_2 \right) dx_1 \\ &= \int_{I_0 \times (0,a)} V(x) dx. \end{aligned} \quad (38)$$

It now remains to find an estimate for $N_-(q_{2,2V,S})$ in (34).

Let $S_n := J_n \times I$, $n \in \mathbb{Z}$, where J_n and I are as defined in (21). Then the variational principle (see, e.g., [11, Ch.6, § 2.1, Theorem 4]) implies

$$N_-(q_{2,2V,S}) \leq \sum_{n \in \mathbb{Z}} N_-(q_{2,2V,S_n}), \quad (39)$$

where

$$\begin{aligned} q_{2,2V,S_n}[w] &:= \int_{S_n} |\nabla w(x)|^2 dx \\ &\quad - 2 \int_{S_n} V(x) |w(x)|^2 dx, \\ \text{dom}(q_{2,2V,S_n}) &= \left\{ w \in W_2^1(S_n) \right. \\ &\quad \left. \cap L^2(S_n, V(x) dx) : \int_{S_n} w(x) dx = 0 \right\}. \end{aligned} \quad (40)$$

Lemma 4 (cf. [6, Lemma 7.8]). *There exists $C_1 > 0$ such that*

$$N_-(q_{2,2V,S_n}) \leq C_1 \|V\|_{L_1(J_n, L_{\mathcal{B}}(I))}, \quad \forall V \geq 0 \quad (41)$$

(see (13)).

Proof. Let $I = \mathbb{I}$, the unit interval. Then it follows from [6, Lemma 7.7] that there is a constant $d_1 > 0$ such that

$$N_-(q_{2,2V,S_n}) \leq d_1 \|V\|_{L_1(J_n, L_{\mathcal{B}}(\mathbb{I}))} + 1, \quad \forall V \geq 0. \quad (42)$$

Similarly to (62) in [6] and using the Poincaré inequality (see, e.g., [12, 1.1.11]), there is a constant $d_2 > 0$ such that

$$\begin{aligned} 2 \int_{S_n} V(x) |w(x)|^2 dx \\ \leq d_2 \|V\|_{L_1(J_n, L_{\mathcal{B}}(\mathbb{I}))} \int_{S_n} |\nabla w(x)|^2 dx \end{aligned} \quad (43)$$

for all $w \in W_2^1(S_n) \cap L^2(S_n, V(x) dx)$ such that $\int_{S_n} w(x) dx = 0$. If $\|V\|_{L_1(J_n, L_{\mathcal{B}}(\mathbb{I}))} \leq 1/d_2$, then $N_-(q_{2,2V,S_n}) = 0$. Otherwise (42) implies

$$N_-(q_{2,2V,S_n}) \leq C_1 \|V\|_{L_1(J_n, L_{\mathcal{B}}(\mathbb{I}))}, \quad \forall V \geq 0, \quad (44)$$

where $C_1 := d_1 + d_2$. Hence (41) follows by the scaling $x_2 \mapsto ax_2$. $\square$

Proof of Theorem 3

Proof. If $\mathcal{D}_n < 1/C_1$, then $N_-(q_{2,2V,S_n}) = 0$ and one can drop this term from the sum (39). Hence, for any $c < 1/C_1$, (39) and Lemma 4 imply that

$$N_-(q_{2,2V,S}) \leq C_1 \sum_{\{n \in \mathbb{Z}, \mathcal{D}_n > c\}} \mathcal{D}_n \quad \forall V \geq 0. \quad (45)$$

This together with (34) and (36) imply (22). $\square$

One can easily show that (22) is an improvement of the estimates obtained by A. Grigor'yan and N. Nadirashvili [3, Theorem 7.9] with a different c and that (22) is strictly sharper. Indeed, let $B_n := \|V\|_{\mathcal{B}, S_n}$. Then it follows from the embedding $L^p(S_n) \hookrightarrow L_{\mathcal{B}}(S_n)$ that there is a constant $C(p)$, $p > 1$ such that

$$\begin{aligned} B_n &= \|V\|_{\mathcal{B}, S_n} \leq C(p) \left(\int_{S_n} V(x)^p dx \right)^{1/p} \\ &= C(p) b_n(V), \end{aligned} \quad (46)$$

where $b_n(V) := \left(\int_{S_n} V(x)^p dx \right)^{1/p}$ (see [6, Remark 6.3]). Now it follows from the known theory of embeddings of mixed-norm Orlicz spaces (see, e.g., [13, 14]) that

$$\mathcal{D}_n \leq C(p) b_n(V). \quad (47)$$

Hence

$$N_-(q_{2,2V,S_n}) \leq C_2 b_n(V), \quad \forall V \geq 0, \quad (48)$$

where $C_2 := C_1 C(p)$. The scaling $V \mapsto tV$, $t > 0$, allows one to extend the above inequality to an arbitrary $V \geq 0$. Thus, for any $c < 1/C_2$, (22) implies [3, Theorem 7.9].

