Hindawi
Journal of Mathematics
Volume 2018, Article ID 7856594, 5 pages
https://doi.org/10.1155/2018/7856594

Research Article

Subordinate Semimetric Spaces and Fixed Point Theorems

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Received 12 March 2018; Accepted 10 May 2018; Published 10 June 2018

Academic Editor: Nan-Jing Huang

We introduce the concept of subordinate semimetric space. Such notion includes the concept of RS-space introduced by Roldán and Shahzad; therefore the concepts of Branciari's generalized metric space and Jleli and Samet's generalized metric space are particular cases. For such spaces we prove a version of Matkowski's fixed point theorem, and introducing the concept of q-contraction we get a fixed point theorem of Kannan-Ćirić type. Moreover, using such result we characterize complete subordinate semimetric spaces.

1. Introduction

Fixed point theory has an enormous number of applications in ordinary (or partial) differential equations, game theory, functional analysis, calculus of several variables (the classical implicit function theorem), etc. This is one of the reasons why we always try to have a fixed point result in a general context, since it could be applied more broadly.

Generalizations of the concept of metric space are currently one of the most active branches of functional analysis (see [1–4] or [5], where the notion of quasimetric space is studied). In fact, in Table 1 we present a nonexhaustive review of some generalizations.

Our work is related to the notion of generalized metric space introduced by Jleli and Samet [1]. Such concept was immediately generalized by Roldán and Shahzad [2] as follows.

Definition 1. An RS-space is a pair (E, d) where E is a nonempty set and $d : E \times E \to [0, \infty]$ is a function such that the following properties are fulfilled:

(i) if $d(x, y) = 0$ then $x = y$,

(ii) $d(x, y) = d(y, x)$ for every $x, y \in E$,

(iii) there exists $c > 0$ such that if $x, y \in E$ are two points and (x_n) is an infinite sequence with $\lim_{n,m\to\infty} d(x_n, x_m) = 0$ and $\lim_{n\to\infty} d(x_n, x) = 0$ then

$$d(x, y) \le c \limsup_{n\to\infty} d(x_n, y). \tag{1}$$

The notions of modular space, quasimetric space, dislocated metric space, and generalized metric space (in the Branciari sense) are particular cases of the notion of RS-space (see [1] or [2]). In this paper we introduce the notion of subordinate semimetric space; such concept is a generalization of the notion of RS-space. On the other hand, we give in Example 5 a pair (E, d) which is a subordinate semimetric space that is not an RS-space. In the context of subordinate semimetric spaces we prove a version of Matkowski's fixed point theorem and introducing the notion of q-contraction we obtain a version of the fixed point theorem of Kannan-Ćirić. Moreover, we prove that if in a subordinate semimetric space every q-contraction has a fixed point then this must be complete.

2. Subordinate Metric Spaces

We start with a generalization of the common notion of semimetric space (see [6]).

Definition 2. A semimetric space is a pair (E, d) where E is a nonempty set and $d : E \times E \to [0, \infty]$ is a function that satisfies the following:

(i) for every $(x, y) \in E \times E$, we have

$$d(x, y) = 0 \longrightarrow x = y, \tag{2}$$

(ii) for every $(x, y) \in E \times E$, we have

$$d(x, y) = d(y, x). \tag{3}$$

Table 1

Year	Author	Concept
1931	W.A Wilson [7]	quasi-metric space
1959	H. Nakano [8]	modular space
1993	S. Czerwik [9]	b-metric space
2000	P. Hitzler, A. K. Seda [10]	dislocated metric space
2000	A. Branciari [11]	generalized metric space
2015	M. Jleli, B. Samet [1]	generalized metric space
2016	A. Roldán, N. Shahzad [2]	RS-space

Some important concepts can be introduced with this general notion.

Definition 3. Let (E, d) be a semimetric space.

(i) A sequence (x_n) in E converges to $x \in E$ if

$$\lim_{n\to\infty} d(x_n, x) = 0. \tag{4}$$

(ii) A sequence (x_n) in E is a Cauchy sequence if

$$\lim_{m,n\to\infty} d(x_n, x_m) = 0. \tag{5}$$

(iii) (E, d) is complete if every Cauchy sequence in E is convergent.

In order to get a rich mathematical structure we introduce a substitute of triangle inequality (for a metric).

Definition 4. We say that a semimetric space (E, d) is subordinate if there is a function $\zeta : [0, \infty] \to [0, \infty]$ such that

(i) ζ is nondecreasing; $\lim_{x\to 0}\zeta(x) = 0$,

(ii) for every $(x, y) \in E \times E$, with $x \neq y$, and (x_n) being an infinity Cauchy sequence in E such that (x_n) converges to x we have

$$d(x, y) \le \zeta\left(\limsup_{n\to\infty} d(x_n, y)\right). \tag{6}$$

In this case we say that (E, d) is subordinate to ζ or that (E, d) is a subordinate semimetric space.

