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Research Article

Common Fixed Points in Generalized Metric Spaces with a Graph

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The aim of this paper is to prove the existence and uniqueness of points of coincidence and common fixed points for a pair of self-mappings defined on generalized metric spaces with a graph. Our results improve and extend several recent results of metric fixed point theory.

1. Introduction

The study of common fixed points of mappings satisfying certain contractive conditions has been at the center of vigorous research activity. In 1976, Jungck [1] proved a common fixed point theorem for commuting maps such that one of them is continuous. In 1982, Sessa [2] generalized the concept of commuting maps to weakly commutating pair of self-mappings. In 1986, Jungck generalized this idea, first to compatible mappings [3] and then in 1996 to weakly compatible mappings [4]. Using the weakly compatibility, several authors established coincidence points results for various classes of mappings on metric spaces with Fatou property (see [5, 6]). In 2011, Haghi et al. showed that some coincidence point and common fixed point generalizations in fixed point theory are not real generalizations as they could easily be obtained from the corresponding fixed point theorems.

Recently, Jleli and Samet introduced a new concept of generalized metric spaces (also known as JS-metric spaces) recovering various topological spaces including standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces (see [7]).

Motivated by the ideas given in some recent works on metric space with a graph [8–21], we extend some common fixed point theorems of Banach, Chatterjea, and Kannan contractions in standard metric spaces, b-metric spaces, dislocated metric spaces, and modular spaces (see, for example, [20, 22–24]) to common fixed point theorems in generalized metric spaces with a graph. As corollaries, we obtain some results in generalized metric spaces for these contractions. Finally, some examples are given to illustrate our results.

2. Some Basic Concepts

In this section, we give some basic notations, definitions, and useful results in generalized metric spaces endowed with a graph.

Let X be a nonempty set and $D : X \times X \longrightarrow [0, +\infty]$ be a given mapping.

For every $x \in X$, let us define the set

$$\mathscr{C}(D, X, x) = \left\{ \{x_n\} \subset X : \lim_{n \longrightarrow \infty} D(x_n, x) = 0 \right\}. \quad (1)$$

Definition 1 (see [7]). We say that D is a generalized metric on X if it satisfies the following conditions:

(D_1) for every $(x, y) \in X \times X$, we have $D(x, y) = 0 \Longrightarrow x = y$;

(D_2) for every $(x, y) \in X \times X$, we have $D(x, y) = D(y, x)$;

(D_3) there exists $C > 0$ such that if $(x, y) \in X \times X$, $\{x_n\} \in \mathscr{C}(D, X, x)$, then

$$D(x, y) \leq C \limsup_{n \longrightarrow \infty} D(x_n, y). \tag{2}$$

In this case, we say that the pair (X, D) is a generalized metric space.

Obviously, if the set $\mathscr{C}(D, X, x)$ is empty for every $x \in X$, then (X, D) is a generalized metric space if and only if (D_1) and (D_2) are satisfied. A sequence $\{x_n\}$ in a generalized metric space (X, D) is said to be D-convergent to $x \in X$ if $\{x_n\} \in \mathscr{C}(D, X, x)$. Note that if the set $\mathscr{C}(D, X, x)$ is not empty for some $x \in X$, then $D(x, x) = 0$.

A sequence $\{x_n\}$ in a generalized metric space (X, D) is said to be a D-Cauchy sequence if $\lim_{m,n \longrightarrow \infty} D(x_n, x_m) = 0$. Note that, in generalized metric spaces, a sequence has at most one limit and a D-convergent sequence may not be D-Cauchy sequence. Moreover, (X, D) is said to be D-complete if every D-Cauchy sequence in X is D-convergent to some element in X.

The next example shows that a D-convergent sequence may not be a D-Cauchy sequence.

Example 2. Let $X = \mathbb{R}_+ \cup \{0, \infty\}$, and let $D : X \times X \longrightarrow [0, \infty]$ be defined as follows:

$$D(x, y) = \begin{cases} x + y & \text{if at least one of x or y is 0} \\ 1 + \dfrac{x + y}{3} & \text{otherwise.} \end{cases} \tag{3}$$

Now we check the axioms of a generalized metric space:

(D_1) $D(x, y) = 0 \Longrightarrow$ either $x + y = 0$ or $1 + (x + y)/3 = 0$. Thus, $x + y = 0$ and $x = y = 0$.

(D_2) It is clear that, for all $x, y \in X$, $D(x, y) = D(y, x)$.

(D_3) For all $x \neq 0$, we have $\mathscr{C}(D, X, x) = \varnothing$.

If $x = 0$, then we can always find a sequence $\{x_n\}$ such that $\lim_{n \longrightarrow \infty} D(x_n, x) = 0$. Taking $C \geq 3$, we have $D(0, y) = y \leq C \limsup_{n \longrightarrow \infty} D(x_n, y)$.

Now, in this structure, let us consider the sequence $\{x_n\}$ where $x_n = 1/n$ for all n in $\mathbb{N}^*$. Then,

$$\lim_{n \longrightarrow \infty} D(x_n, 0) = \lim_{n \longrightarrow \infty} \left(\frac{1}{n} + 0\right) = 0 \Longrightarrow \{x_n\} \ D\text{-converges to } 0. \tag{4}$$

But,

$$\lim_{m,n \longrightarrow \infty} D(x_n, x_m) = \lim_{m,n \longrightarrow \infty} \left(1 + \frac{1/n + 1/m}{3}\right) \neq 0. \tag{5}$$

This shows that $\{x_n\}$ is a D-convergent sequence but not a D-Cauchy sequence. Note that for any $x \in X$ such that $x \neq 0$, $D(x, x) \neq 0$.

