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Research Article

Two-Dimensional Quaternion Linear Canonical Transform: Properties, Convolution, Correlation, and Uncertainty Principle

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A definition of the two-dimensional quaternion linear canonical transform (QLCT) is proposed. The transform is constructed by substituting the Fourier transform kernel with the quaternion Fourier transform (QFT) kernel in the definition of the classical linear canonical transform (LCT). Several useful properties of the QLCT are obtained from the properties of the QLCT kernel. Based on the convolutions and correlations of the LCT and QFT, convolution and correlation theorems associated with the QLCT are studied. An uncertainty principle for the QLCT is established. It is shown that the localization of a quaternion-valued function and the localization of the QLCT are inversely proportional and that only modulated and shifted two-dimensional Gaussian functions minimize the uncertainty.

1. Introduction

The quaternion Fourier transform (QFT) is a nontrivial generalization of the classical Fourier transform (FT) using the quaternion algebra. The QFT has been shown to relate to the other quaternion signal analysis tools, such as quaternion wavelet transform [1–3], fractional quaternion Fourier transform [4, 5], quaternionic windowed Fourier transform [6–9], and quaternion Wigner transform [10]. Because of the noncommutative property of quaternion multiplication, we obtain at least three different kinds of two-dimensional QFTs as follows (see [11–15]):

$$\begin{aligned}
\mathcal{F}_q^{\mathrm{I}}\{f\}(\nu) &= \int_{\mathfrak{R}^2} e^{-\mu_1 \nu\cdot z} f(z)dz, && \nu\cdot z = \nu_1 z_1 + \nu_2 z_2,\\
\mathcal{F}_q^{\mathrm{II}}\{f\}(\nu) &= \int_{\mathfrak{R}^2} f(z) e^{-\mu_1 \nu\cdot z} dz, && \nu\cdot z = \nu_1 z_1 + \nu_2 z_2,\\
\mathcal{F}_q^{\mathrm{III}}\{f\}(\nu) &= \int_{\mathfrak{R}^2} e^{-\mu_1 \nu_1 z_1} f(z) e^{-\mu_2 \nu_2 z_2} dz, && dz = dz_1 dz_2,
\end{aligned} \tag{1}$$

where μ_1 and μ_2 are any two unit pure quaternions ($\mu_1^2 = \mu_2^2 = -1$) that are orthogonal to each other. These three QFTs are so-called *left-sided*, *right-sided*, and *double-sided* QFTs or type I, II, and III QFTs, respectively. As is well known, the linear canonical transform (LCT) is a general form of the FT, and the quaternion linear canonical transform (QLCT) is a generalization of the QFT in the LCT domain.

In the recent years, the LCT and QLCT have received much attention. Kou et al. [16, 17], for instance, constructed the windowed LCT, with a local window function. It can reveal the local LCT-frequency contents and enjoys high concentrations and eliminates the cross term. It is used to study the generalized prolate spheroidal wave functions and the connection with energy concentration problems. In [18–20], Kou et al. also proposed the (right-sided) quaternion linear canonical transform (QLCT) which is a generalization of the quaternion Fourier transform (QFT) in the LCT domain. This generalization is obtained by replacing the Fourier kernel with the right-sided QFT (type III QFT) kernel in the LCT definition. Some important properties such as Parseval's theorem, reconstruction formula, uncertainty principles, and asymptotic behaviour are discussed. Like the uncertainty principle for the QFT [3], they also showed that only a two-dimensional Gaussian signal

minimizes the uncertainty. However, the convolution theorem is an important result of the QLCT which does not hold using this construction because of the noncommutative property of the right-sided quaternion Fourier kernel.

Our attention in this article is to introduce a definition of the QLCT (compared to [21–23]). This definition is obtained by substituting the Fourier kernel with the *type II QFT kernel* (see [24]) in the LCT definition, which is essentially different from the kernel of the type III QFT. We investigate some important properties such as linearity, shift, and modulation and Plancherel's and Parseval's theorems. We study the convolution theorems associated with the QLCT, which can be useful in digital signal and image processing. Based on the convolution definitions of the LCT [25–27] and QFT, we also propose a new correlation definition for the QLCT and obtain its correlation theorems. We finally establish an uncertainty principle for the QLCT and show that only modulated and shifted two-dimensional Gaussian functions minimize the uncertainty.

We display here the organization of the paper: Section 2 describes the preliminaries about the quaternion Fourier transform and its basic properties, which will be used in the next section. The construction of the QLCT is presented in Section 3; its important properties are also discussed in this section. The definition of the convolution in the QLCT domain is introduced and the theorem on the QLCT of the convolution of two quaternion-valued functions is constructed in Section 4. Section 5 provides the definition of the correlation in the QLCT domain and discusses the theorem on the QLCT of a correlation of two quaternion functions. Section 6 establishes an uncertainty principle for the QLCT, which shows that the spread of a quaternion-valued function and its QLCT are inversely proportional. It is shown that only modulated and shifted two-dimensional Gaussian functions minimize the uncertainty.

Throughout this paper, $\mathfrak{R}$ is used to denote a set of real numbers.

2. Preliminaries

2.1. Quaternion. The quaternion, which is a type of hypercomplex number, was originally invented by William Hamilton in 1843 [28]. It is a generalization of a complex number to a 4D algebra and is denoted by $\mathbb{H}$. Every element of $\mathbb{H}$ can be written in the hypercomplex form as follows:

$$\mathbb{H} = \{q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 : q_0, q_1, q_2, q_3 \in \mathfrak{R}\}. \tag{2}$$

Here, the three different imaginary parts satisfy the following multiplication rules:

$$\begin{aligned} \mathbf{ij} &= -\mathbf{ji} = \mathbf{k}, \\ \mathbf{jk} &= -\mathbf{kj} = \mathbf{i}, \\ \mathbf{ki} &= -\mathbf{ik} = \mathbf{j}, \\ \mathbf{i}^2 &= \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1. \end{aligned} \tag{3}$$

For a quaternion $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3 \in \mathbb{H}$, q_0 is called the *scalar part* of q denoted by $\mathrm{Sc}(q)$ and $\mathbf{q} = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ is called the *pure part* of q denoted by $\mathrm{Vec}(q) = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$. A quaternionic conjugation is given by

$$\overline{q} = q_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3. \tag{4}$$

Any quaternion q can be represented in the polar form as (see [29])

$$q = |q|e^{\mu\theta}, \quad |q| = \sqrt{q\overline{q}} = \sqrt{q_0^2 + q_1^2 + q_2^2 + q_3^2}, \tag{5}$$

where $\theta = \arctan|\mathrm{Sc}(q)|/\mathrm{Vec}(q)$, $0 \le \theta \le \pi$, is the eigen angle or phase of q and μ is an arbitrarily fixed unit quaternion such that $\mu^2 = -1$. When $|q| = 1$, q is a unit quaternion. Note that Euler's formula holds for quaternions; that is, $e^{\mu\theta} = \cos\theta + \mu\sin\theta$. We also have $|e^{\mu\theta}| = 1$. Hereinafter, besides the quaternion units $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ and the vector part $\mathbf{q}$ of a quaternion $q \in \mathbb{H}$, we shall use the following real vector notation:

$$\begin{aligned} z &= (z_1, z_2) \in \mathfrak{R}^2, \\ f(z) &= f(z_1, z_2), \\ |z|^2 &= z_1^2 + z_2^2, \\ z \cdot y &= z_1 y_1 + z_2 y_2, \\ zy &= (z_1 y_1, z_2 y_2), \end{aligned} \tag{6}$$

and so on when there is no confusion. We may define the left quaternion inner product for two functions f and g by

$$(f, g)_{L^2(\mathfrak{R}^2;\mathbb{H})} = \int_{\mathfrak{R}^2} f(z)\overline{g(z)}dz. \tag{7}$$