It is clear that every RS-space is a subordinate semimetric space (just take $\zeta(x) = cx$), and we give an example of a subordinate semimetric space that is not an RS-space.

Example 5. Let $E_0 = \{0\} \cup \mathbb{N} \cup (\mathbb{N} \times \mathbb{N})$ and define $d_0 : E_0 \times E_0 \to [0, \infty]$ as

$$\begin{aligned} d_0((r,s),(n,m)) &= d_0((n,m),(r,s)) \\ &= \begin{cases} \infty, & m \neq s, \\ \dfrac{1}{n+r}, & n \neq r,\ m = s, \\ 0, & n = r,\ m = s, \end{cases} \\ d_0(k,(n,m)) &= d_0((n,m),k) \\ &= \begin{cases} \infty, & k \in \mathbb{N},\ m \neq k, \\ \dfrac{1}{n}, & m = k, \\ m, & k = 0, \end{cases} \\ d_0(m,n) &= d_0(n,m) \\ &= \begin{cases} \infty, & m,n \in \mathbb{N},\ m \neq n, \\ n^2, & n \in \mathbb{N},\ m = 0, \\ 0, & m = n. \end{cases} \end{aligned} \tag{7}$$

Since

$$\begin{aligned} \lim_{n,r\to\infty} d_0((n,m),(r,m)) &= \lim_{n,r\to\infty} \frac{1}{n+r} = 0, \\ \lim_{n\to\infty} d_0((n,m),m) &= \lim_{n\to\infty} \frac{1}{n} = 0, \end{aligned} \tag{8}$$

the sequence $\{(n,m)\}_{n\in\mathbb{N}}$ is an infinite Cauchy sequence that converges to m. Suppose that there is a constant $c > 0$ such that

$$m^2 = d_0(m,0) \le c \limsup_{n\to\infty} d_0((n,m),0) = cm, \tag{9}$$

then $c \ge m$, for all $m \in \mathbb{N}$. In this way (E_0, d_0) is not an RS-space. However, the semimetric space (E_0, d_0) is subordinate, for example, to the function $\zeta(x) = x^2$.

With the subordination concept we will see that we are able to prove some important fixed point theorems. Then a natural question arises:

What conditions implies that a semimetric space is subordinate?

The answer will give us fixed point theorems in a more general scheme.

3. Fixed Point Theorems

We now introduce a type of Kannan-Ćirić contraction condition.

Definition 6. Let (E, d) be a semimetric space. A mapping $f : E \to E$ is said to be a q-contraction, with $q \in (0, 1)$, if

$$d(f(x), f(y)) \le q \max\{d(x, f(x)), d(y, f(y))\} \tag{10}$$

holds for every $(x, y) \in E \times E$.

Let $f : E \to E$ be a function. For each $x \in E$ we define $f^{[n]}(x)$ recursively as $f^{[0]}(x) = x$ and $f^{[n+1]}(x) = f(f^{[n]}(x))$, for each $n \in \mathbb{N}\cup\{0\}$. With the above notation we have a version of the Kannan-Ćirić's fixed point theorem.

Theorem 7. *Let $f : E \to E$ be a q-contraction on a complete semimetric space (E, d).*

(i) If there is an $x_0 \in E$ such that

$$\limsup_{n\to\infty} d\left(f^{[n]}(x_0), f^{[n+1]}(x_0)\right) < \infty, \tag{11}$$

then $(f^{[n]}(x_0))$ converge to some $\hat{x} \in E$.

(ii) Suppose that (E, d) is subordinate to ζ and

$$\zeta(t) < \frac{t}{q}, \quad \text{for all } 0 < t < \infty. \tag{12}$$

If $d(\hat{x}, f(\hat{x})) < \infty$ then $\hat{x}$ is the unique fixed point of f.

Proof. (i) Let us set $x_n = f^{[n]}(x_0)$, for each $n \in \mathbb{N}$. The q-contraction property of f implies

$$\begin{aligned}
\limsup_{n\to\infty} d(x_n, x_{n+1}) &= \limsup_{n\to\infty} d(f(x_{n-1}), f(x_n)) \\
&\le \limsup_{n\to\infty} q \max\{d(x_{n-1}, f(x_{n-1})), d(x_n, f(x_n))\} \\
&= q \limsup_{n\to\infty} \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\} \\
&\le q \limsup_{n\to\infty} d(x_n, x_{n+1}).
\end{aligned} \tag{13}$$

The hypothesis (i) yields $\limsup_{n\to\infty} d(x_n, x_{n+1}) = 0$.