Now, we recall some preliminaries from graph theory which are needed for the sequel. The basic concepts related to a graph may be found in any textbook on graph theory, see, for example, [25, 26]. A directed graph or digraph G is determined by a nonempty set $V(G)$ of its vertices and the set $E(G) \subset V(G) \times V(G)$ of its directed edges. Let Δ denote the diagonal of the Cartesian product $V(G) \times V(G)$. A digraph is said to be reflexive if the set $E(G)$ of its edges contains all loops; i.e., $\Delta \subset E(G)$. By G^{-1} we denote the graph obtained from G by reversing the direction of edges and by $\widetilde{G}$ we denote the undirected graph obtained from G by ignoring direction of the edges.

Throughout this paper, let (X, D) be a generalized metric space. Consider a directed graph G such that the set $V(G)$ of its vertices coincides with X, and the set $E(G)$ of its edges contains all loops. We assume that G has no parallel edges. Moreover, we will use the concept of increasing or decreasing sequences in the sense of a digraph. Therefore, the following definitions are needed.

Definition 3 (see [27]). Let G be a digraph. A sequence $\{x_n\} \in V(G)$ is said to be

(1) G-increasing, if $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$;

(2) G-decreasing, if $(x_{n+1}, x_n) \in E(G)$ for all $n \in \mathbb{N}$;

(3) G-monotone, if it is either G-increasing or G-decreasing.

We will need to assume a property introduced in [28, 29] in partially ordered sets and in [19] in metric spaces with a graph. The digraph G is said to satisfy the (JNRL) property, if for any G-increasing sequence (resp., G-decreasing sequence) $\{x_n\}$ which D-converges to some $x \in V(G)$, we have $(x_n, x) \in E(G)$ (resp., $(x, x_n) \in E(G)$) for any $n \in \mathbb{N}$. Let T and S be two self-mappings on X. The following definitions and proposition will be needed in the sequel.

Definition 4 (see [4, 30]). If there exists $x \in X$ such that $Sx = Tx = y$, then x is called a coincidence point of S and T, while y is called a point of coincidence (or coincidence value) of T and S. If $Sx = Tx = x$, then x is called a common fixed point of S and T.

The pair of mappings T and S is said to be weakly compatible if they commute at their coincidence points.

The digraph G is said to satisfy the property (P) for T and S, if x^*, y^* are points of coincidence of T and S in $V(G)$, then $(x^*, y^*) \in E(\widetilde{G})$ and $D(x^*, y^*) < \infty$.

Proposition 5 (see [30]). *Let S and T be weakly compatible self-mappings on a nonempty set X. If S and T have a unique coincidence point x, then x is the unique common fixed point of S and T.*

Now, we introduce G-Banach, G-Chatterjea, and G-Kannan S-contractions mappings in generalized metric spaces.

Definition 6. Let $T, S : X \longrightarrow X$ be two self-mappings. We say that T is

(1) G-Banach S-contraction if there exists $k \in [0, 1)$ such that for every $x, y \in V(G)$

$$(Sx, Sy) \in E(\widetilde{G}) \Longrightarrow D(Tx, Ty) \leq kD(Sx, Sy); \tag{6}$$

(2) G-Chatterjea S-contraction if there exists $k \in [0, 1/2)$ such that for every $x, y \in V(G)$

$$\begin{aligned}(Sx, Sy) \in E(\widetilde{G}) &\Longrightarrow D(Tx, Ty) \\ &\leq k(D(Tx, Sy) + D(Sx, Ty));\end{aligned} \tag{7}$$

(3) G-Kannan S-contraction if there exists $k \in [0, 1/2)$ such that for every $x, y \in V(G)$

$$\begin{aligned}(Sx, Sy) \in E(\widetilde{G}) &\Longrightarrow D(Tx, Ty) \\ &\leq k(D(Tx, Sx) + D(Sy, Ty)).\end{aligned} \tag{8}$$

The number k is called the constant of T.

3. Main Results

In this section we establish common fixed point theorems for a pair of weakly compatible self-mappings S and T such that T is a G-Banach, G-Chatterjea, or G-Kannan S-contraction in the framework of generalized metric spaces with a reflexive digraph G.

Throughout this section we assume that (X, D) is a generalized metric space with $C \geq 1$, and G is a reflexive directed graph such that $V(G) = X$, $E(G) \supseteq \Delta$ and the graph G has no parallel edges.

Let T and S be two self-mappings on X such that $T(X) \subseteq S(X)$.