We see that, for $f = g$, we obtain the norm induced by the above inner product as

$$\|f\|_{L^2(\mathfrak{R}^2;\mathbb{H})} = \left(\int_{\mathfrak{R}^2} |f(z)|^2 dz\right)^{1/2}. \tag{8}$$

We find it convenient to introduce the space as follows:

$$L^2(\mathfrak{R}^2; \mathbb{H}) = \left\{f \mid f : \mathfrak{R}^2 \to \mathbb{H}, \|f\|_{L^2(\mathfrak{R}^2;\mathbb{H})} < \infty\right\}. \tag{9}$$

2.2. Convolution Associated with QFT. The quaternion Fourier transform (QFT) was originally proposed by Ell [11] and applied in quaternion color image processing [12, 30]. Some results related to the fundamental properties of the QFT can be found in [31–33]. We introduce a definition of the QFT as follows.

Definition 1. The QFT of $f \in L^2(\mathfrak{R}^2; \mathbb{H})$ is the function $\mathcal{F}_q\{f\} \in L^2(\mathfrak{R}^2, \mathbb{H})$ given by

$$\mathcal{F}_q\{f\}(v) = \int_{\mathfrak{R}^2} f(z)e^{-\mu v \cdot z}dz, \tag{10}$$

where μ is an arbitrarily fixed pure quaternion such that $\mu^2 = -1$. The inverse transform of the QFT is given by

$$\begin{aligned} f(z) &= \mathcal{F}_q^{-1}\left[\mathcal{F}_q\{f\}\right](z) \\ &= \frac{1}{(2\pi)^2}\int_{\mathfrak{R}^2} \mathcal{F}_q\{f\}(v)e^{\mu v \cdot z}dv, \end{aligned} \tag{11}$$

provided that the integral exists.

Definition 2. The convolution of $f \in L^2(\mathfrak{R}^2;\mathbb{H})$ and $g \in L^2(\mathfrak{R}^2;\mathbb{H})$, denoted by $f \star g$, is defined by

$$(f \star g)(z) = \int_{\mathfrak{R}^2} f(y)g(z-y)dy. \tag{12}$$

We have the following important result on the QFT of a convolution of two quaternion functions.

Theorem 1 [34]. *Let f, $g \in L^2(\mathfrak{R}^2;\mathbb{H})$ have the following representations:*

$$\begin{aligned} f(z) &= f_0(z) + \mathbf{i}f_1(z) + \mathbf{j}f_2(z) + \mathbf{k}f_3(z), \\ g(z) &= g_0(z) + \mathbf{i}g_1(z) + \mathbf{j}g_2(z) + \mathbf{k}g_3(z). \end{aligned} \tag{13}$$

Then, we have

$$\begin{aligned} \mathcal{F}_q\{f \star g\}(v) = &\mathcal{F}_q\{g\}(v)\mathcal{F}_q\{f_0\}(v) \\ &+ \mathbf{i}\mathcal{F}_q\{g\}(v)\mathcal{F}_q\{f_1\}(v) \\ &+ \mathbf{j}\mathcal{F}_q\{g\}(v)\mathcal{F}_q\{f_2\}(v) \\ &+ \mathbf{k}\mathcal{F}_q\{g\}(v)\mathcal{F}_q\{f_3\}(v). \end{aligned} \tag{14}$$

It is easily seen that the convolution theorem in the FT domain is a special case of Theorem 1.

3. Quaternion Linear Canonical Transform

The linear canonical transform (LCT) is a linear integral transformation with three free parameters which has widely been used in various fields such as spectral analysis, image processing, and optical system analysis [35, 36]. Several famous transforms such as the Fourier transform, the fractional Fourier transform, and the other transformations are special cases. This section will consider generalizing the LCT using the quaternion algebra. This extension is then called the quaternion linear canonical transform (QLCT).

3.1. Definition of QLCT. Based on the definition of the type II QFT, we obtain a definition of the QLCT by replacing the kernel of the FT with the kernel of the type II QFT in the classical LCT definition. The special linear group of degree 2 over $\mathfrak{R}$, that is, the group of all real 2×2 matrices with determinant one, is denoted by $\mathrm{SL}(2,\mathfrak{R})$. Let

$$A_s = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} \in \mathrm{SL}(2,\mathfrak{R}), \quad s = 1, 2. \tag{15}$$

When $b_1 b_2 \neq 0$, we define the kernels K_{A_s}, $s = 1, 2$, of the QLCT by

$$K_{A_s}(z_s, v_s) = \frac{1}{\sqrt{2\pi\mu b_s}} e^{(1/2)\mu\left((a_s/b_s)z_s^2 - (2/b_s)z_s v_s + (d_s/b_s)v_s^2\right)}, \quad s = 1, 2. \tag{16}$$

Definition 3. The QLCT of $f \in L^1(\mathfrak{R}^2;\mathbb{H})$ is defined by

$$L^{\mathbb{H}}_{A_1,A_2}\{f\}(v) = \begin{cases} \displaystyle\int_{\mathfrak{R}^2} f(z)K_{A_1}(z_1, v_1)K_{A_2}(z_2, v_2)dz, & b_1 b_2 \neq 0, \\ \sqrt{d_1 d_2}\, f(d_1 v_1, d_2 v_2)e^{\mu(c_1 d_1/2)v_1^2} e^{\mu(c_2 d_2/2)v_2^2}, & b_1 b_2 = 0, \end{cases} \tag{17}$$

where $e^{\mu(c_s d_s/2)v_s^2}$, $s = 1, 2$, are called *chirp signals* in signal processing. Here after, we will deal with the case when $b_1 b_2 \neq 0$ because $L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)$ is trivial for $b_1 b_2 = 0$. When $A_1 = A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, the QLCT reduces to the QFT, that is,

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v) &= \int_{\mathfrak{R}^2} f(z)\frac{1}{\sqrt{2\pi\mu}}e^{-\mu v_1 z_1}\frac{1}{\sqrt{2\pi\mu}}e^{-\mu v_2 z_2}dz \\ &= \int_{\mathfrak{R}^2} f(z)e^{-\mu v\cdot z}\,dz\,\frac{1}{2\pi\mu} \\ &= \mathcal{F}_q\{f\}(v)\frac{1}{2\pi\mu}. \end{aligned} \tag{18}$$

The polar form of the QLCT (17) can be found in [37].

The following describes the general relationship between the QLCT and the type II QFT of a signal f.

