Hindawi
Journal of Mathematics
Volume 2018, Article ID 8175935, 6 pages
https://doi.org/10.1155/2018/8175935

Research Article

High-Order Iterative Methods for the DMP Inverse

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Received 1 December 2017; Revised 4 March 2018; Accepted 26 March 2018; Published 7 May 2018

Academic Editor: Shwetabh Srivastava

We investigate two iterative methods for computing the DMP inverse. The necessary and sufficient conditions for convergence of our schemes are considered and the error estimate is also derived. Numerical examples are given to test the accuracy and effectiveness of our methods.

1. Introduction

Let $\mathbb{C}_r^{m\times n}$ be the set of all $m \times n$ complex matrices with rank r. For any given a matrix $A \in \mathbb{C}^{m\times n}$, let $R(A)$, $N(A)$, and $\|A\|$ be the range space, the null space, and the Frobenius norm of matrix A, respectively. For a nonnegative integer l, if $\operatorname{rank}(A^{l+1}) = \operatorname{rank}(A^l)$, then l is called the index of A and $\operatorname{ind}(A) = l$. In recent years, the generalized inverse has been applied in many fields of engineering and technology, such as control [1], the least squares problem [2, 3], matrix decomposition [4], image restoration, statistics (see [5]), and preconditioning [6–8]. In particular, {2}-inverse plays an important role in stable approximations of ill-posed problems (see [1, 9]) and in linear and nonlinear problems [6, 10]. In [11], Baksalary and Trenkle investigate the core inverse.

For a given matrix $A \in \mathbb{C}^{m\times n}$, there exists matrix $X \in \mathbb{C}^{n\times m}$ satisfying (see [8])

(1) $AXA = A$,
(2) $XAX = X$,
(3) $(AX)^* = AX$,
(4) $(XA)^* = XA$,

where X is called the Moore-Penrose inverse of A, denoted by $X = A^\dagger$, and it is unique. For a given matrix $A \in \mathbb{C}^{n\times n}$, there exists a matrix $X \in \mathbb{C}^{n\times n}$ satisfying

(1) $XAX = X$,
(2) $A^{l+1}X = A^l$,
(3) $AX = XA$,

where X is called the Drain inverse of A, denoted by $X = A^D$, and it is unique. Based on the Drazin (D) inverse and the Moore-Penrose (MP) inverse, a new generalized inverse is defined in [12] as (see also [13, 14]): for a matrix $A \in \mathbb{C}_r^{n\times n}$, there is matrix $X \in \mathbb{C}^{n\times n}$ satisfying (see [12])

(1) $XAX = X$,
(2) $XA = A^DA$,
(3) $A^lX = A^lA^\dagger$,

where X is called DMP inverse, denoted by $X = A^{D,\dagger}$, and it is unique. It is shown that $X = A^DAA^\dagger$ in [12]. In [15], Yu and Deng get some characterizations of DMP inverse in a Hilbert space. By using idempotent element, some new properties of DMP inverse are given in [16].

So far, there are few results on computation of the DMP inverse by the iterative methods given in [17–22]. Recently, a family of higher-order convergent iterative methods are developed in [23] and applied to compute the Moore-Penrose inverse; the method is extended to compute the generalized inverse in [20]. In this paper, we develop two iterative methods to compute the DMP inverse of a given matrix $A \in \mathbb{C}^{n\times n}$. The proposed method (I) is higher-order and the proposed method (II) can be implemented easily.

The paper is organized as follows. The proposed iterative methods for computing DMP inverse are given and some lemmas used for its convergence analysis are given in Section 2. The stability and convergence analysis of our scheme (1) and (4) are given, and numerical examples are given to test the corresponding theoretical results in Sections 3 and 4, respectively.

2. Preliminaries and Iterative Scheme

Lemma 1 (see [23]). *If $A \in \mathbb{C}^{n\times n}$ and $\mathbb{C}^n = L \bigoplus M$, then*

(i) $P_{L,M}A = A \Leftrightarrow R(A) \subseteq L$,

(ii) $AP_{L,M} = A \Leftrightarrow N(A) \supseteq M$.