If $x_0 \in X$ is arbitrary, we can choose a point x_1 in X such that $Tx_0 = Sx_1$. Continuing in this way, for a value x_n in X, we can find $x_{n+1} \in X$ such that

$$Tx_n = Sx_{n+1}, \quad n = 0, 1, 2, \ldots \tag{9}$$

By $C(T, S)$, we denote the set of all elements x_0 of X such that $(Sx_n, Sx_m) \in E(\widetilde{G})$ for $n, m = 1, 2, \ldots$. The following notation is useful in the sequel:

$$\delta(D, S, T, x_0) = \sup\{D(Sx_p, Sx_1) : p \geq 2\}. \tag{10}$$

Theorem 7. *Let (X, D) be a generalized metric space endowed with a reflexive digraph G such that $V(G) = X$ and G has no parallel edges and satisfies the (JNRL) property. Let T and S be two self-mappings on X such that T is a G-Banach S-contraction, $S(X)$ is a D-complete subspace of X, and $T(X) \subseteq S(X)$.*

*Suppose that there exists $x_0 \in C(T, S)$ such that $\delta(D, S, T, x_0) < \infty$; then T and S have a point of coincidence $x^* \in X$. Moreover, T and S have a unique point of coincidence in X if the digraph G has the property **(P)** for T and S.*

Furthermore, if T and S are weakly compatible, then T and S have a unique common fixed point in X.

Proof. Suppose that there exists x_0 in $C(T, S)$ such that $\delta(D, S, T, x_0) < \infty$.

Let m and n be in $\mathbb{N}$ such that $n \geq m \geq 2$. Since $(Sx_{n-1}, Sx_{m-1}) \in E(\widetilde{G})$ and T is a G-Banach S-contraction, then

$$\begin{aligned}D(Tx_{n-1}, Tx_{m-1}) &= D(Sx_n, Sx_m) \\ &\leq kD(Sx_{n-1}, Sx_{m-1}),\end{aligned} \tag{11}$$

and then, by induction on m, we can get

$$\begin{aligned}D(Sx_n, Sx_m) &\leq k^{m-1} D(Sx_{n-m+1}, Sx_1) \\ &\leq k^{m-1} \delta(D, S, T, x_0).\end{aligned} \tag{12}$$

Then $D(Sx_n, Sx_m) \longrightarrow 0$ when $n, m \longrightarrow \infty$, which implies that $\{Sx_n\}$ is a D-Cauchy sequence in $S(X)$.

Since $S(X)$ is a D-complete subspace of X, then the sequence $\{Sx_n\}D$-converges to some $x^* \in S(X)$. Thus there exists $a \in X$ such that $x^* = Sa$.

Let $n \geq 2$. Since G satisfies the (JNRL) property, then $(Sx_{n-1}, Sa) \in E(\widetilde{G})$ and since T is a G-Banach S-contraction, then

$$\begin{aligned}D(Sx_n, Ta) = D(Tx_{n-1}, Ta) &\leq kD(Sx_{n-1}, Sa) \\ &= kD(Sx_{n-1}, x^*).\end{aligned} \tag{13}$$

Since $\lim_{n\longrightarrow\infty} D(Sx_{n-1}, x^*) = 0$, then $\lim_{n\longrightarrow\infty} D(Sx_n, Ta) = 0$.

With the uniqueness of the limit we get $Ta = x^* = Sa$. Thus x^* is a point of coincidence of S and T (a is a coincidence point of S and T).

Suppose that there exists another point of coincidence $y^* \in S(X)$ and $b \in X$ such that $y^* = Sb = Tb$. With the property (P), we have $(Sa, Sb) \in E(\widetilde{G})$ and $D(x^*, y^*) < \infty$.

Since T is a G-Banach S-contraction, then we have $D(Sa, Sb) = D(Ta, Tb) \leq kD(Sa, Sb)$; thus $(1-k)D(Sa, Sb) \leq 0$; then $D(Sa, Sb) = 0$, proving that $x^* = Sa = Sb = y^*$. Since S and T are weakly compatible, then by Proposition 5, x^* is the unique common fixed point of S and T. □

In the next theorem, we establish a common fixed point result for G-Chatterjea S-contraction.

Theorem 8. *Let (X, D) be a generalized metric space endowed with a reflexive digraph G such that $V(G) = X$ and G has no parallel edges and satisfies the (JNRL) property. Let T and S be two self-mappings on X such that T is a G-Chatterjea S-contraction and $S(X)$ is a D-complete subspace of X and $T(X) \subseteq S(X)$.*

(1) *Suppose that there exists $x_0 \in C(T, S)$ such that $\delta(D, S, T, x_0) < \infty$; then the sequence $\{Sx_n\}$ defined by (9) D-converges to $x^* = Sa$ with $a \in X$. Moreover, if $D(Tx_0, Ta) < \infty$, then x^* is a point of coincidence of T and S in X.*

(2) *Moreover, T and S have a unique point of coincidence in X if the digraph G has the property **(P)** for T and S. Furthermore, if T and S are weakly compatible, then T and S have a unique common fixed point in X.*

Proof. (i) Suppose that there exists x_0 in $C(T, S)$ such that $\delta(D, S, T, x_0) < \infty$. Let m and n be in $\mathbb{N}$ such that $n \geq m \geq 2$. Let us prove that

$$\begin{aligned}D(Sx_n, Sx_m) \leq \Bigg(&\sum_{j=m-1}^{n+m-3} k^j \binom{j-1}{m-2} e_{n+m-1-j} \\ &+ \sum_{j=n-1}^{n+m-3} k^j \binom{j-1}{n-2} e_{n+m-1-j}\Bigg),\end{aligned} \tag{14}$$

with $e_{n+m-1-j} = D(Sx_{n+m-1-j}, Sx_1)$.

We prove statement (14) by two-dimensional induction on $p = n + m$ for every $p \geq 4$.