Next we will discuss different forms of (22).

Remark 5. Note that

$$\sum_{\{n \in \mathbb{Z}, \mathcal{G}_n > c\}} \sqrt{\mathcal{G}_n} \leq \frac{2}{\sqrt{c}} \|(\mathcal{G}_n)_{n \in \mathbb{Z}}\|_{1,w} \quad (49)$$

(see (49) in [6]). Estimate (22) implies the following estimate:

$$N_-(q_{V,S}) \leq 1 + C_3 \left(\|(\mathcal{G}_n)_{n \in \mathbb{Z}}\|_{1,w} + \|V\|_{L_1(\mathbb{R}, L_{\mathcal{B}}(I))} \right), \quad (50)$$

$\forall V \geq 0.$

This follows from (49) and

$$\begin{aligned} \sum_{\{n \in \mathbb{Z}, \mathcal{D}_n > c\}} \mathcal{D}_n &\leq \sum_{n \in \mathbb{Z}} \mathcal{D}_n = \int_{\mathbb{R}} \|V(x_1, \cdot)\|_{\mathcal{B}, I} dx_1 \\ &= \|V\|_{L_1(\mathbb{R}, L_{\mathcal{B}}(I))}. \end{aligned} \quad (51)$$

Equation (50) in turn implies the following:

$$\begin{aligned} N_-(q_{V,S}) &\leq 1 + C_4 \left(\|(\mathcal{G}_n)_{n \in \mathbb{Z}}\|_{1,w} \right. \\ &\quad \left. + \int_{\mathbb{R}} \left(\int_I |V(x)|^p dx_2 \right)^{1/p} dx_1 \right), \quad \forall V \geq 0, \end{aligned} \quad (52)$$

which is equivalent to

$$\begin{aligned} N_-(q_{V,S}) &\leq 1 + C_5 \left(\|(\mathcal{G}_n)_{n \in \mathbb{Z}}\|_{1,w} \right. \\ &\quad \left. + \int_{\mathbb{R}} \left(\int_I |V_*(x)|^p dx_2 \right)^{1/p} dx_1 \right), \quad \forall V \geq 0, \end{aligned} \quad (53)$$

where $V_*(x) = V(x) - \tilde{V}(x_1)$, $\tilde{V}(x_1) = (1/a) \int_0^a V(x) dx_2$.
Indeed,

$$\begin{aligned} &\left| \int_{\mathbb{R}} \left(\int_I |V(x)|^p dx_2 \right)^{1/p} dx_1 \right. \\ &\quad \left. - \int_{\mathbb{R}} \left(\int_I |V_*(x)|^p dx_2 \right)^{1/p} dx_1 \right| \\ &\leq \int_{\mathbb{R}} \left(\int_I |\tilde{V}(x_1)|^p dx_2 \right)^{1/p} dx_1 \\ &= a^{1/p} \int_{\mathbb{R}} \tilde{V}(x_1) dx_1 \\ &= a^{1/p} \sum_{n \in \mathbb{Z}} \int_{I_n \times (0,a)} V(x) dx \leq a^{1/p} \sum_{n \in \mathbb{Z}} 2^{-|n|+1} \mathcal{G}_n \\ &\leq \text{const} \sup_{n \in \mathbb{Z}} \mathcal{G}_n \leq \text{const} \|(\mathcal{G}_n)_{n \in \mathbb{Z}}\|_{1,w}. \end{aligned} \quad (54)$$

Thus (52) and (53) are equivalent.