Lemma 2 (see [12]). *Let $A \in \mathbb{C}^{n\times n}$, $\operatorname{ind}(A) = l$. If $A^{D,\dagger}$ exist, then*

(i) $AA^{D,\dagger} = P_{R(A^l),N(A^lA^\dagger)}$, $R(A^{D,\dagger}) \subseteq R(A^l)$,

(ii) $A^{D,\dagger}A = P_{R(A^l),N(A^l)}$, $N(A^{D,\dagger}) \supseteq N(A^lA^\dagger)$.

As in [20], we develop a iterative scheme to compute the DMP inverse as follows.

Scheme I:

$$\begin{aligned} X_n = X_{n-1}\Big[&C_t^1 I - C_t^2 AX_{n-1} + \cdots \\ &+ (-1)^{t-1} C_t^t (AX_{n-1})^{t-1}\Big], \end{aligned} \tag{1}$$

where $\alpha \in \mathbb{C}\backslash\{0\}$, $R(Y) \subseteq R(A^l)$, and $X_0 = \alpha Y$. Following the line [21], we develop the iterative following scheme as

$$\begin{aligned} Y_n &= V_n - F'\left(V_n^{-1}\right)F(V_n), \\ Z_n &= V_n - \left[2^{-1}F'(V_n)\right]\left[F'(V_n)^{-1} + F'(Y_n)^{-1}\right], \\ V_{n+1} &= Z_n - \left[2^{-1}F(V_n)F'(Y_n)^{-1}\right] \\ &\quad \cdot \left[2 + F''(W_n)F(W_n)F'(W_n)^{-2}\right]. \end{aligned} \tag{2}$$

Let $\psi_n = AV_n$ and, by $V^{-1} - A = O$, we have

$$\begin{aligned} V_{n+1} &= -\frac{1}{8}V_n\left(-7I + 9AV_n - 5(AV_n)^2 + (AV_n)^3\right) \\ &\quad \times \left(12I - 42AV_n + 103(AV_n)^2 - 156(AV_n)^3\right. \\ &\quad + 157(AV_n)^4 - 104(AV_n)^5 + 43(AV_n)^6 \\ &\quad \left. - 10(AV_n)^7 + (AV_n)^8\right) = \frac{1}{8}V_n\left(84I - 402\psi_n\right. \\ &\quad + 1159\psi_n^2 - 2241\varphi_n^3 + 3060\psi_n^4 - 3024\psi_n^5 + 2178\psi_n^6 \\ &\quad \left. - 1134\varphi_n^7 + 416\psi_n^8 - 102_n\psi_n^9 + 15\psi_n^{10} - \psi_n^{11}\right) = \frac{1}{8} \\ &\quad \cdot V_n\phi(V_n). \end{aligned} \tag{3}$$

Thus, an efficient high-order iterative method can be written as follows.

Scheme II:

$$\begin{aligned} \psi_n &= AV_n, \\ X_n &= -7I + \psi_n\left(9I + \psi_n\left(5I - \psi_n\right)\right), \\ \theta_n &= \psi_n X_n, \\ V_{n+1} &= -\frac{1}{8}V_nX_n\left(12I + \theta_n\left(6I + \theta_n\right)\right). \end{aligned} \tag{4}$$

The iterative method given in (4) is applied to compute the Drazin inverse by [21]. Here, we use the sequence of iterative $\{V_n\}_{n=0}^{n=\infty}$ to compute the DMP inverse.

3. Scheme I for the DMP Inverse

In this section, we consider the numerical analysis of Scheme I (1) and present a numerical example to test our numerical theoretical results.

3.1. Stability and Convergence Analysis

Theorem 3. *Let $A \in \mathbb{C}^{m\times m}$ with $\operatorname{ind}(A) = l$. For an arbitrary positive integer $t \geq 2$, the sequence (1) converges if only if $\rho(P_{R(A^l),N(A^lA^\dagger)} - AX_0) < 1$. Moreover, we have*

$$\left\|A^{D,\dagger} - X_n\right\| \leq \left\|A^{D,\dagger}\right\| \left\|P_{R(A^l),N(A^lA^\dagger)} - AX_0\right\|^{t^n}. \tag{5}$$

Proof. Let $P = P_{R(A^l),N(A^lA^\dagger)}$ for convenience. If $X_0 = \alpha Y$, $R(Y) \subseteq R(A^l)$, then $R(AX_0) = R(\alpha AY) \subseteq R(AA^l) \subseteq R(A^l)$. From Lemma 2, we attain $PAX_0 = AX_0$.