Lemma 1. *The QLCT of a signal f with $A_s = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} \in SL(2,\mathfrak{R})$, $s = 1, 2$, can be seen as the QFT of a signal f in the form*

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v) = &\mathcal{F}_q\left\{f(z)e^{\mu(a_1/2b_1)z_1^2}e^{\mu(a_2/2b_2)z_2^2}\right\} \\ &\cdot\left(\frac{v_1}{b_1}, \frac{v_2}{b_2}\right)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(d_1/2b_1)v_1^2}e^{\mu(d_2/2b_2)v_2^2}. \end{aligned} \tag{19}$$

Proof. By a straightforward computation, it follows from Definition 3 that

$$
\begin{aligned}
&L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\\
&= \int_{\mathfrak{R}^2} f(z)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{(1/2)\mu\left(\left(a_1/b_1\right)z_1^2-\left(2/b_1\right)z_1v_1+\left(d_1/b_1\right)v_1^2\right)}e^{(1/2)\mu\left(\left(a_2/b_2\right)z_2^2-\left(2/b_2\right)z_2v_2+\left(d_2/b_2\right)v_2^2\right)}dz\\
&= \int_{\mathfrak{R}^2}\left(f(z)e^{\mu\left(a_1/2b_1\right)z_1^2}e^{\mu\left(a_2/2b_2\right)z_2^2}\right)e^{-\mu z_1\left(v_1/b_1\right)}e^{-\mu z_2\left(v_2/b_2\right)}dz\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu\left(d_1/2b_1\right)v_1^2}e^{\mu\left(d_2/2b_2\right)v_2^2}\\
&= \mathcal{F}_q\left\{f(z)e^{\mu\left(a_1/2b_1\right)z_1^2}e^{\mu\left(a_2/2b_2\right)z_2^2}\right\}\left(\frac{v_1}{b_1},\frac{v_2}{b_2}\right)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu\left(d_1/2b_1\right)v_1^2}e^{\mu\left(d_2/2b_2\right)v_2^2},
\end{aligned}
\tag{20}
$$

thus proving the theorem.

Theorem 2. *If $f \in L^1(\mathfrak{R}^2;\mathbb{H})$ and $L^{\mathbb{H}}_{A_1,A_2} \in L^1(\mathfrak{R}^2;\mathbb{H})$, then the inversion formula of the QLCT is given by*

$$
\begin{aligned}
f(z) &= \int_{\mathfrak{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)K_{A_1^{-1}}(v_1,z_1)K_{A_2^{-1}}(v_2,z_2)dv\\
&= \int_{\mathfrak{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\overline{K_{A_1}(z_1,v_1)K_{A_2}(z_2,v_2)}dv,
\end{aligned}
\tag{21}
$$

which is equivalent to

$$
\begin{aligned}
f(z) = \int_{\mathfrak{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\frac{1}{\sqrt{-2\pi\mu b_1}}e^{-(1/2)\mu\left(\left(a_1/b_1\right)z_1^2-\left(2/b_1\right)z_1v_1+\left(d_1/b_1\right)v_1^2\right)}\frac{1}{\sqrt{-2\pi\mu b_2}}\\
\times e^{-(1/2)\mu\left(\left(a_2/b_2\right)z_2^2-\left(2/b_2\right)z_2v_2+\left(d_2/b_2\right)v_2^2\right)}dv,
\end{aligned}
\tag{22}
$$

where $A_s^{-1} = \begin{pmatrix} d_s & -b_s \\ -c_s & a_s \end{pmatrix}$, $s = 1, 2$.

Proof. A direct calculation gives

$$
\begin{aligned}
f(y) &= \int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2} f(y)\frac{1}{\sqrt{-2\pi\mu b_1}}\frac{1}{\sqrt{-2\pi\mu b_2}}e^{-(1/2)\mu\left(\left(a_1/b_1\right)y_1^2-\left(2/b_1\right)y_1v_1+\left(d_1/b_1\right)v_1^2\right)}e^{-(1/2)\mu\left(\left(a_2/b_2\right)y_2^2-\left(2/b_2\right)y_2v_2+\left(d_2/b_2\right)v_2^2\right)}\\
&\quad\times\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{(1/2)\mu\left(\left(a_1/b_1\right)z_1^2-\left(2/b_1\right)z_1v_1+\left(d_1/b_1\right)v_1^2\right)}e^{(1/2)\mu\left(\left(a_2/b_2\right)z_2^2-\left(2/b_2\right)z_2v_2+\left(d_2/b_2\right)v_2^2\right)}dv\,dy\\
&= \int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2} f(y)\frac{1}{2\pi|b_1|}e^{-\left(a_1/2b_1\right)\mu\left(y_1^2-z_1^2\right)}e^{\left(\mu/b_1\right)\left(y_1-z_1\right)v_1}\times\frac{1}{2\pi|b_2|}e^{-\left(a_2/2b_2\right)\mu\left(y_2^2-z_2^2\right)}e^{\left(\mu/b_2\right)\left(y_2-z_2\right)v_2}dv\,dy\\
&= \int_{\mathfrak{R}^2} f(y)e^{-\left(a_1/2b_1\right)\mu\left(y_1^2-z_1^2\right)}\delta\left(y_1-z_1\right)e^{-\left(a_2/2b_2\right)\mu\left(y_2^2-z_2^2\right)}\delta\left(y_2-z_2\right)dy,
\end{aligned}
\tag{23}
$$

thus proving the theorem.

The following proposition will be very useful when proving the uncertainty principle for the QLCT.

Proposition 1. *Let the kernel functions $K_{A_s}(z_s,v_s)$, $s = 1, 2$, be defined by (16). Then, we have*

$$
\int_{\mathfrak{R}} v_s^k K_{A_s}(z_s,v_s)\overline{K_{A_s}(y_s,v_s)}dv_s = b_s^k(-\mu)^{-k}\delta^{(k)}(z_s-y_s),
\qquad k\in\mathbb{N}\cup\{0\},\quad s=1,2,
\tag{24}
$$

where $\delta^{(k)}$ denotes the k-th derivative of the delta function δ.

Proof. For $k \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
&\int_{\mathfrak{R}} v_s^k K_{A_s}(z_s, v_s)\overline{K_{A_s}(y_s, v_s)}dv_s \\
&= \frac{1}{2\pi b_s}\int_{\mathfrak{R}} v_s^k e^{\mu\left((a_s/2b_s)z_s^2-(1/b_s)z_s v_s+(d_s/2b_s)v_s^2\right)} e^{-\mu\left((a_s/2b_s)y_s^2-(1/b_s)y_s v_s+(d_s/2b_s)v_s^2\right)}dv_s \\
&= \frac{1}{2\pi b_s}\int_{\mathfrak{R}} v_s^k e^{\mu(a_s/2b_s)(z_s^2-y_s^2)}\frac{b_s^k}{v_s^k}(-\mu)^{-k}\left(\frac{\partial}{\partial z_s}\right)^k e^{\mu(1/b_s)v_s(y_s-z_s)}dv_s \\
&= e^{\mu(a_s/2b_s)(z_s^2-y_s^2)}b_s^k(-\mu)^{-k}\left(\frac{\partial}{\partial z_s}\right)^k \frac{1}{2\pi b_s}\int_{\mathfrak{R}} e^{\mu(1/b_s)v_s(y_s-z_s)}dv_s \\
&= e^{\mu(a_s/2b_s)(z_s^2-y_s^2)}b_s^k(-\mu)^{-k}\left(\frac{\partial}{\partial z_s}\right)^k \delta(z_s-y_s) \\
&= b_s^k(-\mu)^{-k}e^{\mu(a_s/2b_s)(z_s^2-y_s^2)}\delta^{(k)}(z_s-y_s).
\end{aligned} \tag{25}$$