Now let us see that (x_n) is a Cauchy sequence. Let $\varepsilon > 0$, then there is $k \in \mathbb{N}$ such that

$$d(x_n, x_{n+1}) < \frac{\varepsilon}{q}, \quad \text{for all } n \ge k. \tag{14}$$

Therefore, if $m, n \ge k + 1$, then

$$\begin{aligned}
d(x_n, x_m) &= d(f(x_{n-1}), f(x_{m-1})) \\
&\le q \max\{d(x_{n-1}, x_n), d(x_{m-1}, x_m)\} \\
&\le q\left(\frac{\varepsilon}{q}\right) = \varepsilon.
\end{aligned} \tag{15}$$

Then there is $\hat{x} \in E$ such that $\lim_{n\to\infty} d(x_n, \hat{x}) = 0$.

(ii) Let us see that $\hat{x}$ is a fixed point for f. If the set $\{x_n : n \in \mathbb{N}\}$ is finite, the Cauchy property of (x_n) implies that there exists $n_0 \in \mathbb{N}$ such that $x_n = x_{n_0} = \hat{x}$, for all $n \ge n_0$, then $f(x) = f(x_{n_0}) = x_{n_0+1} = \hat{x}$. On the other hand, if $\{x_n : n \in \mathbb{N}\}$ is an infinite set then there is an infinite Cauchy subsequence (x_{n_j}) of (x_n) such that $\lim_{j\to\infty} d(x_{n_j}, \hat{x}) = 0$. If $d(\hat{x}, f(\hat{x})) > 0$, then there is $n_0 \in \mathbb{N}$ such that

$$d(x_n, x_m) < \frac{1}{2} d(\hat{x}, f(\hat{x})), \quad \text{for all } n, m \ge n_0, \tag{16}$$

and in this way

$$\begin{aligned}
d(\hat{x}, f(\hat{x})) &\le \zeta\left(\limsup_{j\to\infty} d\left(x_{n_j}, f(\hat{x})\right)\right) \\
&\le \zeta\left(q \limsup_{j\to\infty} \max\left\{d\left(x_{n_j-1}, x_{n_j}\right), d(\hat{x}, f(\hat{x}))\right\}\right) \\
&\le \zeta\left(q \max\left\{\frac{1}{2} d(\hat{x}, f(\hat{x})), d(\hat{x}, f(\hat{x}))\right\}\right) \\
&= \zeta(q d(\hat{x}, f(\hat{x}))).
\end{aligned} \tag{17}$$

If $t := q d(\hat{x}, f(\hat{x}))$, then $t \le q\zeta(t)$; therefore $t = 0$ or $t = \infty$, but this is not possible, because $0 < d(\hat{x}, f(\hat{x})) < \infty$. Thus $\hat{x}$ is a fixed point. If $\hat{y}$ is other fixed point, then

$$\begin{aligned}
d(\hat{x}, \hat{y}) &= d(f(\hat{x}), f(\hat{y})) \\
&\le q \max\{d(\hat{x}, f(\hat{x})), d(\hat{y}, f(\hat{y}))\} = 0.
\end{aligned} \tag{18}$$

From this the uniqueness of $\hat{x}$ follows. □

Now let us give an example where condition (12) is necessary to obtain a fixed point.

Example 8. Let $\zeta : [0, \infty) \to [0, \infty)$ be defined as $\zeta(x) = x^\alpha$, with $0 < \alpha < 1$. Given $q_0 \in (0, 1)$ there is $t_0 = (q_0/2)^{1/(1-\alpha)} \in (0, 1)$ such that $\zeta(t_0) \ge t_0/q_0$. Let us consider the set $E = \{0\} \cup \mathbb{N}$ with the semimetric

$$d(m, n) = d(n, m) = \begin{cases} t_0^{\min\{n,m\}}, & (n, m) \in \mathbb{N} \times \mathbb{N}, \\ \zeta(t_0^m), & (n, m) \in \{0\} \times \mathbb{N}. \end{cases} \tag{19}$$

Since the sequence $(n)_{n\in\mathbb{N}}$ is an infinite Cauchy sequence that converge to 0 (this follows from $\lim_{m\to\infty} d(m, 0) = \lim_{m\to\infty} \zeta(t_0^m) = 0$) we have

$$\begin{aligned}
d(0, m) &\le \zeta(t_0^m) = \zeta\left(\limsup_{n\to\infty} t_0^{\min\{n,m\}}\right) \\
&= \zeta\left(\limsup_{n\to\infty} d(n, m)\right).
\end{aligned} \tag{20}$$