Now, we test $PAX_n = AX_n$ by using mathematical induction. Assume that $PAX_n = AX_n$ for any positive integer $n > 0$. By Lemma 2, we have

$$\begin{aligned} &PAX_{n+1} \\ &\quad = PAX_n\left[C_t^1 I - C_t^2 AX_n + \cdots + (-1)^{t-1} C_t^t (AX_n)^{t-1}\right] \\ &\quad = C_t^1 AX_n - C_t^2 (AX_n)^2 + \cdots + (-1)^{t-1}(AX_n)^t \\ &\quad = AX_{n+1}. \end{aligned} \tag{6}$$

Let $E_n = P - AX_n$; then

$$\begin{aligned} E_{n+1} &= P - C_t^1 AX_n + C_t^2 (AX_n)^2 + \cdots + (-1)^t (AX_n)^t \\ &= E_n^t. \end{aligned} \tag{7}$$

Similarly, by Lemma 2 and (7), we derive $A^{D,\dagger}AX_n = P_{R(A^l),N(A^l)}X_n = X_n$ and

$$\begin{aligned} A^{D,\dagger} - X_n &= A^{D,\dagger}AA^{D,\dagger} - A^{D,\dagger}AX_n \\ &= A^{D,\dagger}\left(P - AX_n\right) = A^{D,\dagger}E_n. \end{aligned} \tag{8}$$

Thus, we have

$$\left\|A^{D,\dagger} - X_n\right\| \leq \left\|A^{D,\dagger}\right\| \left\|E_0\right\|^{t^n}. \tag{9}$$

Next, we investigate the necessary and sufficient condition for convergent property of Scheme I (1). Assume that the sequence $\{X_n\}$ converges to $A^{D,\dagger}$. Thus, $E_n = E_0^{t^n} = P - AX_n \to 0$, while $n \to \infty$. Therefore, we have $\rho(E_0) < 1$.

Conversely, let $\rho(P_{R(A^l),N(A^lA^\dagger)} - AX_0) < 1$ for some scalar $\alpha \in \mathbb{C}\backslash\{0\}$. Then $A^{D,\dagger} - X_n = A^{D,\dagger}E_n = A^{D,\dagger}E_0^{t^n}$. Thus, we have $A^{D,\dagger} - X_n \to 0$ and

$$\left\|A^{D,\dagger} - X_n\right\| \le \left\|A^{D,\dagger}\right\| \left\|P_{R(A^l),N(A^lA^\dagger)} - AX_0\right\|^{t^n}. \tag{10}$$

□

As in [24], we show that Scheme I (1) is asymptotically stable as the following result.

Theorem 4. *Let $A \in \mathbb{C}^{m\times m}$ and let the sequence $\{X_n\}_{n=0}^{n=\infty}$ be generated by (1) with an initial $X_0 = \alpha Y$. If $\alpha \in \mathbb{C}\backslash\{0\}$, $R(Y) \subseteq R(A^l)$, and $\rho(P_{R(A^l),N(A^lA^\dagger)} - AX_0) < 1$, then Scheme I (1) is asymptotically stable.*

Proof. Let $\triangle X_n$ be the numerical perturbation of X_n in Scheme I (1). Thus, it can be written into as $X_n = \widetilde{X_n} + \triangle X_n$. Here, we perform a first-order error analysis; that is, we formally neglect quadratic or higher terms. The manipulation is meaningful, while $\triangle X_n$ is sufficiently small. Further, we have

$$X_nA \triangle X_n \approx \triangle X_nAX_n. \tag{11}$$

Let $\phi(X_n) = \sum_{i=1}^{t}(-1)^{i-1}C_t^i(AX_n)^{i-1}$; we have

$$\begin{gathered} X_{n+1} = X_n\phi(X_n), \\ (AX_n + A\triangle X_n)^t \approx (AX_n)^t + t(AX_n)^{t-1}(A\triangle X_n). \end{gathered} \tag{12}$$