Observe that

$$\begin{aligned} D(Sx_2, Sx_2) &\leq k(D(Sx_1, Sx_2) + D(Sx_2, Sx_1)) \\ &\leq \sum_{j=1}^{1} k^j \binom{j-1}{0} e_{3-j} \\ &\quad + \sum_{j=1}^{1} k^j \binom{j-1}{0} e_{3-j}. \end{aligned} \tag{15}$$

Thus, inequality (14) holds for $p = 4$ with $(m, n) = (2, 2)$.

Now, assume that inequality (14) holds for any $(m', n') \in \mathbb{N}^2$ such that $n' \geq m' \geq 2$ and $n' + m' = p$, and let $(m, n) \in \mathbb{N}^2$ such that $n \geq m \geq 2$ and $n + m = p + 1$.

Since T is a G-Chatterjea S-contraction and $(Sx_{n-1}, Sx_{m-1}) \in E(\widetilde{G})$, then

$$\begin{aligned} D(Sx_n, Sx_m) &= D(Tx_{n-1}, Tx_{m-1}) \\ &\leq k(D(Sx_{n-1}, Tx_{m-1}) + D(Tx_{n-1}, Sx_{m-1})) \\ &= k(D(Sx_{n-1}, Sx_m) + D(Sx_n, Sx_{m-1})). \end{aligned} \tag{16}$$

Thus,

$$D(Sx_n, Sx_m) \leq k(D(Sx_{n-1}, Sx_m) + D(Sx_n, Sx_{m-1})). \tag{17}$$

Since $n + (m-1) = p$ and $(n-1) + m = p$, the inductive hypothesis gives

$$\begin{aligned} kD(Sx_{n-1}, Sx_m) &\leq \sum_{j=m-1}^{n+m-4} \binom{j-1}{m-2} k^{j+1} e_{n+m-2-j} \\ &\quad + \sum_{j=n-2}^{n+m-4} \binom{j-1}{n-3} k^{j+1} e_{n+m-2-j} \\ kD(Sx_n, Sx_{m-1}) &\leq \sum_{j=m-2}^{n+m-4} \binom{j-1}{m-3} k^{j+1} e_{n+m-2-j} \\ &\quad + \sum_{j=n-1}^{n+m-4} \binom{j-1}{n-2} k^{j+1} e_{n+m-2-j}. \end{aligned} \tag{18}$$

From inequality (17), we get

$$\begin{aligned} &D(Sx_n, Sx_m) \\ &\quad \leq \sum_{j=m-1}^{n+m-4} \left(\binom{j-1}{m-3} + \binom{j-1}{m-2} \right) k^{j+1} e_{n+m-2-j} \\ &\qquad + k^{m-1} e_n \\ &\qquad + \sum_{j=n-1}^{n+m-4} \left(\binom{j-1}{n-3} + \binom{j-1}{n-2} \right) k^{j+1} e_{n+m-2-j} \\ &\qquad + k^{n-1} e_m \\ &\quad \leq \sum_{j=m-1}^{n+m-4} \binom{j}{m-2} k^{j+1} e_{n+m-2-j} \\ &\qquad + \sum_{j=n-1}^{n+m-4} \binom{j}{n-2} k^{j+1} e_{n+m-2-j} + k^{m-1} e_n \\ &\qquad + k^{n-1} e_m \\ &\quad \leq \sum_{j=m}^{n+m-3} \binom{j-1}{m-2} k^j e_{n+m-1-j} \\ &\qquad + \sum_{j=n}^{n+m-3} \binom{j-1}{n-2} k^j e_{n+m-1-j} + k^{m-1} e_n + k^{n-1} e_m \\ &\quad \leq \sum_{j=m-1}^{n+m-3} \binom{j-1}{m-2} k^j e_{n+m-1-j} \\ &\qquad + \sum_{j=n-1}^{n+m-3} \binom{j-1}{n-2} k^j e_{n+m-1-j}, \end{aligned} \tag{19}$$

proving that inequality (14) holds for $(n, m) \in \mathbb{N}^*$ such that $n + m = p + 1$.

Since $e_{n+m-1-j} \leq \delta_0 = \delta(D, S, T, x_0)$, $\binom{j-1}{m-2} \leq 2^{j-1}$, and $\binom{j-1}{n-2} \leq 2^{j-1}$, then, for any $n, m \in \mathbb{N}$ such that $n \geq m \geq 2$, we have

$$\begin{aligned} D(Sx_n, Sx_m) &\leq \frac{1}{2}\delta_0 \left(\sum_{j=m-1}^{n+m-3} (2k)^j + \sum_{j=n-1}^{n+m-3} (2k)^j \right) \\ &\leq \frac{1}{2}\delta_0 \left(\frac{(2k)^{m-1}}{1-2k} + \frac{(2k)^{n-1}}{1-2k} \right) \\ &\leq \delta_0 \frac{(2k)^{m-1}}{1-2k}. \end{aligned} \tag{20}$$

Then $D(Sx_n, Sx_m) \longrightarrow 0$ when n and $m \longrightarrow \infty$, which implies that $\{Sx_n\}$ is a D-Cauchy sequence. Since $S(X)$ is a D-complete subspace of X, then the sequence $\{Sx_n\}$ D-converges to some $x^* \in S(X)$; that is, there exists $a \in X$ such that $x^* = Sa$. Since G satisfies the (JNRL) property, then $(Sx_{n-1}, Sa) \in E(\widetilde{G})$. Since T is a G-Chatterjea S-contraction, then for any $n \geq 2$ we have

$$\begin{aligned} D(Sx_n, x^*) &\leq C \limsup_{p \longrightarrow \infty} D(Sx_n, Sx_p) \\ &\leq C \limsup_{p \longrightarrow \infty} \delta_0 \frac{(2k)^{n-1}}{1-2k} = C\delta_0 \frac{(2k)^{n-1}}{1-2k} \end{aligned} \tag{21}$$