Similarly,

$$\begin{aligned} &\left| \|V\|_{L_1(\mathbb{R}, L_{\mathcal{B}}(I))} - \|V_*\|_{L_1(\mathbb{R}, L_{\mathcal{B}}(I))} \right| \\ &= \left| \int_{\mathbb{R}} \|V(x_1, \cdot)\|_{\mathcal{B}, I} dx_1 - \int_{\mathbb{R}} \|V_*(x_1, \cdot)\|_{\mathcal{B}, I} dx_1 \right| \\ &\leq \int_{\mathbb{R}} \|\tilde{V}(x_1)\|_{\mathcal{B}, I} dx_1 = \text{const} \int_{\mathbb{R}} |\tilde{V}(x_1)| dx_1 \\ &\leq \text{const} \sup_{n \in \mathbb{Z}} \mathcal{G}_n \leq \text{const} \|(\mathcal{G}_n)_{n \in \mathbb{Z}}\|_{1,w}. \end{aligned} \quad (55)$$

Hence (50) is equivalent to the following estimate:

$$N_-(q_{V,S}) \leq 1 + C_6 \left(\|(\mathcal{G}_n)_{n \in \mathbb{Z}}\|_{1,w} + \|V_*\|_{L_1(\mathbb{R}, L_{\mathcal{B}}(I))} \right), \quad (56)$$

$\forall V \geq 0.$

Note the last term on the right-hand side of (56) (and (53)) drops out if V does not depend on x_2 .

Let $\alpha > 0$ be given. It is well known that the lowest possible (semiclassical) rate of growth of $N_-(q_{\alpha V, S})$ is

$$N_-(q_{\alpha V, S}) = O(\alpha) \quad \text{as } \alpha \rightarrow +\infty \quad (57)$$

(see, e.g., [2, 4]). It turns out that the finiteness of the first term in (22) is necessary for $N_-(q_{\alpha V, S}) = O(\alpha)$ as $\alpha \rightarrow +\infty$ to hold (see the next theorem).

Theorem 6. Let $V \geq 0$. If $N_-(q_{\alpha V, S}) = O(\alpha)$ as $\alpha \rightarrow +\infty$, then $\|\mathcal{G}_n\|_{1,w} < \infty$.

Proof. Consider the function

$$w_n(x_1) := \begin{cases} 0, & x_1 \leq 2^{n-1} \text{ or } x_1 \geq 2^{n+2}, \\ 4(x_1 - 2^{n-1}), & 2^{n-1} < x_1 < 2^n, \\ 2^{n+1}, & 2^n \leq x_1 \leq 2^{n+1}, \\ 2^{n+2} - x_1, & 2^{n+1} < x_1 < 2^{n+2}, \end{cases} \quad (58)$$

for $n > 0$. Let $u_n(x) = w_n(x_1)u_1(x_2)$, where u_1 is the first eigenfunction of the one-dimensional second-order differential operator on $L^2((0, a))$ which is identically equal to 1. Then we have

$$\begin{aligned} \int_S |\nabla u_n(x)|^2 dx &= a \int_{\mathbb{R}} |w'_n(x_1)|^2 dx_1 \\ &= a \left(\int_{2^{n-1}}^{2^n} |w'_n(x_1)|^2 dx_1 + \int_{2^n}^{2^{n+1}} |w'_n(x_1)|^2 dx_1 \right. \\ &\quad \left. + \int_{2^{n+1}}^{2^{n+2}} |w'_n(x_1)|^2 dx_1 \right) = C_7 2^{n+1}, \end{aligned} \quad (59)$$

where

$$C_7 := 5a. \quad (60)$$

Now

$$\begin{aligned} \int_S V(x) |u_n(x)|^2 dx &\geq \int_{2^n}^{2^{n+1}} 2^{2(n+1)} dx_1 \int_0^a V(x) dx_2 \\ &\geq \int_{2^n}^{2^{n+1}} 2^{n+1} dx_1 \int_0^a |x_1| V(x) dx_2 \\ &= \frac{1}{C_7} \left(\int_S |\nabla u_n(x)|^2 dx \right) \mathcal{G}_n. \end{aligned} \quad (61)$$

If $\mathcal{G}_n > C_7$, then $q_{V,S}[u_n] < 0$. The auxiliary functions w_n can be defined similarly for $n \leq 0$. Since u_n and u_k have disjoint supports for $|n - k| \geq 3$, then

$$N_-(q_{V,S}) \geq \frac{1}{3} \text{card} \{n \in \mathbb{Z} : \mathcal{G}_n > C_7\} \quad (62)$$