Similarly, we have $\widetilde{X}_{n+1} = \widetilde{X}_n\phi(\widetilde{X}_n)$ and

$$\begin{aligned} \widetilde{X}_{n+1} &= (X_n + \triangle X_n)\sum_{i=1}^{t}(-1)^{i-1}C_t^i(AX_n + A\triangle X_n)^{i-1} \\ &= tX_n + t\triangle X_n \\ &\quad + \sum_{i=2}^{t}(-1)^{i-1}C_t^i(X_n + \triangle X_n)(AX_n)^{i-1} \\ &\quad + \sum_{i=2}^{t}(-1)^{i-1}C_t^i(X_n + \triangle X_n)(i-1)(AX_n)^{i-2}A \\ &\quad \cdot \triangle X_n \approx X_{n+1} + t\triangle X_n + \sum_{i=2}^{t}(-1)^{i-1}C_t^i \\ &\quad \cdot \triangle X_n(AX_n)^{i-1} \\ &\quad + \sum_{i=2}^{t}(-1)^{i-1}C_t^i(i-1)(X_n)(AX_n)^{i-2}A\triangle X_n. \end{aligned} \tag{13}$$

By (13) and $PAX_n = AX_n$, we derive

$$\begin{aligned} \widetilde{X}_{n+1} &= X_{n+1} + t\triangle X_n \\ &\quad + \sum_{i=2}^{t}(-1)^{i-1}iC_t^i\triangle X_n(AX_n)^{i-1} \\ &= X_{n+1} + t\triangle X_n(P - AX_n)^{t-1}. \end{aligned} \tag{14}$$

By (7), we obtain

$$\triangle X_n = t\triangle X_n(P - AX_n)^{t-1} = t\triangle X_n(E_n)^{t-1}. \tag{15}$$

Thus, we derive

$$\begin{aligned} \|\triangle X_n\| &\le t\|\triangle X_n\|\left\|(E_n)^{t-1}\right\| \\ &\le t^{n+1}\|\triangle X_0\|\left\|(E_0)\right\|^{t^n-1}. \end{aligned} \tag{16}$$

We can conclude that the perturbation at the iterate $n+1$ is bounded. Therefore, the sequence $\{X_n\}_{n=0}^{n=\infty}$ generated by (1) is asymptotically stable. □

3.2. Numerical Example. Here is an example for computing DMP inverse in the iterative method (1).

Example 1. Let

$$\begin{gathered} A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \\ Y = A^lA^* = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \end{gathered} \tag{17}$$

where $l = \mathrm{ind}(A) = 2$ and $X_0 = \alpha Y$. Thus, $R(Y) \subseteq R(A^l)$. To test the high accuracy and efficiency of Scheme I (1), the DMP inverse of A is given as

$$A^{D,\dagger} = \begin{bmatrix} 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{18}$$

Here, we apply Scheme I given in (1) to compute the DMP inverse with $\alpha = 0.2$. The errors $\|A^{D,\dagger} - X_n\|$ and $\|X_n - X_{n-1}\|$ are given in Table 1. From the numerical results note that (1) converges to $A^{D,\dagger}$ and it has high-order accuracy.

4. Scheme II for the DMP Inverse

Here, the numerical analysis of Scheme II is derived and a numerical example is given to test our numerical theoretical results. Note that it is difficult to construct a projection P given in Theorems 3 and 4 with satisfying $\rho(P_{R(A^l),N(A^lA^\dagger)} - AX_0) < 1$.

Table 1: Numerical results of Scheme I with $\alpha = 0.2$ in Example 1.

Order	Steps	$\left\|A^{D,\dagger} - X_n\right\|$	$\left\|X_n - X_{n-1}\right\|$	cpu time
$t = 5$	4	0	$3.3307e-16$	0.000380
$t = 6$	4	$6.6613e-16$	$1.7764e-15$	0.001192
$t = 7$	4	$2.4425e-15$	$7.1054e-15$	0.000473
$t = 8$	3	$9.7700e-15$	$1.9429e-14$	0.000606
$t = 9$	3	$3.9968e-15$	$1.0658e-14$	0.000516

4.1. Stability and Convergence Analysis

Theorem 2. *Let $A \in \mathbb{C}^{m\times m}$ with $\operatorname{ind}(A) = l$; the sequence $\{V_n\}_{n=0}^{n=\infty}$ defined by Scheme II. If initial approximation V_0 satisfies $\|I - AV_0\| < 1$ and $R(V_0) \subseteq R(A^l)$, then Scheme II converges to $A^{D,\dagger}$ and*

$$\frac{\left\|A^{D,\dagger} - V_n\right\|}{\left\|A^{D,\dagger}\right\|} \le \left\|I - AV_0\right\|^{9^n}. \tag{19}$$