Since the distributional support of $\delta^{(k)}(z_s - y_s)$ as a distribution of z_s is $\{y_s\}$, we have

$$\begin{aligned}
e^{\mu(a_s/2b_s)(z_s^2-y_s^2)}\delta^{(k)}(z_s-y_s) &= e^{\mu(a_s/2b_s)(z_s^2-z_s^2)}\Big|_{z_s=y_s} \\
&\quad\cdot \delta^{(k)}(z_s-y_s) \\
&= \delta^{(k)}(z_s-y_s),
\end{aligned} \tag{26}$$

in the distributional sense. With (25) and (26), we have (24).

3.2. Useful Properties of the QLCT. Parseval's theorem for Fourier transform can be generalized to the QLCT as well. Let us now formulate Parseval's theorem in the QLCT domain.

Theorem 3 (QLCT Parseval). *Let $f, g \in L^2(\mathfrak{R}^2; \mathbb{H})$. Then, we have*

$$(f, g) = \left(L^{\mathbb{H}}_{A_1,A_2}\{f\}, L^{\mathbb{H}}_{A_1,A_2}\{g\}\right). \tag{27}$$

Proof. For $f, g \in L^2(\mathfrak{R}^2, \mathbb{H})$, we have

$$\begin{aligned}
(f, g) &= \int_{\mathfrak{R}^2} f(z)\overline{g(z)}dz \\
&= \int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)K_{A_1^{-1}}(v_1, z_1)K_{A_2^{-1}}(v_2, z_2)dv\,\overline{g(z)}dz \\
&= \int_{\mathfrak{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\int_{\mathfrak{R}^2}\overline{K_{A_1}(z_1, v_1)K_{A_2}(z_2, v_2)\overline{g(z)}}dz\,dv \\
&= \int_{\mathfrak{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\overline{\int_{\mathfrak{R}^2} g(z)K_{A_1}(z_1, v_1)K_{A_2}(z_2, v_2)dz}\,dv \\
&= \int_{\mathfrak{R}^2} L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\overline{L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)}dv \\
&= \left(L^{\mathbb{H}}_{A_1,A_2}\{f\}, L^{\mathbb{H}}_{A_1,A_2}\{g\}\right).
\end{aligned} \tag{28}$$

In the second equality in Theorem 3, we have replaced the quaternion function f with its inverse QLCT expression in Theorem 2. In the third equality, we have interchanged the order of integration. In the fourth equality, we have applied the quaternion conjugation rule $\overline{pq} = \overline{q}\,\overline{p}$ for $p, q \in \mathbb{H}$, thus proving the theorem.

Due to the noncommutativity of the kernel of the QLCT, we only have a left linearity property with quaternion constants; that is,

$$L^{\mathbb{H}}_{A_1,A_2}\{\alpha f + \beta g\}(v) = \alpha L^{\mathbb{H}}_{A_1,A_2}\{f\}(v) + \beta L^{\mathbb{H}}_{A_1,A_2}\{g\}(v). \tag{29}$$

In the following, we will summarize the important properties of the QLCT. We will see that the results are generalizations of the basic properties of the LCT. Let $f \in L^2(\mathfrak{R}^2; \mathbb{H})$. The translation operator τ_k by $k \in \mathfrak{R}^2$ is defined by

$$\tau_k f(z) = f(z-k). \tag{30}$$

The modulation operator $\mathbb{M}_{\mathbf{v}_0}$ by $v_0 = (u_0, v_0) \in \mathfrak{R}^2$ is defined by

$$\begin{aligned}
\mathbb{M}_{v_0} f(z) &= f(z)e^{\mu z\cdot v_0}, \\
z\cdot v_0 &= z_1 u_0 + z_2 v_0.
\end{aligned} \tag{31}$$

We now begin with the shift property of the QLCT.

Theorem 4 (shift property). *For $f \in L^2(\mathfrak{R}^2; \mathbb{H})$, we have*

$$\begin{aligned}
L^{\mathbb{H}}_{A_1,A_2}\{\tau_k f\}(v) &= L^{\mathbb{H}}_{A_1,A_2}\{f\}(v_1 - a_1k_1, v_2 - a_2k_2) \\
&\quad\cdot e^{-(1/2)\mu\left(a_1c_1k_1^2+a_2c_2k_2^2\right)}e^{\mu\left(c_1k_1v_1+c_2k_2v_2\right)}.
\end{aligned} \tag{32}$$

Next, we are concerned about the behavior of the QLCT under modulation.

Theorem 5 (modulation property). *For $f \in L^2(\mathfrak{R}^2; \mathbb{H})$, we have*

$$L^{\mathbb{H}}_{A_1,A_2}\left\{\mathbb{M}_{\mathbf{v}_0}f\right\}(v) = L^{\mathbb{H}}_{A_1,A_2}\{f(z)e^{\mu z\cdot v_0}\}(v) = L^{\mathbb{H}}_{A_1,A_2}\{f\}(v-bv_0)e^{-(1/2)\mu\left(b_1d_1u_0^2+b_2d_2v_0^2\right)} \cdot e^{\mu\left(d_1u_0v_1+d_2v_0v_2\right)}. \tag{33}$$

Theorem 6 (time-frequency shift). *For $f \in L^2(\mathfrak{R}^2;\mathbb{H})$, we have*

$$L^{\mathbb{H}}_{A_1,A_2}\left\{\mathbb{M}_{\mathbf{v}_0}\tau_k f\right\}(v) = L^{\mathbb{H}}_{A_1,A_2}\{f(z-k)e^{\mu z\cdot v_0}\}(v) = L^{\mathbb{H}}_{A_1,A_2}\{f\}(v-\mathrm{ak}-bv_0)e^{-(1/2)\mu\left(a_1c_1k_1^2+a_2c_2k_2^2+b_1d_1u_0^2+b_2d_2v_0^2\right)} \times e^{\mu\left(c_1k_1+d_1u_0\right)v_1}e^{\mu\left(c_2k_2+d_2v_0\right)v_2}e^{-\mu\left(b_1c_1k_1u_0+b_2c_2k_2v_0\right)}. \tag{34}$$

We present the following result, which is needed for deriving the uncertainty principle related to the QLCT.