(see [6, Theorem 9.1]). If, for some constant $C_8 > 0$, $N_-(q_{\alpha V,S}) \leq C_8 \alpha$, then

$$\frac{1}{3} \text{card} \{n \in \mathbb{Z} : \alpha \mathcal{G}_n > C_7\} \leq C_8 \alpha, \quad (63)$$

and so

$$\text{card} \left\{ n \in \mathbb{Z} : \mathcal{G}_n > \frac{C_7}{\alpha} \right\} \leq 3C_8 \alpha. \quad (64)$$

With $s = C_7/\alpha$ we have

$$\text{card} \{n \in \mathbb{Z} : \mathcal{G}_n > s\} \leq C_9 s^{-1}, \quad s > 0, \quad (65)$$

where $C_9 := 3C_7C_8$. $\square$

4. Estimates Involving Norms of the Potential Supported by a Lipschitz Curve inside a Strip

In this section we obtain estimates analogous to those in the previous section when the potential V is strongly singular, i.e., when V is supported by a Lipschitz curve ℓ embedded in S . When dealing with function spaces on ℓ , we will always assume that ℓ is equipped with the arc length measure. Before we introduce the estimates, let us first look at the following operator that we shall need in the sequel.

Consider a one-dimensional Schrödinger operator $H_{X,\alpha}$, with point δ -interactions on a countable set $X = \{x_k\}_{k=1}^\infty$ of points, called points of interaction, and intensities $\alpha = \{\alpha_k\}_{k=1}^\infty$, defined by the differential expression $-d^2/dx^2$ on functions $w(x)$ that belong to the Sobolev space $W_2^2(\mathbb{R} \setminus X)$ satisfying, in the points of the set X , the following conjugation conditions:

$$\begin{aligned} w(x_k + 0) &= w(x_k - 0), \\ w'(x_k + 0) - w'(x_k - 0) &= \alpha_k w(x_k). \end{aligned} \quad (66)$$

Since, for each k , (x_k, x_{k+1}) is an open interval, then any function in $W_2^2((x_k, x_{k+1}))$ and its derivative have well defined

(one-sided) values at the end-points. The operator $H_{X,\alpha}$ has the following representation:

$$H_{X,\alpha} := -\frac{d^2}{dx^2} + \sum_{k=1}^\infty \alpha_k \delta(x - x_k), \quad (67)$$

where δ is Dirac's delta distribution. We shall assume that $H_{X,\alpha}$ is self-adjoint (see, e.g., [15] in case the set X is finite). One can also define operator (67) via its quadratic form as follows:

$$q[w] := \int_{\mathbb{R}} |w'(x)|^2 dx - \sum_{k=1}^\infty \alpha_k |w(x_k)|^2, \quad (68)$$

$$\forall w \in W_2^1(\mathbb{R}).$$

Lemma 7. *Given an infinite sequence of positive numbers (α_k) , there is a sequence of points (x_k) in X such that (67) has infinitely many negative eigenvalues.*

Proof. Let $\psi \in C_0^\infty(\mathbb{R})$ such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < \frac{1}{2}, \\ 0 & \text{if } |x| \geq 1. \end{cases} \quad (69)$$

Assume that $x_{k+1} - x_k > x_k - x_{k-1}$, for all $k = 1, 2, \dots$ and let $\varphi_k(x) := \psi((2/(x_k - x_{k-1}))(x - x_k))$. Then

$$\varphi_k(x) = \begin{cases} 1 & \text{if } |x - x_k| < \frac{x_k - x_{k-1}}{4}, \\ 0 & \text{if } |x - x_k| \geq \frac{x_k - x_{k-1}}{2}. \end{cases} \quad (70)$$

Let $\mathcal{L}$ be a linear subspace of $W_2^1(\mathbb{R})$ defined by

$$\mathcal{L} := \left\{ w \in W_2^1(\mathbb{R}) : w = \sum_{k=1}^\infty a_k \varphi_k, \quad a_k \in \mathbb{C} \right\}. \quad (71)$$