Proof. Similar to the proof of Theorem 9 in [21], let $F_n = I - AV_n$ for $n \ge 0$; then

$$\begin{aligned} F_{n+1} &= I - AV_{n+1} = I - A\Big(-\frac{1}{8}V_n\big(-7I + 9AV_n \\ &\quad - 5(AV_n)^2 + (AV_n)^3\big) \times \big(12I - 42AV_n \\ &\quad + 103(AV_n)^2 - 156(AV_n)^3 + 157(AV_n)^4 \\ &\quad - 104(AV_n)^5 + 43(AV_n)^6 - 10(AV_n)^7 \\ &\quad + (AV_n)^8\big)\Big) = \frac{1}{8}(I + F_n)^3 F_n^9 = \frac{1}{8}\big(F_n^9 + 3F_n^{10} \\ &\quad + 3F_n^{11} + F_n^{12}\big). \end{aligned} \tag{20}$$

Thus, we have

$$\left\|F_{n+1}\right\| \le \frac{1}{8}\left(\left\|F_n\right\|^9 + 3\left\|F_n\right\|^{10} + 3\left\|F_n\right\|^{11} + \left\|F_n\right\|^{12}\right). \tag{21}$$

Since $\|F_0\| < 1$, we obtain that

$$\begin{aligned} \left\|F_1\right\| &\le \frac{1}{8}\left(\left\|F_0\right\|^9 + 3\left\|F_0\right\|^{10} + 3\left\|F_0\right\|^{11} + \left\|F_0\right\|^{12}\right) \\ &\le \left\|F_0\right\|^9 < 1. \end{aligned} \tag{22}$$

Similarly, $\|F_2\| < \|F_1\|^9 < 1$. So we can get the conclusion $\|F_n\| \le 1$, $\|F_{n+1}\| \le \|F_n\|^9 \le \|F_0\|^{9^{n+1}}$.

Since $R(V_0) \subseteq R(A^l)$, $R(V_n) \subseteq R(V_{n-1})$, we attain $R(V_n) \subseteq R(A^l)$. By Lemma 2, we have $A^{D,\dagger}AV_n = V_n$ and

$$\delta_n = A^{D,\dagger} - A^{D,\dagger}AV_n = A^{D,\dagger}(I - AV_n) = A^{D,\dagger}F_n. \tag{23}$$

So, we have $\|\delta_n\| \le \|A^{D,\dagger}\|\|F_n\| \le \|A^{D,\dagger}\|\|F_0\|^{9^n}$.

If the condition $\|I - AV_0\| < 1$ is satisfied, then $\|\delta_n\| \to 0$ when $n \to \infty$. In other words, $V_n \to A^{D,\dagger}$, the scheme $\{V_n\}_{n=0}^{\infty}$ satisfies the following error estimate:

$$\frac{\left\|A^{D,\dagger} - V_n\right\|}{\left\|A^{D,\dagger}\right\|} \le \left\|I - AV_0\right\|^{9^n}. \tag{24}$$

□

Theorem 3. *Let $A \in \mathbb{C}^{m\times m}$ be a singular square matrix with $\operatorname{ind}(A) = l$ and the sequence $\{V_n\}_{n=0}^{n=\infty}$ defined by Scheme II. If the initial approximation V_0 is chosen such that $\|I - AV_0\| < 1$ and $R(V_0) \subseteq R(A^l)$, then the order of convergence of the sequence $\{V_n\}_{n=0}^{n=\infty}$ is nine.*

Proof. Let $F_0 = I - AV_0$ and $F_n = I - AV_n$; we have $F_n = (1/8)(F_n^9 + 3F_n^{10} + 3F_n^{11} + F_n^{12})$.