Theorem 7. *Let $A_s = \begin{pmatrix} a_s & b_s \\ c_s & d_s \end{pmatrix} \in SL(2,\mathfrak{R})$ and $b_s \neq 0$ for $s = 1, 2$. Assume that $f \in C^1(\mathfrak{R}^2;\mathbb{H})$, that f, $(\partial f/\partial z_i) \in L^2(\mathfrak{R}^2;\mathbb{H})$, and that $v_i L^{\mathbb{H}}_{A_1,A_2}\{f\} \in L^2(\mathfrak{R}^2;\mathbb{H})$, $i = 1, 2$. Then, we have*

$$\int_{\mathfrak{R}^2} v_i^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\right|^2 dv = b_i^2\int_{\mathfrak{R}^2}\left|\frac{\partial}{\partial z_i}f(z)\right|^2 dz, \quad i = 1, 2. \tag{35}$$

Proof. Recall the quaternion conjugation rule $\overline{pq} = \overline{q}\,\overline{p}$ for p, $q \in \mathbb{H}$. The distributional calculation gives

$$\begin{aligned}\int_{\mathfrak{R}^2} v_1^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\right|^2 dv &= \int_{\mathfrak{R}^2} v_1^2\left(\int_{\mathfrak{R}^2} f(z)K_{A_1}(z_1,v_1)K_{A_2}(z_2,v_2)dz\right) \cdot \left(\int_{\mathfrak{R}^2}\overline{f(y)K_{A_1}(y_1,v_1)K_{A_2}(y_2,v_2)}dy\right)dv \\ &= \int_{\mathfrak{R}^4}\int_{\mathfrak{R}} f(z)v_1^2K_{A_1}(z_1,v_1) \cdot \left(\int_{\mathfrak{R}} K_{A_2}(z_2,v_2)\overline{K_{A_2}(y_2,v_2)}dv_2\right)\overline{K_{A_1}(y_1,v_1)}dv_1\overline{f(y)}dy\,dz.\end{aligned} \tag{36}$$

Using $k = 0$ in Proposition 1, we get

$$\begin{aligned}\int_{\mathfrak{R}^2} v_1^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\right|^2 dv &= \int_{\mathfrak{R}^4} f(z)\int_{\mathfrak{R}} v_1^2K_{A_1}(z_1,v_1)\delta(z_2-y_2)\overline{K_{A_1}(y_1,v_1)}dv_1 \cdot \overline{f(y)}dy_2dy_1dz \\ &= \int_{\mathfrak{R}^4} f(z)\left(\int_{\mathfrak{R}} v_1^2f(z)K_{A_1}(z_1,v_1)\overline{K_{A_1}(y_1,v_1)}dv_1\right) \cdot \overline{f(y_1,z_2)}dy_1dz.\end{aligned} \tag{37}$$

Again taking $k = 2$ in Proposition 1 gives

$$\int_{\mathfrak{R}^2} v_1^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\right|^2 dv = b_1^2(-\mu)^{-2}\int_{\mathfrak{R}^3} f(z)\delta^{(2)}(z_1-y_1)\overline{f(y_1,z_2)}dy_1dz. \tag{38}$$

Since the delta function is even function and $(-\mu)^{-2} = -1$, we have

$$\begin{aligned}(-\mu)^{-2}\delta^{(2)}(z_1-y_1) &= -\left(\frac{\partial}{\partial z_1}\right)\left(-\frac{\partial}{\partial y_1}\right)\delta(z_1-y_1) \\ &= \left(-\frac{\partial}{\partial z_1}\right)\left(-\frac{\partial}{\partial y_1}\right)\delta(y_1-z_1).\end{aligned} \tag{39}$$

Then, integration by parts in the sense of distribution implies

$$\begin{aligned}(-\mu)^{-2}\int_{\mathfrak{R}^3} f(z)\delta^{(2)}(z_1-y_1)\overline{f(y_1,z_2)}dy_1dz &= \int_{\mathfrak{R}^3} f(z)\left(-\frac{\partial}{\partial z_1}\right)\left(-\frac{\partial}{\partial y_1}\right)\delta(y_1-z_1)\overline{f(y_1,z_2)}dy_1dz \\ &= \int_{\mathfrak{R}^2}\left(\frac{\partial}{\partial z_1}\right)f(z)\int_{\mathfrak{R}}\delta(y_1-z_1)\left(\frac{\partial}{\partial y_1}\right)\overline{f(y_1,z_2)}dy_1dz \\ &= \int_{\mathfrak{R}^2}\left(\frac{\partial}{\partial z_1}\right)f(z)\left(\frac{\partial}{\partial y_1}\right)\overline{f(y_1,z_2)}\Big|_{y_1=z_1}dz \\ &= \int_{\mathfrak{R}^2}\left(\frac{\partial}{\partial z_1}\right)f(z)\overline{\left(\frac{\partial}{\partial z_1}\right)f(z)}dz,\end{aligned} \tag{40}$$

which implies the desired result.

The above properties of the QLCT are summarized in Table 1.

Table 1: Useful properties of the QLCT.

Property	Quaternion func.	QLCT
Left linearity	$\alpha f+\beta g$	$\alpha L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)+\beta L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)$
Shift	$f(z-k)$	$L^{\mathbb{H}}_{A_1,A_2}\{f\}(v_1-a_1k_1,v_2-a_2k_2)\times e^{-(1/2)\mu(a_1c_1k_1^2+a_2c_2k_2^2)}e^{\mu(c_1k_1v_1+c_2k_2v_2)}$
Modulation	$f(z)\,e^{\mu z\cdot v_0}$	$L^{\mathbb{H}}_{A_1,A_2}\{f\}(v-bv_0)e^{-(1/2)\mu(b_1d_1u_0^2+b_2d_2v_0^2)}\times e^{\mu(d_1u_0v_1+d_2v_0v_2)}$
Scaling	$f(z/\alpha)$	$\alpha^2L^{\mathbb{H}}_{A_1',A_2'}\{f\}(v/\alpha)$, $A_i'=\begin{pmatrix} a_i & b_i/\alpha^2\\ c_i\alpha^2 & d_i\end{pmatrix},\quad i=1,2$
Time-frequency shift	$f(z-k)\,e^{\mu z\cdot v_0}$	$L^{\mathbb{H}}_{A_1,A_2}\{f\}(v-\mathrm{ak}-bv_0)\times e^{-(1/2)\mu(a_1c_1k_1^2+a_2c_2k_2^2+b_1d_1u_0^2+b_2d_2v_0^2)}\times e^{\mu(c_1k_1+d_1u_0)v_1}\ e^{\mu(c_2k_2+d_2v_0)v_2}\times e^{-\mu(b_1c_1k_1u_0+b_2c_2k_2v_0)}$
Formula		
Parseval's theorem		$\lVert f\rVert=\lVert L^{\mathbb{H}}_{A_1,A_2}\{f\}\rVert$
Plancherel's theorem		$(f,g)=(L^{\mathbb{H}}_{A_1,A_2}\{f\},L^{\mathbb{H}}_{A_1,A_2}\{g\})$
Reconstruction		$f(z)=\int_{\mathfrak{R}^2}K_{A_1^{-1}}(v_1,z_1)L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)K_{A_2^{-1}}(v_2,z_2)dv$ $\int_{\mathfrak{R}}v_s^kK_{A_s}(z_s,v_s)\overline{K_{A_s}(y_s,v_s)}dv_s=b_s^k(-\mu)^{-k}\delta^{(k)}(z_s-y_s),k\in\mathbb{N}\cup\{0\}$ $\int_{\mathfrak{R}^2}v_i^2\lvert L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\rvert^2dv=b_i^2\int_{\mathfrak{R}^2}\lvert(\partial/\partial z_i)f(z)\rvert^2dz,\quad i=1,2$

4. Convolution Associated with QLCT

As we know, convolution is one of the fundamental results in the Fourier transform and LCT. Because the QLCT is a generalization of the LCT using the quaternion algebra, it makes possible to build the convolution theorem in the QLCT domain. For this reason, we introduce the following definition.

