Research Article

On Nearly Prime Submodules of Unitary Modules

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The aim of this paper is to introduce the notion of nearly prime submodules as a generalization of prime submodules. We investigate some of their basic properties and point out the similarities between these submodules and the prime submodules. We also indicate some applications of nearly prime submodules. These applications show how nearly prime submodules control the structure of modules and they recover earlier relative theorems.

1. Introduction and Preliminary

Modules over associative rings play important roles in the investigation of ring constructions (see [1, 2]). Modules are very important and have been actively investigated (see, for example, [3–7]). Throughout this paper, all rings are associative with identity and all modules are unitary right R -modules. For a right R -module M , we denote $S = \text{End}_R(M)$ for its endomorphism ring. A submodule X of M is called a *fully invariant submodule* of M if, for any $f \in S$, we have $f(X) \subset X$. A right R -module M is called a *self-generator* if it generates all submodules.

In 2008, Sanh et al. [8] introduced the new notion of prime and semiprime submodules. Following that, a prime submodule X of a right R -module M is a proper fully invariant submodule of M with the property that, for any ideal I of $S = \text{End}_R(M)$ and any fully invariant submodule U of M , $I(U) \subset X$ implies $I(M) \subset X$ or $U \subset X$. We can say that this new approach is nontrivial, creative, and well-posed. We already got many results using those new notions that are unparalleled. As an extension of this work, we generalize the notion of a prime submodule.

Many people generalize the notion of a prime submodule. To do that, there are several ways but we put our attention to replace a weaker condition that X is invariant under φS

instead of requiring the submodule X to be fully invariant, and we called it nearly prime submodule. Using this new definition, we proved many meaningful properties of nearly prime submodules which are similar to that of prime submodules and also prime ideals.

General background materials can be found in [9–12].

2. Main Results

We introduce the definition of a nearly prime submodule by a weaker condition that X is invariant under φS , instead of requiring the submodule X to be fully invariant.

Definition 1. A proper submodule X of a right R -module M is called a *nearly prime submodule* if, for any $\varphi \in S$ and for any $m \in M$, if $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$, then either $m \in X$ or $\varphi(M) \subset X$. Particularly, a proper right ideal P of R is a *nearly prime right ideal* if for $a, b \in R$ such that $aRb \subset P$ and $aRP \subset P$, then either $a \in P$ or $b \in P$.

From these definitions, any prime submodule of a right R -module M is nearly prime.

In the following theorem and its corollary, we can see that a proper right ideal P of R is nearly prime if for any right ideals $A, B \subset R$ such that $AP \subset P$ and $AB \subset P$, then either $A \subset P$

or $B \subset P$. Note that Koh [13] gave this definition and used the terminology *prime right ideals*.

Theorem 2. *Let X be a proper submodule of M . The following conditions are equivalent.*

- (1) X is a nearly prime submodule of M .
- (2) For any right ideal I of S , any submodule U of M , if $I(U) \subset X$ and $I(X) \subset X$, then either $I(M) \subset X$ or $U \subset X$.
- (3) For any $\varphi \in S$ and fully invariant submodule U of M , if $\varphi(U) \subset X$ and $\varphi S(X) \subset X$, then either $\varphi(M) \subset X$ or $U \subset X$.

Corollary 3. *Let I be a proper right ideal of a ring R . The following conditions are equivalent.*

- (1) I is a nearly prime right ideal of R .
- (2) For any right ideal A, B of R , if $AB \subset I$ and $AI \subset I$, then either $A \subset I$ or $B \subset I$.
- (3) For any $a \in R$ and any ideal B of R , if $aB \subset I$ and $aRI \subset I$, then either $a \in I$ or $B \subset I$.

Next, we give some examples and remark of nearly prime submodules and nearly prime right ideals; we maintain the notion and terminology as in [8].