Now, let $\delta_n = A^{D,\dagger} - V_n$; we have

$$\begin{aligned} A\delta_{n+1} &= AA^{D,\dagger} - I + I - AV_{n+1} \\ &= \frac{1}{8}\left(\left(F_n^9 + 3F_n^{10} + 3F_n^{11} + F_n^{12}\right) + 8\left(AA^{D,\dagger} - I\right)\right) \\ &= \frac{1}{8}\left(\left(F_n^9 + AA^{D,\dagger} - I\right) + 3\left(F_n^{10} + AA^{D,\dagger} - I\right)\right. \\ &\quad \left. + 3\left(F_n^{11} + AA^{D,\dagger} - I\right) + \left(F_n^{12} + AA^{D,\dagger} - I\right)\right). \end{aligned} \tag{25}$$

Since DMP inverse is a special of {2}-inverse, by citing the proof of Theorem 3.1 in [25], we have $(I - AA^{D,\dagger})^t = 0$ and $(F_n)^t + AA^{D,\dagger} - I = (A\delta_n)^t$ and $A\delta_{n+1} = (1/8)((A\delta_n)^9 + 3(A\delta_n)^{10} + 3(A\delta_n)^{11} + (A\delta_n)^{12})$.

Since $\|A\delta_n\| \le 1$ when $n \to \infty$, we have $\|A\delta_{n+1}\| \le (1/8)(\|A\delta_n\|^9 + 3\|A\delta_n\|^{10} + 3\|A\delta_n\|^{11} + \|A\delta_n\|^{12}) \le \|A\delta_n\|^9$.

It would be easy to find the error inequality of the high-order iterative as follows:

$$\left\|\delta_{n+1}\right\| = \left\|V_{n+1} - A^{D,\dagger}\right\| \le \left\|A^{D,\dagger}\right\|\|A\|^9\left\|\delta_n\right\|^9. \tag{26}$$

Thus, the sequence converges to DMP inverse and the convergence order is nine. □

It is easy to find V_0 with satisfying $\|I - AV_0\| < 1$. So the method to compute DMP inverse is more convenient than another.

In what follows, we investigate the stability of the iterative method (4).

Theorem 4. *Let $A \in \mathbb{C}^{m\times m}$; the sequence $\{V_n\}_{n=0}^{n=\infty}$ defined by Scheme II with the initial approximation V_0 satisfies $\|I - AV_0\| < 1$ that is asymptotically stable for computing DMP inverse.*

Proof. Let $\triangle n$ be the numerical perturbation introduced in Scheme II. Next, the modified value

$$\widetilde{V_n} = V_n + \triangle_n \tag{27}$$

Table 2: Numerical results of Scheme II (28) in Example 1 in Section 4.2.

Steps	$\left\|A^{D,\dagger} - X_n\right\|$	$\left\|X_n - X_{n-1}\right\|$	CPU time
5	$2.6038e-14$	$2.5121e-14$	0.002925

appears instead of the exact value V_n. Here, we formally neglect quadratic or higher terms such as $(\triangle_n)^2$. This formal manipulation is meaningful if $\triangle n$ is sufficiently small and further has $V_n\triangle_n \approx \triangle_n V_n$, then

$$\begin{aligned}\widetilde{V_{n+1}} &= \frac{1}{8}\widetilde{V_n}\phi\left(\widetilde{V_n}\right) = \frac{1}{8}\left(V_n + \triangle_n\right)\phi\left(V_n + \triangle_n\right)\\ &\approx V_{n+1} + \frac{1}{8}\left(84I - 2\times 402\psi + 3\times 1159\psi^2 - 4\right.\\ &\times 2241\varphi^3 + 5\times 3060\psi^4 - 6\times 3024\psi^5 + 7\\ &\times 2178\psi^6 - 8\times 1134\varphi^7 + 9\times 416\psi^8 - 10\\ &\left.\times 102\psi^9 + 11\times 15\psi^{10} - 12\times\psi^{11}\right)\triangle_n = V_{n+1}\\ &+ \frac{1}{8}\left[9\times F^8 + 10\times 3F^9 + 11\times 3F^{10} + 12\times F^{11}\right]\\ &\cdot\triangle_n,\end{aligned} \tag{28}$$

$V_n + (\triangle_n)^t = V_n^t + tV_n^{t-1}\triangle_n$, $\triangle_{n+1} = \widetilde{V_{n+1}} - V_{n+1}$, when $k\to\infty$, $V_n \to A^{D,\dagger}$.