Definition 4. For $f,\ g\in L^2(\mathfrak{R}^2;\mathbb{H})$, we define the convolution operator of the QLCT by

$$(f\star g)(z)=\int_{\mathfrak{R}^2}f(t)g(z-t)e^{\mu(a_1/b_1)t_1(t_1-z_1)}e^{\mu(a_2/b_2)t_2(t_2-z_2)}dt. \tag{41}$$

The above definition implies the following important theorem, which describes how the convolution of two quaternion-valued functions interacts with their QLCTs.

Theorem 8. *Let*

$$\begin{aligned} f(z)&=f_0(z)+\mathbf{i}f_1(z)+\mathbf{j}f_2(z)+\mathbf{k}f_3(z),\\ g(z)&=g_0(z)+\mathbf{i}g_1(z)+\mathbf{j}g_2(z)+\mathbf{k}g_3(z), \end{aligned} \tag{42}$$

belong to $L^2(\mathfrak{R}^2;\mathbb{H})$. Then, the QLCTs of the convolution of f and g are given by

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(v)=\ &\big(L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_0\}(v)\\ &+\mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_1\}(v)\\ &+\mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\\ &+\mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_3\}(v)\big)\\ &\cdot\sqrt{2\pi b_1\mu}\sqrt{2\pi b_2\mu}e^{-\mu(d_1v_1^2/2b_1)}e^{-\mu(d_2v_2^2/2b_2)}. \end{aligned} \tag{43}$$

Proof. Let $L^{\mathbb{H}}_{A_1,A_2}\{f\}$ and $L^{\mathbb{H}}_{A_1,A_2}\{g\}$ denote the QLCTs of f and g, respectively. Expanding the QLCT of the left-hand side of the above identity, we obtain

$$\begin{aligned} L^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(v)&\overset{(7)}{=}\int_{\mathfrak{R}^2}(f\star g)(z)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}\\ &\cdot e^{(1/2)\mu\left((a_1/b_1)z_1^2-(2/b_1)z_1v_1+(d_1/b_1)v_1^2\right)}\\ &\times e^{(1/2)\mu\left((a_2/b_2)z_2^2-(2/b_2)z_2v_2+(d_2/b_2)v_2^2\right)}\mathrm{dz}\\ &=\int_{\mathfrak{R}^2}\left[\int_{\mathfrak{R}^2}f(t)g(z-t)e^{\mu(a_1/b_1)t_1(t_1-z_1)}e^{\mu(a_2/b_2)t_2(t_2-z_2)}\mathrm{dt}\right]\\ &\times\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{(1/2)\mu\left((a_1/b_1)z_1^2-(2/b_1)z_1v_1+(d_1/b_1)v_1^2\right)}\\ &\cdot e^{(1/2)\mu\left((a_2/b_2)z_2^2-(2/b_2)z_2v_2+(d_2/b_2)v_2^2\right)}\mathrm{dz}. \end{aligned} \tag{44}$$

Changing variables $y = z - t$ in the above expression, we have

$$\begin{aligned}
&L^{\mathbb{H}}_{A_1,A_2}\{f \star g\}(v) \\
&= \int_{\mathfrak{R}^2}\left[\int_{\mathfrak{R}^2} f(t)g(y)e^{\mu(a_1/b_1)(-t_1y_1)}e^{\mu(a_2/b_2)(-t_2y_2)}dt\right] \times \frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{(1/2)\mu\left((a_1/b_1)(y_1+t_1)^2-(2/b_1)(y_1+t_1)v_1+(d_1/b_1)v_1^2\right)} \\
&\quad \times e^{(1/2)\mu\left((a_2/b_2)(y_2+t_2)^2-(2/b_2)(y_2+t_2)v_2+(d_2/b_2)v_2^2\right)}dy \\
&= \int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2} f(t)g(y)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)y_1^2}e^{\mu(a_1/2b_1)t_1^2}e^{-\mu(1/b_1)(y_1v_1)}e^{-\mu(1/b_1)(t_1v_1)}e^{\mu(d_1/2b_1)v_1^2} \\
&\quad \times e^{\mu(a_2/2b_2)y_2^2}e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(1/b_2)(y_2v_2)}e^{-\mu(1/b_2)(t_2v_2)}e^{\mu(d_2/2b_2)v_2^2}dy\,dt \\
&= \int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2} f(t)g(y)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{(1/2)\mu\left((a_1/b_1)y_1^2-(2/b_1)y_1v_1+(d_1/b_1)v_1^2\right)}e^{(1/2)\mu\left((a_2/b_2)y_2^2-(2/b_2)y_2v_2+(d_2/b_2)v_2^2\right)} \\
&\quad \times e^{\mu(a_1/2b_1)t_1^2}e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_1v_1/b_1)}e^{-\mu(t_2v_2/b_2)}dy\,dt.
\end{aligned} \tag{45}$$

Applying the QLCT definition yields

$$\begin{aligned}
L^{\mathbb{H}}_{A_1,A_2}\{f \star g\}(v) = \int_{\mathfrak{R}^2} f(t)L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)e^{\mu(a_1/2b_1)t_1^2} \\
\cdot e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_1v_1/b_1)}e^{-\mu(t_2v_2/b_2)}dt.
\end{aligned} \tag{46}$$

The noncommutativity of the quaternion multiplication requires us to decompose $f(t)$ into $f_0(t) + \mathbf{i}f_1(t) + \mathbf{j}f_2(t) + \mathbf{k}f_3(t)$. This gives

$$\begin{aligned}
L^{\mathbb{H}}_{A_1,A_2}\{f \star g\}(v) &= \int_{\mathfrak{R}^2}\left[f_0(t) + \mathbf{i}f_1(t) + \mathbf{j}f_2(t) + \mathbf{k}f_3(t)\right]L^{\mathbb{H}}_{A_1,A_2}\{g\}(v) \\
&\quad \times e^{\mu(a_1/2b_1)t_1^2}e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_1v_1/b_1)}e^{-\mu(t_2v_2/b_2)}dt \\
&= \int_{\mathfrak{R}^2}\left[L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_0(t) + \mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_1(t) + \mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_2(t)\right. \\
&\quad \left. + \mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_3(t)\right]e^{\mu(a_1/2b_1)t_1^2}e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_1v_1/b_1)}e^{-\mu(t_2v_2/b_2)}dt.
\end{aligned} \tag{47}$$

Multiplying both sides of the above identity by $1/\sqrt{2\pi\mu b_1}e^{\mu(d_1v_1^2/2b_1)}$ and $1/\sqrt{2\pi\mu b_2}e^{\mu(d_2v_2^2/2b_2)}$, we obtain

$$\begin{aligned}
&L^{\mathbb{H}}_{A_1,A_2}\{f\star g\}(v)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(d_1v_1^2/2b_1)}e^{\mu(d_2v_2^2/2b_2)}\\
&=\int_{\mathfrak{R}^2}\left[L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_0(t)+\mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_1(t)+\mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_2(t)+\mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_3(t)\right]\\
&\quad\cdot\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{-\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt\\
&=L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_0(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{-\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\quad\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt+\mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_1(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{-\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\quad\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt+\mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_2(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{-\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\quad\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt+\mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_3(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{-\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\quad\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt,
\end{aligned}\tag{48}$$

thus proving the theorem.