Now we prove that for any integer $n \geq 2$ we have

$$D(Sx_n, Ta) \leq k^{n-1} D(Tx_0, Ta) + \frac{C\delta_0}{1-2k} (2k)^{n-1} \tag{22}$$

For $n = 2$,

$$\begin{aligned} D(Sx_2, Ta) &= D(Tx_1, Ta) \\ &\le k(D(Sx_1, Ta) + D(Tx_1, Sa)) \\ &\le k(D(Tx_0, Ta) + D(Sx_2, x^*)) \\ &\le kD(Tx_0, Ta) + \frac{C\delta_0}{1-2k}(2k)^1. \end{aligned} \tag{23}$$

Let $n \ge 2$. Suppose that inequality (22) holds for n and prove that it holds for $n+1$.

$$\begin{aligned} D(Sx_{n+1}, Ta) &= D(Tx_n, Ta) \\ &\le k(D(Sx_n, Ta) + D(Tx_n, Sa)) \\ &\le k(D(Sx_n, Ta) + D(Sx_{n+1}, Sa)) \\ &\le k^n D(Tx_0, Ta) + k\frac{C\delta_0}{1-2k}(2k)^{n-1} \\ &\quad + k\frac{C\delta_0}{1-2k}(2k)^n \\ &\le k^n D(Tx_0, Ta) + \frac{C\delta_0}{1-2k}(2k)^n. \end{aligned} \tag{24}$$

Since $D(Tx_0, Ta) < \infty$, then $\lim_{n\longrightarrow\infty} D(Sx_n, Ta) = 0$. With the uniqueness of the limit we get $Ta = x^* = Sa$. Thus x^* is a point of coincidence of S and T (a is a coincidence point of S and T).

(ii) Assume that there exists another point of coincidence $y^* \in S(X)$ and $b \in X$ such that $y^* = Sb = Tb$, $(Sa, Sb) \in E(\widetilde{G})$, and $D(x^*, y^*) < \infty$.

Since T is a G-Chatterjea S-contraction, then we have

$$\begin{aligned} D(Sa, Sb) &= D(Ta, Tb) \\ &\le k(D(Sa, Tb) + D(Ta, Sb)) \\ &\le k(D(Sa, Sb) + D(Sa, Sb)), \end{aligned} \tag{25}$$

and thus $(1-2k)D(Sa, Sb) \le 0$, and then $D(Sa, Sb) = 0$ and $x^* = Sa = Sb = y^*$. Since S and T are weakly compatible, then, by Proposition 5, x^* is the unique common fixed point of S and T. □

In the next theorem, we establish a common fixed point result for G-Kannan S-contraction.

Theorem 9. *Let (X, D) be a generalized metric space endowed with a reflexive digraph G such that $V(G) = X$ and G has no parallel edges and satisfies the (JNRL) property. Let T and S be two self-mappings on X such that T is a G-Kannan S-contraction with constant $k \in [0, \inf\{1/2, 1/C\})$ and $S(X)$ is a D-complete subspace of X and $T(X) \subseteq S(X)$.*

(1) *Suppose that there exists $x_0 \in C(T, S)$ such that $\delta(D, S, T, x_0) < \infty$; then the sequence $\{Sx_n\}$ defined by (9) D-converges to $x^* = Sa$ with $a \in X$. Moreover, if $D(Sa, Ta) < \infty$ then x^* is a point of coincidence of T and S in X.*

(2) *Moreover, T and S have a unique point of coincidence in X if the digraph G has the property **(P)** for T and S. Furthermore, if T and S are weakly compatible, then T and S have a unique common fixed point in X.*

Proof. (i) Suppose that there exists x_0 in $C(T, S)$ such that $\delta_0 = \delta(D, S, T, x_0) < \infty$. Let m and n be in $\mathbb{N}$ such that $n \ge m \ge 2$.

Since T is a G-Kannan S-contraction and $(Sx_{n-1}, Sx_{m-1}) \in E(\widetilde{G})$, then we have

$$\begin{aligned} D(Sx_n, Sx_{n-1}) &= D(Tx_{n-1}, Tx_{n-2}) \\ &\le k(D(Sx_{n-1}, Tx_{n-1}) + D(Tx_{n-2}, Sx_{n-2})) \\ &\le k(D(Sx_{n-1}, Sx_n) + D(Sx_{n-1}, Sx_{n-2})), \end{aligned} \tag{26}$$

which implies that

$$D(Sx_n, Sx_{n-1}) \le \frac{k}{1-k} D(Sx_{n-1}, Sx_{n-2}). \tag{27}$$

Then, by induction, we can get

$$\begin{aligned} D(Sx_n, Sx_{n-1}) &\le \left(\frac{k}{1-k}\right)^{n-2} D(Sx_2, Sx_1) \\ &\le \beta^{n-2} D(Sx_2, Sx_1), \end{aligned} \tag{28}$$

where $\beta = k/(1-k)$. Now by using (8) we get

$$\begin{aligned} D(Sx_n, Sx_m) &\le k(D(Sx_{n-1}, Sx_n) + D(Sx_m, Sx_{m-1})) \\ &\le k(\beta^{n-2} + \beta^{m-2}) D(Sx_2, Sx_1). \end{aligned} \tag{29}$$

Since $D(Sx_2, Sx_1) < \delta_0$, then $D(Sx_n, Sx_m) \longrightarrow 0$ when n and $m \longrightarrow \infty$, which implies that $\{Sx_n\}$ is a D-Cauchy sequence. Since $S(X)$ is a D-complete subspace of X, then the sequence $\{Sx_n\}$ D-converges to some $x^* \in S(X)$. Thus there exists $a \in X$ such that $x^* = Sa$.