Example 4. (1) Following Sanh et al. [8], a fully invariant is a prime submodule if, for any ideal I of $S = \text{End}_R(M)$, any fully invariant submodule U of M , if $I(U) \subset X$, then either $I(M) \subset X$ or $U \subset X$. By our definition, any prime submodule of M is nearly prime.

(2) Also by Sanh et al. [8], if X is a maximal fully invariant submodule of M , then X is prime. We now show that any maximal submodule X of M is nearly prime. In fact, let $\varphi(U) \subset X$, where U is a submodule of M and $\varphi \in S$ with $\varphi S(X) \subset X$. Suppose that $U \not\subset X$. Then there is an $u \in U$ such that $X + uR = M$. This follows that $\varphi(M) = \varphi(X) + \varphi(uR) = \varphi(X) + \varphi(u)R \subset X$ since $\varphi(U) \subset X$. This shows that X is nearly prime. Note that, in general, a maximal submodule of a right R -module M does not need to be fully invariant. Therefore the class of nearly prime submodules of a given right R -module M is larger than that of prime submodules. As a consequence, every maximal right ideal is a nearly prime right ideal.

(3) The following example is due to Reyes [14]. Let D be a division ring and let R be the following subring of $M_3(D)$:

$$R := \begin{pmatrix} D & D & D \\ 0 & D & 0 \\ 0 & 0 & D \end{pmatrix}. \quad (1)$$

Let $P \subset R$ be the right ideal consisting of matrices in R whose first row is zero, i.e., $P := \begin{pmatrix} 0 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$. Now, we assume

that $aRb \subset P$ and $aRP \subset P$ for arbitrary $a, b \in R$. Since $aRP \subset P$, we get $a_{11} + a_{12} = 0$ and $a_{11} + a_{13} = 0$. And hence

$$aRb = \begin{pmatrix} a_{11}b_{11}D & a_{11}b_{12}D & a_{11}b_{13}D \\ 0 & a_{22}b_{22}D & 0 \\ 0 & 0 & a_{33}b_{33}D \end{pmatrix} \subset P. \quad (2)$$

This would imply that either $a \in P$ or $b \in P$, so P is a nearly prime right ideal of R .

One useful generalization of nearly prime submodule is obtained by replacing the condition “ $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$ ”; we called it *nearly strongly prime submodule*.

Definition 5 (see [15]). A proper submodule X of a right R -module M is called a *nearly strongly prime submodule* if, for any $\varphi \in S$ and $m \in M$, if $\varphi(m) \subset X$ and $\varphi(X) \subset X$, then either $m \in X$ or $\varphi(M) \subset X$.

To see the relationship between a nearly prime submodule and nearly strongly prime submodule, we will need the following terminology.

Definition 6. A submodule X of a right R -module M is called to have *insertion factor property* (briefly, an IFP-submodule), if for all $\varphi \in S = \text{End}_R(M)$ and $m \in M$, $\varphi(m) \in X$, then $\varphi S(m) \subseteq X$.

Then the relationship between a nearly prime submodule and nearly strongly prime submodule is as follows.

Proposition 7. *Let M be a right R -module and X a submodule of M . If X is a nearly strongly prime submodule of M , then X is a nearly prime submodule of M .*

Proof. The proof is immediate. $\square$

Proposition 8. *Let M be a right R -module and X a submodule of M . If X is a nearly prime submodule of M and has insertion factor property, then X is a nearly strongly prime submodule of M .*

Proof. Let $\varphi \in S = \text{End}_R(M)$ and $m \in M$ such that $\varphi(m) \in X$ and $\varphi(X) \subseteq X$. Since X is an IFP-submodule of M , $\varphi S(m) \subseteq X$. Let $x \in X$. Since $\varphi(X) \subseteq X$, $\varphi(x) \in X$. But X is an IFP-submodule of M ; we have $\varphi S(x) \subseteq X$. So $\varphi S(X) \subseteq X$. Since X is a nearly prime submodule of M , $m \in M$ or $\varphi(M) \subseteq X$. Therefore X is a nearly strongly prime submodule of M . $\square$