Simplifying (28) we attain

$$\begin{aligned}\triangle_{n+1} &= \frac{1}{8}\left[9\times F^8 + 10\times 3F^9 + 11\times 3F^{10} + 12\right.\\ &\left.\times F^{11}\right]\triangle_n \approx \frac{1}{8}\left[9\times\left(I - AA^{D,\dagger}\right)^8 + 10\right.\\ &\times 3\left(I - AA^{D,\dagger}\right)^9 + 11\times 3\left(I - AA^{D,\dagger}\right)^{10} + 12\\ &\left.\times\left(I - AA^{D,\dagger}\right)^{11}\right]\triangle_n.\end{aligned} \tag{29}$$

Using the matrix identity, we have $(I - AA^{D,\dagger})^t = I - AA^{D,\dagger}$, $t \geq 1$,

$$\begin{aligned}\triangle_{n+1} &\approx \frac{1}{8}(9 + 30 + 33 + 12)\left(I - AA^{D,\dagger}\right)\\ \triangle_n &\approx \frac{84}{8}\left(I - AA^{D,\dagger}\right)\triangle_n,\end{aligned} \tag{30}$$

then

$$\left\|\triangle_{n+1}\right\| \leq \left(\frac{84}{8}\right)^{n+1}\left\|I - AA^{D,\dagger}\right\|\left\|\triangle_0\right\|. \tag{31}$$

We can conclude that the perturbation at the iterate $n+1$ is bounded. Therefore, the sequence $\{V_n\}_{n=0}^{n=\infty}$ generated by Scheme II is asymptotically stable. □

4.2. Numerical Example. The numerical examples are worked out by using high level language Matlab R2013a on an Intel(R) core running on Windows 10 Professional Version.

Example 1. Let

$$A = \begin{bmatrix} 1 & 1 & 0\\ 0 & 1 & 1\\ 0 & 0 & 1\end{bmatrix}. \tag{32}$$

Take

$$V_0 = \begin{bmatrix} 0.2 & -0.5 & 0.1\\ 0 & 0.5 & -0.1\\ 0 & 0 & 0.1\end{bmatrix}, \tag{33}$$

where $\operatorname{ind}(A) = l = 1, R(V_0) \subseteq R(A^l)$. Thus, $\|I - AV_0\| < 1$. To test the efficiency and accuracy of our scheme, we present the DMP inverse of A as

$$A^{D,\dagger} = \begin{bmatrix} 1 & -1 & 1\\ 0 & 1 & -1\\ 0 & 0 & 1\end{bmatrix}. \tag{34}$$

In Table 2, we give the errors $\|A^{D,\dagger} - X_n\|$, $\|X_n - X_{n-1}\|$. The results show that the proposed method (4) converges to $A^{D,\dagger}$ and has high-order accuracy.

5. Conclusions

We have developed two iterative methods for computing the DMP inverse. The proposed scheme has high-order accuracy and Scheme II can be implemented without constructing the projection $P_{R(A^l),N(A^lA^\dagger)}$. The stability, convergence analysis, and the error estimate of our schemes are given. Numerical examples show that our schemes have high-order accuracy and effectiveness. It is more interesting that we shall extend these methods to compute other generalized inverse, such as {2}-generalized inverse [18, 20, 22].

Conflicts of Interest

No potential conflicts of interest were reported by the authors.

Acknowledgments

This work is supported partially by the National Natural Science Foundation of China (Grant no. 11361009), the Special Fund for Scientific and Technological Bases and Talents of Guangxi (Grant no. 2016AD05050), and the Special Fund for Bagui Scholars of Guangxi.

References

[1] H. Zhang and H. Yin, "Conjugate gradient least squares algorithm for solving the generalized coupled Sylvester matrix

equations," *Computers & Mathematics with Applications. An International Journal*, vol. 73, no. 12, pp. 2529–2547, 2017.

[2] I. Kyrchei, "Weighted singular value decomposition and determinantal representations of the quaternion weighted Moore-Penrose inverse," *Applied Mathematics and Computation*, vol. 309, pp. 1–16, 2017.

[3] I. Kyrchei, "Explicit representation formulas for the minimum norm least squares solutions of some quaternion matrix equations," *Linear Algebra and Its Applications*, vol. 438, no. 1, pp. 136–152, 2013.

[4] H. Wang, "Core-EP decomposition and its applications," *Linear Algebra and Its Applications*, vol. 508, pp. 289–300, 2016.