Since G satisfies the (JNRL) property, then $(Sx_{n-1}, Sa) \in E(\widetilde{G})$. Since T is a G-Kannan S-contraction, then for any $n \ge 2$ we have

$$\begin{aligned} D(Sx_n, Ta) &= D(Tx_{n-1}, Ta) \\ &\le k(D(Tx_{n-1}, Sx_{n-1}) + D(Ta, Sa)) \\ &\le k(D(Sx_n, Sx_{n-1}) + D(Ta, Sa)) \\ &\le k(\delta_0\beta^{n-2} + D(Ta, Sa)). \end{aligned} \tag{30}$$

Taking superior limit as $n \longrightarrow \infty$, we get

$$D(Sa, Ta) \le C \limsup_{n\longrightarrow\infty} D(Sx_n, Ta) \le kCD(Sa, Ta). \tag{31}$$

Thus $(1-kC)D(Sa, Ta) \le 0$. Since $D(Sa, Ta) < \infty$, then $Ta = x^* = Sa$. Thus x^* is a point of coincidence of S and T (a is a coincidence point of S and T).

(ii) Suppose that there exists another point of coincidence $y^* \in S(X)$ and $b \in X$ such that $y^* = Sb = Tb$. With the property (P), we have $(Sa, Sb) \in E(\widetilde{G})$ and $D(x^*, y^*) < \infty$.

Since T is a G-Kannan S-contraction, then we have

$$\begin{aligned} D(Sa, Sb) &= D(Ta, Tb) \\ &\leq k(D(Ta, Sa) + D(Sb, Tb)) \leq 0, \end{aligned} \tag{32}$$

and then $Sa = Sb = x^* = y^*$.

Since S and T are weakly compatible, then, by Proposition 5, x^* is the unique common fixed point of S and T. □

Example 10. Let $X = \mathbb{R}$ and define $D : X \times X \longrightarrow \mathbb{R}_+$ by $D(x, y) = |x - y|^2$ for all $x, y \in X$. Then (X, D) is a D-complete generalized metric space with the coefficient $C \geq 2$. Consider the graph G on X defined by $V(G) = X$ and

$$\begin{aligned} &E(G) \\ &= \Delta \cup \left\{\left(0, \frac{1}{10^n}\right) : n = 0, 1, \ldots\right\} \\ &\quad \cup \left\{\left(\frac{1}{10^m}, \frac{1}{10^n}\right) : m > n \text{ and } n, m = 0, 1, \ldots\right\}. \end{aligned} \tag{33}$$

The digraph G satisfies the (JNRL) property; in fact let $\{y_n\}$ be a G-monotone sequence in $V(G)$ which D-converges to some $y \in V(G)$.

(1) Let $\{y_n\}$ be a G-increasing sequence; i.e., $(y_n, y_{n+1}) \in E(G)$ for any $n \in \mathbb{N}$; then we have the following:

(a) If $y_0 = a$ such that $a \neq 0$ and $a \neq 1/10^p, \forall p \in \mathbb{N}$, then $y_n = a$ for all $n \in \mathbb{N}$.

(b) If $y_0 = 0$, then $\{y_n\}$ is a null sequence or $\{y_n\}$ is a stationary sequence.

(c) If $y_0 = 1/10^p$ with $p \in \mathbb{N}$, then $\{y_n\}$ is a stationary sequence.

In the three cases, we have $(y_n, y) \in E(G)$.

(2) Let $\{y_n\}$ be a G-decreasing sequence; i.e., $(y_{n+1}, y_n) \in E(G)$ for any $n \in \mathbb{N}$; then we have the following:

(a) If $y_0 = a$ such that $a \neq 1/10^p, \forall p \in \mathbb{N}$, then $\{y_n\}$ is a constant sequence.

(b) If $y_0 = 1/10^p$ with $p \in \mathbb{N}$, then $\{y_n\}$ is a stationary sequence or $y_n = 1/10^{p_n}$ for any $n \in \mathbb{N}$ such that $\{p_n\}$ is a increasing integer sequence and $\{y_n\}$ D-converges to 0.

In both cases, we have $(y, y_n) \in E(G)$.

Let T and S be two self-mappings on X defined by

$$Tx = \begin{cases} \dfrac{x}{2} & \text{if } x \neq \dfrac{2}{7} \\ 1 & \text{if } x = \dfrac{2}{7} \end{cases}. \tag{34}$$

and $Sx = 5x$ for all $x \in X$. Obviously, $T(X) \subseteq S(X) = X$.

Let $x, y \in X$ such that $(Sx, Sy) \in E(G)$.

(1) If $x = y$, then $Sx = Sy$ and so $(Sx, Sy) \in E(G)$. Thus $D(Tx, Ty) = 0$.