Proposition 9. *If X is a nearly prime submodule of M , then X contains a minimal nearly prime submodule of M .*

Proof. Let $\mathcal{F}$ be the set of all nearly prime submodules of M which are contained in X . Since $X \in \mathcal{F}$, $\mathcal{F}$ is nonempty. By Zorn's Lemma, $\mathcal{F}$ has a minimal element with respect to the inclusion operation provided; we show that any nonempty chain $\mathcal{G} \subset \mathcal{F}$ has a lower bound Q in $\mathcal{F}$. Put $Q = \bigcap_{N \in \mathcal{G}} N$; then $\varphi S(Q) \subset Q$ for any $\varphi \in S$. We will show that Q is a nearly prime submodule of M and $Q \subset X$. Suppose that $\varphi \in S$ and $m \in M/Q$ such that $\varphi S(m) \in Q$. Since $m \notin Q = \bigcap_{N \in \mathcal{G}} N$, there exists $N \in \mathcal{G}$ with $m \notin N$. By the nearly primeness of N ,

we have $\varphi(M) \subset N$. For any $U \in \mathcal{G}$, either $U \subset N$ or $N \subset U$. If $U \subset N$, we see that $m \notin U$, which implies that $\varphi(M) \subset U$ by the nearly primeness of U . If $N \subset U$, we have $\varphi(M) \subset N \subset U$. Thus $\varphi(M) \subset U$ for any $U \in \mathcal{G}$. Hence $\varphi(M) \subset Q$, proving that Q is nearly prime in M . It is clear that $Q \subset X$. Therefore, Q is a lower bound for $\mathcal{G}$. Again by Zorn's Lemma, there exists a nearly prime P^* which is minimal among the nearly prime submodules in $\mathcal{F}$. Since any nearly prime submodule contained in P^* is in $\mathcal{F}$, we conclude that P^* is a minimal nearly prime submodule of M . $\square$

Let X be a submodule of M . Then the set $I_X = \{f \in S \mid f(M) \subset X\}$ is a right ideal of S . In the following theorem, we consider the relation between X and I_X .

Theorem 10. *Let M be a right R -module which is a self-generator and X be a submodule of M . If X is a nearly prime submodule, then $I_X = \{f \in S \mid f(M) \subset X\}$ is a nearly prime right ideal of S . Conversely, if I_X is a nearly prime right ideal of S , then X is a nearly prime submodule of M .*

Proof. Since $\varphi SI_X \subset I_X$, we have $\varphi SI_X(M) \subset I_X(M)$, and since M is a self-generator, we have $I_X(M) = X$ and hence $\varphi S(X) \subset X$. Take any $\psi \in S$. If $\varphi S\psi \subset I_X$, then $\varphi S\psi(M) \subset X$. This follows that $\varphi S\psi(m) \subset X$, for all $m \in M$. Hence $\varphi(M) \subset X$ or $\psi(m) \in X$, showing that $\varphi \in I_X$ or $\psi \in I_X$.

Conversely, suppose that I_X is nearly prime. Let $\varphi S(m) \subset X$ and $\varphi S(X) \subset X$. We have $\varphi S(m)R \subset X$ and it would imply that $\varphi S(mR) \subset X$. Since $mR = \sum_{g \in A} g(M)$ for some subset A of S ,

$$\varphi S(mR) = \varphi S\left(\sum_{g \in A} g(M)\right) = \sum_{g \in A} \varphi Sg(M) \subset X. \quad (3)$$

This would imply that $\varphi Sg \subset I_X$. It follows from the hypothesis that $\varphi \in I_X$ or $g \in I_X$. This shows that $\varphi(M) \subset X$ or $g(M) \subset X$ for all $g \in A$. Hence $\varphi(M) \subset X$ or $m \in X$, proving our theorem. $\square$

Recall from [16] that a module N is called M -generated if there is an epimorphism $M^{(I)} \rightarrow N$ for some index set I . If I is finite, then N is called *finitely M -generated*. In particular, a module N is called M -cyclic if there is an epimorphism from $M \rightarrow N$.