[5] H.-A. Diao, Y. Wei, and P. Xie, "Small sample statistical condition estimation for the total least squares problem," *Numerical Algorithms*, vol. 75, no. 2, pp. 435–455, 2017.

[6] M. Z. Nashed, *Generalized Inverses and Applications*, Academic Press, New York, NY, USA, 1976.

[7] A. J. Getson and F. C. Hsuan, $A_{T,S}^{(2)}$*-Inverses and Their Statistical Applications*, Lecture Notes in Statistics, Springer, Berlin, Germany, 1988.

[8] A. Ben-Israel and T. N. E. Greville, *Generalized Inverses: Theory and Applications*, Springer, New York, NY, USA, 2nd edition, 2003.

[9] B. Zheng and R. B. Bapat, "Generalized inverse $A_{T,S}{}^{(2)}$ and a rank equation," *Applied Mathematics and Computation*, vol. 155, no. 2, pp. 407–415, 2004.

[10] M. Z. Nashed and X. Chen, "Convergence of Newton-like methods for singular operator equations using outer inverses," *Numerische Mathematik*, vol. 66, no. 2, pp. 235–257, 1993.

[11] O. M. Baksalary and G. Trenkler, "Core inverse of matrices," *Linear and Multilinear Algebra*, vol. 58, no. 5-6, pp. 681–697, 2010.

[12] S. B. Malik and N. Thome, "On a new generalized inverse for matrices of an arbitrary index," *Applied Mathematics and Computation*, vol. 226, pp. 575–580, 2014.

[13] A. Herrero, F. J. Ramírez, and N. Thome, "Relationships between different sets involving group and Drazin projectors and nonnegativity," *Linear Algebra and Its Applications*, vol. 438, no. 4, pp. 1688–1699, 2013.

[14] L. Lebtahi and N. Thome, "The spectral characterization of k-generalized projectors," *Linear Algebra and Its Applications*, vol. 20, pp. 572–575, 2007.

[15] A. Yu and C. Deng, "Characterizations of DMP inverse in a Hilbert space," *Calcolo. A Quarterly on Numerical Analysis and Theory of Computation*, vol. 53, no. 3, pp. 331–341, 2016.

[16] C. Deng and A. Yu, *Relationships between DMP Relation and Some Partial Orders*, Elsevier Science, 2015.

[17] W. Li and Z. Li, "A family of iterative methods for computing the approximate inverse of a square matrix and inner inverse of a non-square matrix," *Applied Mathematics and Computation*, vol. 215, no. 9, pp. 3433–3442, 2010.

[18] X. Liu, C. Hu, and Y. Yu, "Further results on iterative methods for computing generalized inverses," *Journal of Computational and Applied Mathematics*, vol. 234, no. 3, pp. 684–694, 2010.

[19] J. Zhong, X. Liu, G. Zhou, and Y. Yu, "A new iterative method for computing the Drazin inverse," *Filomat*, vol. 26, no. 3, pp. 597–606, 2012.

[20] X. Liu, H. Jin, and Y. Yu, "Higher-order convergent iterative method for computing the generalized inverse and its application to Toeplitz matrices," *Linear Algebra and Its Applications*, vol. 439, no. 6, pp. 1635–1650, 2013.

[21] F. Soleymani and P. S. Stanimirović, "A higher order iterative method for computing the Drazin inverse," *The Scientific World Journal*, vol. 2013, Article ID 708647, 11 pages, 2013.

[22] S. Srivastava and D. K. Gupta, "The iterative methods for $A_{T,S}{}^{(2)}$ of the bounded linear operator between Banach spaces," *Applied Mathematics and Computation*, vol. 49, no. 1-2, pp. 383–396, 2015.

[23] H. Chen and Y. Wang, "A family of higher-order convergent iterative methods for computing the Moore-Penrose inverse," *Applied Mathematics and Computation*, vol. 218, no. 8, pp. 4012–4016, 2011.

[24] F. Soleymani and P. S. Stanimirović, "A note on the stability of a th order iteration for finding generalized inverses," *Applied Mathematics Letters*, vol. 28, no. 2, pp. 77–81, 2014.

[25] V. Y. Pan, F. Soleymani, and L. Zhao, "An efficient computation of generalized inverse of a matrix," *Applied Mathematics and Computation*, vol. 316, pp. 89–101, 2018.