(2) If $x = 0$, $y = 1/(5.10^n)$, then $Sx = 0$, $Sy = 1/10^n$ and so $(Sx, Sy) \in E(G)$.

$$\begin{aligned} D(Tx, Ty) &= \frac{1}{10^{2n+2}} \\ D(Sx, Sy) &= \frac{1}{10^{2n}} \\ D(Sx, Ty) + D(fx, gy) &= \frac{101}{10^{2n+2}} \\ D(Tx, Sx) + D(fy, gy) &= \frac{81}{10^{2n+2}} \end{aligned} \tag{35}$$

(3) If $x = 1/(5.10^m)$, $y = 1/(5.10^n)$, then $Sx = 1/10^m$, $Sy = 1/10^n$ and so $(Sx, Sy) \in E(\widetilde{G})$.

$$\begin{aligned} D(Tx, Ty) &= \frac{10^{2n} + 10^{2m} - 2.10^{n+m}}{10^{2(n+m+1)}} \\ D(Sx, Sy) &= \frac{10^{2n} + 10^{2m} - 2.10^{n+m}}{10^{2(n+m)}} \\ &D(Sx, Ty) + D(Tx, Sy) \\ &= \frac{101\left(10^{2n} + 10^{2m}\right) - 40.10^{n+m}}{10^{2(n+m+1)}} \\ D(Tx, Sx) + D(Ty, Sy) &= \frac{81\left(10^{2n} + 10^{2m}\right)}{10^{2(n+m+1)}} \end{aligned} \tag{36}$$

In all cases we have

$$\begin{aligned} &D(Tx, Ty) \leq k_1 D(Sx, Sy) \text{ with } k_1 \in \left[\frac{1}{100}, \frac{1}{2}\right) \\ &D(Tx, Ty) \leq k_2 (D(Sx, Ty) + D(Tx, Sy)) \text{ with } k_2 \\ &\qquad \in \left[\frac{1}{101}, \frac{1}{2}\right) \\ &D(Tx, Ty) \leq k_3 (D(Tx, Sx) + D(Ty, Sy)) \text{ with } k_3 \\ &\qquad \in \left[\frac{1}{81}, \frac{1}{2}\right) \end{aligned} \tag{37}$$

proving that T is a G-Banach, G-Chatterjea, and G-Kannan S-contraction, respectively.

For $x_0 = 1/5$ we have $\delta(D, S, T, x_0) = 1/100 < \infty$ and we can verify that $x_0 \in C(T, S)$. In fact, (9) gives that $Sx_1 = T(1/5) = 1/10$ and so $Sx_2 = Tx_1 = 1/10^2$. Proceeding in this way, we get $Sx_n = 1/10^n$ for $n = 1, 2, \ldots$ and hence $(Sx_m, Sx_n) = (1/10^m, 1/10^n) \in E(G)$ for $m, n = 1, 2, \ldots$. Then $\{Sx_n\}$ D-converges to 0. Since $D(Tx_0, T0) = 1/100 < \infty$ and $D(S0, T0) = 0 < \infty$, then $0 = T0 = S0$ is a point of coincidence of T and S. Obviously G satisfies the property (P) for T and S.

Furthermore, T and S are weakly compatible. Thus, we have all the conditions of Theorems 7, 8, and 9, proving that T and S have a unique common fixed point in X which is 0.

The next remark shows that we cannot relax the weak compatibility condition in Theorem 7.

Remark 11. In Example 10, if we take $Sx = 5x - 9$ for all $x \in X$, and $x_0 = 2$, then we can easily verify that T is a G-Banach S-contraction, $x_0 \in C(T, S)$ and $T(2) = S(2) = 1$; i.e., 2 is a coincidence point of S and T. Since $TS2 \neq ST2$, then T and S are not weakly compatible. However, we can check that all other conditions of Theorem 7 are satisfied. Despite the fact that T and S have a unique point of coincidence 1, they do not have any common fixed point.

Remark 12.

(1) By taking $G = G_0$, where G_0 is the complete graph, i.e., $G_0 = (X, X \times X)$, we get corollaries for Theorems 7, 8, and 9 in the framework of generalized metric space without using graph for Banach, Chatterjea, and Kannan S-contractions.

(2) Since standard metric spaces and b-metric spaces are particular generalized metric spaces, then our results can be viewed as generalizations and extensions of corresponding results in [20, 22–24] and several other comparable results.

(3) By taking $S = I_X$ in Theorems 7, 8, and 9, we obtain versions of Banach, Chatterjea, and Kannan fixed point theorems in generalized metric spaces with a graph.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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References

[1] G. Jungck, "Commuting mappings and fixed points," *The American Mathematical Monthly*, vol. 83, no. 4, pp. 261–263, 1976.

[2] S. Sessa, "On a weak commutativity condition of mappings in fixed point considerations," *Publications de l'Institut Mathématique*, vol. 32, no. 46, pp. 149–153, 1982.

[3] G. Jungck, "Compatible mappings and common fixed points," *International Journal of Mathematics and Mathematical Sciences*, vol. 9, no. 4, pp. 771–779, 1989.

[4] G. Jungck, "Common fixed points for noncontinuous nonself maps on nonmetric spaces," *Far East Journal of Mathematical Sciences*, vol. 4, no. 2, pp. 199–215, 1996.