Since φ_k and φ_j for $k \neq j$ have disjoint supports, then $\dim \mathcal{L} = \infty$. Thus, for all $w \in \mathcal{L} \setminus \{0\}$, it follows from (68) that

$$\begin{aligned} q[w] &= \sum_{k=1}^\infty |a_k|^2 \left(\int_{\mathbb{R}} |\varphi_k'(x)|^2 dx - \alpha_k \right) = \sum_{k=1}^\infty |a_k|^2 \\ &\cdot \left(\frac{4}{|x_k - x_{k-1}|^2} \int_{\mathbb{R}} \left| \psi' \left(\frac{2}{x_k - x_{k-1}} (x - x_k) \right) \right|^2 dx \right. \\ &\left. - \alpha_k \right) = \sum_{k=1}^\infty |a_k|^2 \left(\frac{2}{|x_k - x_{k-1}|} \int_{\mathbb{R}} |\psi'(t)|^2 dt \right. \\ &\left. - \alpha_k \right), \end{aligned} \quad (72)$$

where $t = (2/(x_k - x_{k-1}))(x - x_k)$. Take x_k such that

$$x_k - x_{k-1} > \frac{2}{\alpha_k} \int_{\mathbb{R}} |\psi'(t)|^2 dt, \quad (73)$$

then $q[w] < 0$ and operator (67) has infinitely many negative eigenvalues. $\square$

Let us now return to operator (1) with V supported by and locally integrable on a Lipschitz curve ℓ embedded in S . Let

$$\begin{aligned} q_{V,\ell}[u] &:= \int_S |\nabla u(x)|^2 dx \\ &\quad - \int_\ell V(x) |u(x)|^2 ds(x) \quad (74) \\ \text{dom}(q_{V,\ell}) &= W_2^1(S) \cap L^2(\ell, Vds). \end{aligned}$$

Let $\{x_1^{(k)}\}, k \in \mathbb{N}$ be a sequence of points on $\mathbb{R}$ satisfying conditions in (66). Define

$$\gamma_k := \ell \cap \left(\{x_1^{(k)}\} \times (0, a) \right), \quad x_1 \in \mathbb{R}, \quad k \in \mathbb{N} \quad (75)$$

and

$$\Sigma := \{x_1^{(k)} : |\gamma_k| > 0\}. \quad (76)$$

Then the set Σ is at most countable. Let I be an arbitrary interval in $\mathbb{R}$ and let

$$\nu(I) := \int_{\ell \cap (I \times (0, a))} V(x) ds(x). \quad (77)$$

Furthermore, let

$$\begin{aligned} \mathcal{F}_n &:= \int_{I_n} |x_1| d\nu(x_1), \quad n \neq 0, \\ \mathcal{F}_0 &:= \int_{I_0} d\nu(x_1) \quad (\text{cf. (35)}), \\ \ell_n &:= \ell \cap S_n, \quad S_n := (n, n+1) \times (0, a), \quad n \in \mathbb{N}, \\ \mathcal{C}_n &:= \|V\|_{\mathcal{B}, \ell_n}^{(\text{av})}. \end{aligned} \quad (78)$$

Theorem 8. Suppose that N is the cardinality of Σ . Then there exist constants $c_1, c_2, C_{10}, C_{11} > 0$ such that

$$\begin{aligned} N_-(q_{V,\ell}) &\leq 1 + N + C_9 \sum_{\{n \in \mathbb{Z}, \mathcal{F}_n > c_1\}} \sqrt{\mathcal{F}_n} \\ &\quad + C_{10} \sum_{\{n \in \mathbb{Z}, \mathcal{C}_n > c_2\}} \mathcal{C}_n, \quad \forall V \geq 0. \end{aligned} \quad (79)$$

If Σ is infinite, then $N_-(q_{V,\ell}) = \infty$.

Proof. Let $q_{1,2V,\ell}$ and $q_{2,2V,\ell}$ be the restrictions of the form $q_{V,\ell}$ to the subspaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively (see (27)), and then

$$N_-(q_{V,\ell}) \leq N_-(q_{1,2V,\ell}) + N_-(q_{2,2V,\ell}) \quad (80)$$

(cf. (34)). Let us start by estimating the first term on the right-hand side of (80). On the complement of Σ , $\nu(\{x_1\}) = 0$ for all $x_1 \in \mathbb{R}$. This implies

$$\begin{aligned} &\int_\ell V(x) |u(x)|^2 ds(x) \\ &= \int_{\mathbb{R}} |w(x_1)|^2 d\nu(x_1) \\ &\quad + |w(x_1)|^2 \sum_{k \in \mathbb{N}} \int_{\gamma_k} V(x_2) dx_2 \\ &= \int_{\mathbb{R}} |w(x_1)|^2 d\nu(x_1) + \sum_{k \in \mathbb{N}} c_k |w(x_1)|^2, \end{aligned} \quad (81)$$