5. Correlation Theorem for QLCT

In this section, we define the correlation of the QLCT. Then, we investigate the relationship between the QLCT and the correlation of two quaternion functions.

Definition 5. For $f, g\in L^1(\mathfrak{R}^2;\mathbb{H})$, we define the correlation operator of the QLCT by

$$\begin{aligned}(f\diamond g)(z)=&\int_{\mathfrak{R}^2}\overline{f(t)}g(t+z)e^{\mu(a_1/b_1)t_1(t_1+z_1)}\\&\cdot e^{\mu(a_2/b_2)t_2(t_2+z_2)}dt.\end{aligned}\tag{49}$$

Then, we reap a consequence of the definition above.

Theorem 9. *For $f, g\in L^2(\mathfrak{R}^2;\mathbb{H})$, we have the QLCTs of correlation of f and g by*

$$\begin{aligned}L^{\mathbb{H}}_{A_1,A_2}\{f\diamond g\}(v)=&\Big(L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_0\}(v)-\mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_1\}(-v)\\&-\mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_2\}(-v)-\mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_3\}(-v)\Big).\end{aligned}\tag{50}$$

When $A_1=A_2=\begin{pmatrix}0&1\\-1&0\end{pmatrix}$, the above expression reduces to

$$L^{\mathbb{H}}_{A_1,A_2}\{f\diamond g\}(v)=L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{\overline{f}\}(-v)\sqrt{2\pi\mu b_1}\sqrt{2\pi\mu b_2}.\tag{51}$$

Proof. From the definitions of the QLCT (17) and the convolution, we get

$$\begin{aligned}L^{\mathbb{H}}_{A_1,A_2}\{f\diamond g\}(v)=&\int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2}\left[\overline{f(t)}g(z+t)e^{\mu(a_1/b_1)t_1(t_1+z_1)}e^{\mu(a_2/b_2)t_2(t_2+z_2)}dt\right]\\&\times\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{(1/2)\mu\left((a_1/b_1)z_1^2-(2/b_1)z_1v_1+(d_1/b_1)v_1^2\right)}e^{(1/2)\mu\left((a_2/b_2)z_2^2-(2/b_2)z_2v_2+(d_2/b_2)v_2^2\right)}dz.\end{aligned}\tag{52}$$

Setting $z + t = v$, we get

$$L^{\mathbb{H}}_{A_1,A_2}\{f \diamond g\}(v)$$

$$= \int_{\mathfrak{R}^2}\left[\int_{\mathfrak{R}^2}\overline{f(t)}g(v)e^{\mu(a_1/b_1)(t_1v_1)}e^{\mu(a_2/b_2)(t_2v_2)}dt\right] \times \frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{(1/2)\mu\left((a_1/b_1)(v_1-t_1)^2-(2/b_1)(v_1-t_1)v_1+(d_1/b_1)v_1^2\right)}$$

$$\times e^{(1/2)\mu\left((a_2/b_2)(v_2-t_2)^2-(2/b_2)(v_2-t_2)v_2+(d_2/b_2)v_2^2\right)}dt\,dv$$

$$= \int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2}\overline{f(t)}g(v)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)v_1^2}e^{\mu(a_1/2b_1)t_1^2}e^{-\mu(a_1/b_1)(v_1t_1)}e^{-\mu(1/b_1)(v_1v_1)}e^{\mu(1/b_1)(t_1v_1)}$$

$$\times e^{\mu(a_2/2b_2)v_2^2}e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(a_2/b_2)(v_2t_2)}e^{-\mu(1/b_2)(v_2v_2)}e^{\mu(1/b_2)(t_2v_2)} \times e^{\mu(d_1/2b_1)v_1^2}e^{\mu(d_2/2b_2)v_2^2}e^{\mu(a_1/b_1)(v_1t_1)}e^{\mu(a_2/b_2)(v_2t_2)}dt\,dv$$

$$= \int_{\mathfrak{R}^2}\int_{\mathfrak{R}^2}\overline{f(t)}g(v)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)v_1^2}e^{-\mu(1/b_1)(v_1v_1)}e^{\mu(d_1/2b_1)v_1^2}e^{\mu(a_1/2b_2)v_2^2}e^{-\mu(1/b_2)(v_2v_2)}$$

$$\times e^{\mu(d_2/2b_2)v_2^2}e^{\mu(a_2/2b_1)t_1^2}e^{\mu(1/b_1)(t_1v_1)}e^{\mu(a_2/2b_2)t_2^2}e^{\mu(1/b_2)(t_2v_2)}dt\,dv. \tag{53}$$

Using the QLCT definition, we obtain

$$L^{\mathbb{H}}_{A_1,A_2}\{f \diamond g\}(v) = \int_{\mathfrak{R}^2}\overline{f(t)}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)e^{\mu(a_1/2b_1)t_1^2} \cdot e^{\mu(t_1v_1/b_1)}e^{\mu(a_2/2b_2)t_2^2}e^{\mu(t_2v_2/b_2)}dt. \tag{54}$$

Substituting $\overline{f(t)} = f_0(t) - \mathbf{i}f_1(t) - \mathbf{j}f_2(t) - \mathbf{k}f_3(t)$, we have

$$L^{\mathbb{H}}_{A_1,A_2}\{f \diamond g\}(v) = \int_{\mathfrak{R}^2}\left[f_0(t) - \mathbf{i}f_1(t) - \mathbf{j}f_2(t) - \mathbf{k}f_3(t)\right]L^{\mathbb{H}}_{A_1,A_2}\{g\}(v) \times e^{\mu(a_1/2b_1)t_1^2}e^{\mu(t_1v_1/b_1)}e^{\mu(a_2/2b_2)t_2^2}e^{\mu(t_2v_2/b_2)}dt$$

$$= \int_{\mathfrak{R}^2}\left[L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_0(t) - \mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_1(t) - \mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_2(t) - \mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_3(t)\right]e^{\mu(a_1/2b_1)t_1^2}e^{\mu(t_1v_1/b_1)}e^{\mu(a_2/2b_2)t_2^2}e^{\mu(t_2v_2/b_2)}dt. \tag{55}$$