Therefore, the semimetric space (E, d) is subordinate to ζ. The function $f : E \to E$, defined as

$$f(n) = n + 1, \tag{21}$$

does not have fixed points. On the other hand, for $m \ge 1$,

$$t_0^m \le t_0 \le q_0 \zeta(t_0) \le \zeta(t_0), \tag{22}$$

this implies

$$\begin{aligned}
d(f(0), f(m)) &= d(1, m + 1) = t_0 \le q_0 \zeta(t_0) \\
&= q_0 \max\{\zeta(t_0), t_0^m\} \\
&= q_0 \max\{d(0, f(0)), d(m, f(m))\}
\end{aligned} \tag{23}$$

and, for $n \geq 1$,

$$\begin{aligned} d(f(n), f(m)) &= d(n+1, m+1) = t_0^{\min\{n+1,m+1\}} \\ &= t_0 t_0^{\min\{n,m\}} \leq q_0 t_0^{\min\{n,m\}} = q_0 \max\{t_0^n, t_0^m\} \\ &= q_0 \max\{d(n, f(n)), d(m, f(m))\}. \end{aligned} \tag{24}$$

Therefore, f is a q-contraction on the complete semimetric space (E, d) subordinated to ζ without fixed point.

Right away we will try a version of Matkowski's theorem in the context of subordinate semimetric spaces.

Theorem 9. *Let (E, d) be a complete semimetric space subordinated to ζ and $f : E \to E$. Suppose that there exists a nondecreasing function $\varphi : [0, \infty] \to [0, \infty]$ such that $\lim_{n\to\infty}\varphi^{[n]}(t) = 0$, for all $t \in [0, \infty)$, and*

$$d(f(x), f(y)) \leq \varphi(d(x, y)), \quad \text{for all } x, y \in E. \tag{25}$$

If there is an $x_0 \in E$ such that

$$\delta(d, f, x_0) := \sup\left\{d\left(x_0, f^{[n]}(x_0)\right) : n \in \mathbb{N}\right\} < \infty, \tag{26}$$

then $(f^n(x_0))$ converges to some $\hat{x} \in E$. Moreover, $\hat{x}$ is the unique fixed point of f.

Proof. Let us take $x_n = f^{[n]}(x_0)$. Suppose $m < n$, then

$$\begin{aligned} d(x_n, x_m) &< \varphi(d(x_{n-1}, x_{m-1})) \\ &< \varphi^{[2]}(d(x_{n-2}, x_{m-2})) \\ &\vdots \\ &< \varphi^{[m]}(d(x_{n-m}, x_0)) \\ &\leq \varphi^{[m]}(\delta(d, f, x_0)). \end{aligned} \tag{27}$$

The hypothesis implies that (x_n) is a Cauchy sequence, then there is $\hat{x} \in E$ such that $\lim_{n\to\infty} d(x_n, \hat{x}) = 0$. Suppose that $x_{m_0} = x_{n_0}$ for some $m_0 < n_0$, then $x_{m_0} = f^{[n_0-m_0]}(x_{m_0})$, and this yields

$$\begin{aligned} &d\left(x_{m_0}, f\left(x_{m_0}\right)\right) \\ &\quad = d\left(f^{[n_0-m_0]}\left(x_{m_0}\right), f^{[n_0-m_0]}\left(f\left(x_{m_0}\right)\right)\right) \\ &\quad \leq \varphi^{[n_0-m_0]}\left(d\left(x_{m_0}, f\left(x_{m_0}\right)\right)\right) \\ &\quad < \varphi^{[n_0-m_0-1]}\left(d\left(x_{m_0}, f\left(x_{m_0}\right)\right)\right) \\ &\quad \leq d\left(x_{m_0}, f\left(x_{m_0}\right)\right). \end{aligned} \tag{28}$$

Hence all the terms in the Cauchy sequence (x_n) are different. Moreover

$$\begin{aligned} d(\hat{x}, f(\hat{x})) &\leq \zeta\left(\limsup_{n\to\infty} d(x_n, f(\hat{x}))\right) \\ &\leq \zeta\left(\limsup_{n\to\infty} \varphi(d(x_{n-1}, \hat{x}))\right) \\ &\leq \zeta\left(\limsup_{n\to\infty} d(x_{n-1}, \hat{x})\right) \leq \zeta(0) = 0. \end{aligned} \tag{29}$$

Thus $\hat{x}$ is a fixed point of f. If $\hat{y}$ is other fixed point, then

$$d(\hat{x}, \hat{y}) = d(f(\hat{x}), f(\hat{y})) \leq \varphi(d(\hat{x}, \hat{y})) = 0. \tag{30}$$

From this the uniqueness of $\hat{x}$ follows. □

It is clear that a semimetric space (E, d) for which $\delta(d, f, x) = \infty$, for each $x \in E$, does not have fixed points, then (26) is a necessary condition in order to have a function $f : E \to E$ and a fixed point.