where $c_k := \int_{\gamma_k} V(x_2) dx_2 < \infty$. Hence

$$\begin{aligned} q_{1,2V,\ell}[u] &= \int_S |\nabla u(x)|^2 dx - 2 \int_\ell V(x) |u(x)|^2 dx \\ &= \int_{\mathbb{R}} |w'(x_1)|^2 dx_1 \\ &\quad - 2 \int_{\mathbb{R}} |w(x_1)|^2 d\nu(x_1) \\ &\quad - \sum_{k \in \mathbb{N}} c'_k |w(x_1)|^2, \end{aligned} \quad (82)$$

where $c'_k := 2c_k$. Let

$$\begin{aligned} q_{1,2\nu}[w] &:= \int_{\mathbb{R}} |w'(x_1)|^2 dx_1 \\ &\quad - 2 \int_{\mathbb{R}} |w(x_1)|^2 d\nu(x_1), \\ \text{dom}(q_{1,2\nu}) &= W_2^1(\mathbb{R}) \cap L^2(\mathbb{R}, d\nu), \\ q_{2,c'_k}[w] &:= \int_{\mathbb{R}} |w'(x_1)|^2 dx_1 - \sum_{k \in \mathbb{N}} c'_k |w(x_1)|^2, \\ \text{dom}(q_{2,c'_k}) &= W_2^1(\mathbb{R}). \end{aligned} \quad (83)$$

Then, it follows from [3, Lemma 3.6] that

$$N_-(q_{1,2V,\ell}) \leq N_-(q_{1,2\nu}) + N_-(q_{2,c'_k}). \quad (84)$$

Similarly to (36) one has

$$N_-(q_{1,2\nu}) \leq 1 + 7.16 \sum_{\{n \in \mathbb{Z}, \mathcal{F}_n > 0.046\}} \sqrt{\mathcal{F}_n}. \quad (85)$$

If Σ is finite, then

$$N_-(q_{2,c'_k}) \leq N \quad (86)$$

(see, e.g., [15]). Otherwise, Lemma 7 implies

$$N_-(q_{2,c'_k}) = \infty. \quad (87)$$

Now, it remains to estimate the second term on the right-hand side of (80). Let

$$\begin{aligned} q_{2,2V,\ell_n}[u] &:= \int_{S_n} |\nabla u(x)|^2 dx - \int_{\ell_n} V(x) |u(x)|^2 ds(x) \\ \text{dom}(q_{2,2V,\ell_n}) &= \left\{ u \in W_2^1(S_n) \cap L^2(\ell_n, Vds) : \int_{S_n} u(x) dx = 0 \right\}. \end{aligned} \quad (88)$$

For any $V \in L_{\mathcal{B}}(\ell_n)$, $V \geq 0$ and any $r \in \mathbb{N}$, following a similar argument in the proof of [6, Lemma 7.6], one can show that there exists a finite cover of ℓ_n by rectangles S_{n_k} , $k = 1, \dots, r_0$ such that $r_0 \leq r$ and

$$\begin{aligned} &\int_{\ell_n} V(x) |u(x)|^2 ds(x) \\ &\leq C_{12} r^{-1} \|V\|_{\mathcal{B},\ell_n}^{(\text{av})} \int_{S_n} |\nabla u(x)|^2 dx \end{aligned} \quad (89)$$

for all $u \in W_2^1(S_n) \cap L^2(\ell_n, Vds)$ with $\int_{S_{n_k}} u(x) dx = 0$, where C_{12} is a constant independent of V . Now, using (89) instead of [6, Lemma 7.6] in the proof of [6, Lemma 7.7], one can easily show similarly to [6, Lemma 7.8] that there is a constant $C_{11} > 0$ such that

$$N_-(q_{2,2V,\ell_n}) \leq C_{11} \|V\|_{\mathcal{B},\ell_n}^{(\text{av})}, \quad \forall V \geq 0. \quad (90)$$

If $\|V\|_{\mathcal{B},\ell_n}^{(\text{av})} < 1/C_{11}$, then $N_-(q_{2,2V,\ell_n}) = 0$. Thus, similarly to (39) one has for any $c_2 < 1/C_{11}$ that

$$N_-(q_{2,2V,\ell}) \leq C_{11} \sum_{\{\mathcal{E}_n > c_2, n \in \mathbb{Z}\}} \mathcal{E}_n, \quad \forall V \geq 0. \quad (91)$$

Hence, the statement of the theorem follows from (80), (84), (85), (86), (87), and (91). $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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