Multiplying both sides of the above identity by $1/\sqrt{2\pi\mu b_1}e^{\mu(d_1 v_1^2/2b_1)}$ and $1/\sqrt{2\pi\mu b_2}e^{\mu(d_2 v_2^2/2b_2)}$, we obtain

$$\begin{aligned}
&L^{\mathbb{H}}_{A_1,A_2}\{f\diamond g\}(v)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(d_1 v_1^2/2b_1)}e^{\mu(d_2 v_2^2/2b_2)}\\
&\quad=\int_{\mathfrak{R}^2}\left[L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_0(t)-\mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_1(t)-\mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)f_2(t)-\mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)f_3(t)\right]\\
&\qquad\cdot\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt\\
&\quad=L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_0(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\qquad\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt-\mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_1(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\qquad\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt-\mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_2(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\qquad\times e^{\mu(a_2/2b_2)t_2^2}e^{-\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt-\mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)\int_{\mathfrak{R}^2}f_3(t)\frac{1}{\sqrt{2\pi\mu b_1}}\frac{1}{\sqrt{2\pi\mu b_2}}e^{\mu(a_1/2b_1)t_1^2}e^{\mu(t_1v_1/b_1)}e^{\mu(d_1v_1^2/2b_1)}\\
&\qquad\times e^{\mu(a_2/2b_2)t_2^2}e^{\mu(t_2v_2/b_2)}e^{\mu(d_2v_2^2/2b_2)}dt.
\end{aligned}\tag{56}$$

We finally obtain

$$\begin{aligned}
&L^{\mathbb{H}}_{A_1,A_2}\{f\diamond g\}(v)\\
&=\Big(L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_0\}(-v)-\mathbf{i}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_1\}(-v)-\mathbf{j}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_2\}(-v)\\
&\quad-\mathbf{k}L^{\mathbb{H}}_{A_1,A_2}\{g\}(v)L^{\mathbb{H}}_{A_1,A_2}\{f_3\}(-v)\Big),
\end{aligned}\tag{57}$$

thus proving the theorem.

6. Uncertainty Principle for QLCT

As in the case of the LCT [38, 39], the uncertainty principle is an important result of the QLCT. In [18, 19, 21], the authors have established the uncertainty principle for the two-sided QLCT. Based on the definition of our proposed QLCT in this article, we investigate an uncertainty principle related to the QLCT. The proof of the uncertainty differs from ones mentioned above. This uncertainty principle describes that the spread of a quaternion-valued function and its QLCT are inversely proportional. It is found that modulated and shifted two-dimensional Gaussian functions minimize the uncertainty.

Theorem 10. *Let $f\in C^1(\mathfrak{R}^2;\mathbb{H})\cap L^2(\mathfrak{R}^2;\mathbb{H})$, $(\partial f/\partial z_i)\in L^2(\mathfrak{R}^2;\mathbb{H})$, and $L^{\mathbb{H}}_{A_1,A_2}\{f\}$, $v_i L^{\mathbb{H}}_{A_1,A_2}\{f\}\in L^2(\mathfrak{R}^2;\mathbb{H})$, $i=1,2$, and $F=\int_{\mathfrak{R}^2}|f(z)|^2dz<\infty$. Then, the following inequalities hold:*

$$\int_{\mathfrak{R}^2}z_i^2|f(z)|^2dz\int_{\mathfrak{R}^2}v_i^2\left|L^{\mathbb{H}}_{A_1,A_2}\{f\}(v)\right|^2dv\geq\frac{b_i^2}{4}F^2,\quad i=1,2.\tag{58}$$

Moreover, we have equality in the above expression for $i=1,2$ if and only if f is a modulated and shifted two-dimensional Gaussian function; that is, it has the form

$$f(z)=K_0e^{\mu\mathbf{v}_0\cdot\mathbf{z}}e^{-\left((z_1-k_1)^2/2\alpha_1+(z_2-k_2)^2/2\alpha_2\right)},\tag{59}$$

where K_0 is a quaternion constant and α_1, α_2 are positive real constants.

Proof. We will prove this theorem in two parts. Firstly, we set $i=1$. By inserting Theorem 7 into the left-hand side of (58), we obtain

$$\begin{aligned}
&\int_{\mathfrak{R}^2} z_1^2 |f(z)|^2 dz \int_{\mathfrak{R}^2} v_1^2 \left| L^{\mathbb{H}}_{A_1,A_2}\{f\}(v) \right|^2 dv \\
&= \int_{\mathfrak{R}^2} \left| z_1 f(z) \right|^2 dz \int_{\mathfrak{R}^2} \left| b_1 \frac{\partial f}{\partial z_1}(z) \right|^2 dz \geq \frac{b_1^2}{4} \\
&\quad \cdot \left[\int_{\mathfrak{R}^2} \left(\frac{\partial f(z)}{\partial z_1} z_1 \overline{f(z)} + z_1 f(z) \overline{\frac{\partial f(z)}{\partial z_1}} \right) dz \right]^2 \\
&= \frac{b_1^2}{4} \left(\int_{\mathfrak{R}^2} z_1 \frac{\partial}{\partial z_1} \left[f(z) \overline{f(z)} \right] dz \right)^2 \\
&= \frac{b_1^2}{4} \left(\int_{\mathfrak{R}^2} z_1 \frac{\partial}{\partial z_1} |f(z)|^2 dz \right)^2,
\end{aligned} \tag{60}$$

where the third equality follows from the quaternion Cauchy–Schwarz inequality.

Secondly, for $i = 2$, we can take similar steps as above and have

$$\begin{aligned}
&\int_{\mathfrak{R}^2} z_2^2 |f(z)|^2 dz \int_{\mathfrak{R}^2} v_2^2 \left| L^{\mathbb{H}}_{A_1,A_2}\{f\}(v) \right|^2 dv \\
&\geq \frac{b_2^2}{4} \left(\int_{\mathfrak{R}^2} z_2 \frac{\partial}{\partial z_2} \left[\overline{f(z)} f(z) \right] dz \right)^2.
\end{aligned} \tag{61}$$

Now notice that, for $i = 1, 2$, using integration by parts, we have

$$\begin{aligned}
&\int_{\mathfrak{R}^2} z_i^2 |f(z)|^2 dz \int_{\mathfrak{R}^2} v_i^2 \left| L^{\mathbb{H}}_{A_1,A_2}\{f\}(v) \right|^2 dv \\
&\geq \frac{b_i^2}{4} \left(\int_{\mathfrak{R}^2} z_i \frac{\partial}{\partial z_i} |f(z)|^2 dz \right)^2 \\
&= \frac{b_i^2}{4} \left(\left[\int_{\mathfrak{R}} z_i |f(z)|^2 dz_l, \ (l \neq i) \right]_{z_i=-\infty}^{z_i=\infty} - \int_{\mathfrak{R}^2} |f(z)|^2 dz \right)^2 \\
&= \frac{b_i^2}{4} \left(0 - \int_{\mathfrak{R}^2} |f(z)|^2 dz \right)^2 \\
&= \frac{b_i^2}{4} F^2.
\end{aligned} \tag{62}$$

Note that the equality in equation (58) only holds for a two-dimensional Gaussian function:

$$f(z) = K_0 e^{-\left(\left(z_1^2 / 2\alpha_1 \right) + \left(z_2^2 / 2\alpha_2 \right) \right)}. \tag{63}$$

It is known that modulation and shift properties are valid for the type II QFT (compare to [40]). Moreover, equality in (58) holds if and only if f is a modulated and shifted two-dimensional Gaussian function, thus proving the theorem.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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