Proposition 10. *Let (E, d) be a semimetric space subordinated to ζ. Let (x_n) be a Cauchy sequence in (E, d) with $x_n \neq x_m$ whenever $n \neq m$. If there is a subsequence (x_{n_j}) of (x_n) such that $\lim_{j\to\infty} x_{n_j} = x$, then (x_n) converge to x.*

Proof. Let $\varepsilon > 0$, then there is $\delta > 0$ such that $\zeta(t) < \varepsilon$, for each $0 < t < \delta$. On the other hand, there exists $n_0 \in \mathbb{N}$ for which

$$d(x_n, x_m) < \frac{\delta}{2}, \quad \text{for all } m, n \geq n_0. \tag{31}$$

Therefore,

$$\limsup_{j\to\infty} d\left(x_{n_j}, x_m\right) \leq \frac{\delta}{2}, \quad \text{for all } m \geq n_0. \tag{32}$$

Since ζ is nondecreasing

$$d(x, x_m) \leq \zeta\left(\limsup_{j\to\infty} d\left(x_{n_j}, x_m\right)\right) \leq \zeta\left(\frac{\delta}{2}\right) < \varepsilon, \quad \text{for all } m \geq n_0. \tag{33}$$

Hence $\lim_{n\to\infty} d(x, x_n) = 0$. □

With the next result we characterize when a subordinate semimetric space is complete; the corresponding result for metric spaces is due to Subrahmanyam [12].

Theorem 11. *Let (E, d) be a semimetric space subordinated to ζ. Let $q \in (0, 1)$; if every q-contraction has a fixed point then (E, d) is complete.*

Proof. Let us suppose that (E, d) is not complete, then there is a nonconvergent Cauchy sequence (x_n). The Cauchy property implies that we can take, if necessary, a subsequence with all the elements different, so we suppose that x_n are distinct (see the second part in the proof of Theorem 7). Let us define the sets $A_n = \{x_m : m \geq n\}$, for each $n \in \mathbb{N}$. If $z \notin A_n$ then Proposition 10 implies $d(z, A_n) > 0$. The Cauchy property of (x_n) implies that the sets $M_n(z) = \{k \geq n : d(x_r, x_s) <$

$qd(z, A_n)$, for all $r, s \geq k\}$, are not empty. Let us define $k_n(z) = \min M_n(z)$, then

$$d\left(x_r, x_{k_n(z)}\right) < qd\left(z, A_n\right) \leq qd\left(z, x_t\right), \quad \text{for all } r \geq k_n(z) \text{ and all } t \geq n. \tag{34}$$

For each $z \in E$ there is $n \in \mathbb{N}$ such that $z \notin A_n$; thus $n(z) = \min\{n \in \mathbb{N} : z \notin A_n\}$ is well defined. The function $f : E \to E$, defined as $f(z) = x_{k_{n(z)}(z)}$, does not have fixed points. Indeed, by definition $x_{k_{n(z)}(z)} \in A_{k_{n(z)}(z)}$ and $z \notin A_{k_{n(z)}(z)}$, since $A_{k_{n(z)}(z)} \subset A_{n(z)}$. Moreover, for $z, y \in E$, we assume, without loss of generality, that $k_{n(y)}(y) > k_{n(z)}(z)$, then $k_{n(z)}(z) \geq n(z)$ and (34) implies

$$\begin{aligned} d\left(f(z), f(y)\right) &= d\left(x_{k_{n(y)}(y)}, x_{k_{n(z)}(z)}\right) \\ &< qd\left(z, x_{k_{n(z)}(z)}\right) \\ &\leq q \max\left\{d\left(z, x_{k_{n(z)}(z)}\right), d\left(y, x_{k_{n(y)}(y)}\right)\right\} \\ &= q \max\left\{d\left(z, f(z)\right), d\left(y, f(y)\right)\right\}. \end{aligned} \tag{35}$$

Thus f is a q-contraction without fixed points; this contradicts the hypothesis. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

Acknowledgments

The author was partially supported by Centro de Ciencias Básicas and the Grant PIM18-2 of Universidad Autónoma de Aguascalientes.

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