[5] G. Jungck and N. Hussain, "Compatible maps and invariant approximations," *Journal of Mathematical Analysis and Applications*, vol. 325, no. 2, pp. 1003–1012, 2007.

[6] G. Jungck, "Common fixed points for commuting and compatible maps on compacta," *Proceedings of the American Mathematical Society*, vol. 103, no. 3, pp. 977–983, 1988.

[7] M. Jleli and B. Samet, "A generalized metric space and related fixed point theorems," *Fixed Point Theory and Applications*, vol. 2015, no. 1, article 61, 2015.

[8] S. M. Aleomraninejad, S. Rezapour, and N. Shahzad, "Some fixed point results on a metric space with a graph," *Topology and its Applications*, vol. 159, no. 3, pp. 659–663, 2012.

[9] S. Aleomraninejad, S. Rezapour, and N. Shahzad, "Fixed point results on subgraphs of directed graphs," *Mathematical Sciences*, vol. 7, no. 1, article 41, 2013.

[10] M. R. Alfuraidan and M. A. Khamsi, "Caristi Fixed Point Theorem in Metric Spaces with a Graph," *Abstract and Applied Analysis*, vol. 2014, Article ID 303484, 5 pages, 2014.

[11] H. Alikhani and S. Rezapour, "Two endpoint results for β-shrinking and β-convergent multifunctions with application to an integral equation," *University Politehnica of Bucharest Scientific Bulletin-Series A-Applied Mathematics and Physics*, vol. 78, no. 1, pp. 71–80, 2016.

[12] J. H. Asl, B. Mohammadi, S. Rezapour, and S. M. Vaezpour, "Some fixed point results for generalized quasi-contractive multifunctions on graphs," *Filomat*, vol. 27, no. 2, pp. 311–315, 2013.

[13] I. Beg, A. R. Butt, and S. Radenović, "The contraction principle for set valued mappings on a metric space with a graph," *Computers & Mathematics with Applications*, vol. 60, no. 5, pp. 1214–1219, 2010.

[14] F. Bojor, "Fixed point of φ-contraction in metric spaces endowed with a graph," *Annals of the University of Craiova-Mathematics and Computer Science Series*, vol. 37, no. 4, pp. 85–92, 2010.

[15] F. Bojor, "Fixed points of Bianchini mappings in metric spaces endowed with a graph," *Carpathian Journal of Mathematics*, vol. 28, no. 2, pp. 207–214, 2012.

[16] F. Bojor, "Fixed points of kannan mappings in metric spaces endowed with a graph," *Analele Universitatii" Ovidius" Constanta-Seria Matematica*, vol. 20, no. 1, pp. 31–40, 2012.

[17] R. H. Haghi, S. Rezapour, and N. Shahzad, "Some fixed point generalizations are not real generalizations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 74, no. 5, pp. 1799–1803, 2011.

[18] R. H. Haghi, S. Rezapour, and N. Shahzad, "Fixed points of G-type quasi-contractions on graphs," *Abstract and Applied Analysis*, vol. 2013, Article ID 167530, 5 pages, 2013.

[19] J. Jachymski, "The contraction principle for mappings on a metric space with a graph," *Proceedings of the American Mathematical Society*, vol. 136, no. 4, pp. 1359–1373, 2008.

[20] S. K. Mohanta, "Common fixed points in b-metric spaces endowed with a graph," *Matematicki Vesnik*, vol. 68, no. 2, pp. 140–154, 2016.

[21] M. Samreen, T. Kamran, and N. Shahzad, "Some fixed point theorems in-metric space endowed with graph," *Abstract and Applied Analysis*, vol. 2013, Article ID 967132, 9 pages, 2013.

[22] V. Berinde, "Approximating common fixed points of noncommuting discontinuous weakly contractive mappings in metric spaces," *Carpathian Journal of Mathematics*, vol. 25, no. 1, pp. 13–22, 2009.

[23] V. Berinde, "Common fixed points of noncommuting discontinuous weakly contractive mappings in cone metric spaces," *Taiwanese Journal of Mathematics*, vol. 14, no. 5, pp. 1763–1776, 2010.

[24] Z. Mustafa, M. M. Jaradat, and H. M. Jaradat, "Some common fixed point results of graphs on b-metric space," *Journal of Nonlinear Sciences and Applications. JNSA*, vol. 9, no. 6, pp. 4838–4851, 2016.

[25] J. A. Bondy and U. S. R. Murty, "Graph theory," volume 244 of graduate texts in mathematics, 2008.

[26] W. D. Wallis, *A beginner's guide to graph theory*, Springer Science & Business Media, 2010.

[27] M. R. Alfuraidan, M. Bachar, and M. A. Khamsi, "Almost monotone contractions on weighted graphs," *Journal of Nonlinear Sciences and Applications. JNSA*, vol. 9, no. 8, pp. 5189–5195, 2016.

[28] K. Chaira, A. Eladraoui, and M. Kabil, "Some generalised fixed point theorems in a partially ordered space endowed with two metrics," *Advances in Fixed Point Theory*, vol. 6, no. 4, article no. 498, 2017.

[29] J. J. Nieto and R. Rodríguez-López, "Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations," *Order*, vol. 22, no. 3, pp. 223–239, 2005.

[30] M. Abbas and G. Jungck, "Common fixed point results for noncommuting mappings without continuity in cone metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 341, no. 1, pp. 416–420, 2008.