Lemma 11. *Let M be a quasi-projective module and X be an M -cyclic submodule of M . Then I_X is a principal right ideal of S .*

Proof. Since X is M -cyclic, there exists an epimorphism $\varphi : M \rightarrow X$ such that $X = \varphi(M)$. It follows that $\varphi S \subset I_X$. By the quasi-projective of M , for any $f \in I_X$, we can find a $\psi \in S$ such that $f = \varphi\psi$, proving that $I_X = \varphi S$. Hence I_X is a principal right ideal of S . $\square$

Proposition 12. *Let M be a right R -module and X be a submodule of M . If f is injective for all $0 \neq f \in \text{End}_R(M/X)$, then X is a nearly strongly prime submodule of M .*

Proof. Suppose that f is injective for all $0 \neq f \in \text{End}_R(M/X)$. Let $\varphi \in S = \text{End}_R(M)$ and $m \in M$ such that $\varphi(m) \in X$ and $\varphi(X) \subseteq X$. Assume that $\psi(M) \not\subseteq X$. Then there exists $\hat{m} \in M$ such that $\varphi(\hat{m}) \notin X$. Define $f : M/X \rightarrow M/X$ by $f(x + X) = \varphi(x) + X$ for all $x \in X$. Let $x_1, x_2 \in X$ such that $x_1 + X = x_2 + X$. We have $x_1 - x_2 \in X$ but $\varphi(X) \subseteq X$, $\varphi(x_1 - x_2) \in X$, and hence $\varphi(x_1) + X = \varphi(x_2) + X$. So f is well-defined and it is clear that f is an R -homomorphism. Since $f(\hat{m} + X) = \varphi(\hat{m}) + X \neq X$, $f \neq 0$. By assumption, f is injective. Then $f(m + X) = \varphi(m) + X = X$ but f is an injective, $m + X = X$, and hence $m \in X$. Therefore X is a nearly strongly prime submodule of M . $\square$

3. Applications of Nearly Prime Submodules

The following theorem shows how nearly prime submodules control the structure of a finitely generated module. Moreover, this theorem can be considered as a generalization of Cohen's theorem, a famous theorem in commutative algebra.

Theorem 13. *Let M be a finitely generated right R -module. Then M is a Noetherian module if and only if every nearly prime submodule of M is finitely generated.*

In this section, we will show other applications of nearly prime submodules. The following results had appeared in [13] and we propose them here to use later on.

Theorem 14. *Every right ideal of R is generated by one element if and only if every prime right ideal of R is generated by one element.*

Theorem 15. *Let M be a quasi-projective, finitely generated right R -module which is a self-generator. If all of nearly prime submodules of M are M -cyclic submodules, then every ideal in S is principal.*

Proof. Let P be a prime right ideal of S and $X = P(M)$. Since M is finitely generated and quasi-projective, it follows from [12][18.4] that $P = I_X$ and, therefore, X is a nearly prime submodule of M by Theorem 10. Moreover, by hypothesis and Lemma 11, we can see that P is a principal right ideal of S . It follows from Theorem 14 that every right ideal of R is generated by one element. $\square$

Corollary 16. *Let M be a quasi-projective, finitely generated right R -module which is a self-generator. If all of nearly prime submodules of M are M -cyclic submodules, then every submodule of M is M -cyclic.*

Proof. The proof is immediate. $\square$

For $M = R_R$, the next corollary follows consequently.

Corollary 17. *A ring R is a principal right ideal ring if and only if all of its nearly prime right ideals are principal.*

Particularly, the following corollary is very useful in commutative algebra.

Corollary 18. *An integral domain is a principal ideal domain (PID) if every prime ideal is principal.*

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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