Research Article

Fixed Point and Acute Point Theorems for New Mappings in a Banach Space

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In a Hilbert space, concepts of attractive point and acute point are studied by many researchers. Moreover, these concepts extended to Banach space. In this paper, we introduce a new class of mappings on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. Moreover, we introduce some extensions of acute point and prove some acute point theorems.

1. Introduction

In [1] Takahashi and Takeuchi introduced a concept of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H , and let T be a mapping from C into H . $x \in H$ is called an attractive point of T if

$$\|x - Ty\| \leq \|x - y\| \quad (1)$$

for any $y \in C$. Let

$$\begin{aligned} A(T) \\ = \{x \in H \mid \|x - Ty\| \leq \|x - y\| \text{ for any } y \in C\}. \end{aligned} \quad (2)$$

Moreover, they proved that the Baillon type ergodic theorem [2] for generalized hybrid mappings [3] without convexity of C . A mapping T from C into H is said to be generalized hybrid if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha \|Tx - Ty\|^2 + (1 - \alpha) \|x - Ty\|^2 \\ \leq \beta \|Tx - y\|^2 + (1 - \beta) \|x - y\|^2 \end{aligned} \quad (3)$$

for any $x, y \in C$. Such a mapping is said to be (α, β) -generalized hybrid. The class of all generalized hybrid mappings is a new class of nonlinear mappings including nonexpansive mappings, nonspreading mappings [4], and hybrid

mappings [5]. A mapping T from C into H is said to be nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\| \quad (4)$$

for any $x, y \in C$; nonspreading if

$$2 \|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \quad (5)$$

for any $x, y \in C$; hybrid if

$$3 \|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2 \quad (6)$$

for any $x, y \in C$. Any nonexpansive mapping is $(1, 0)$ -generalized hybrid; any nonspreading mapping is $(2, 1)$ -generalized hybrid; any hybrid mapping is $(3/2, 1/2)$ -generalized hybrid.

Motivated these mappings, in [6] Kawasaki and Takahashi introduced a new very wider class of mappings, called widely more generalized hybrid mappings, than the class of all generalized hybrid mappings. A mapping T from C into H is widely more generalized hybrid if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta \in \mathbb{R}$ such that

$$\begin{aligned} \alpha \|Tx - Ty\|^2 + \beta \|x - Ty\|^2 + \gamma \|Tx - y\|^2 \\ + \delta \|x - y\|^2 + \varepsilon \|x - Tx\|^2 + \zeta \|y - Ty\|^2 \\ + \eta \|(x - Tx) - (y - Ty)\|^2 \leq 0 \end{aligned} \quad (7)$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta)$ -widely more generalized hybrid. This class includes the class of all generalized hybrid mappings and also the class of all k -pseudocontractions [7] for $k \in [0, 1]$. A mapping T from C into H is called a k -pseudocontraction if

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2 \quad (8)$$

for any $x, y \in C$. Any (α, β) -generalized hybrid mapping is $(\alpha, 1 - \alpha, -\beta, \beta - 1, 0, 0, 0)$ -widely more generalized hybrid; any k -pseudocontraction is $(1, 0, 0, -1, 0, 0, -k)$ -widely more generalized hybrid. Moreover, they proved some fixed point theorems and some ergodic theorems; for instance, see [6, 8–10].

There are some studies on Banach space related to these results. In [11] Takahashi, Wong and Yao introduced the generalized nonspreading mapping and the skew-generalized nonspreading mapping in a Banach space. Let E be a smooth Banach space and let C be a nonempty subset of E . A mapping T from C into E is said to be generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ & \leq \varepsilon(\phi(Ty, Tx) - \phi(Ty, x)) \\ & \quad + \zeta(\phi(y, Tx) - \phi(y, x)) \end{aligned} \quad (9)$$

for any $x, y \in C$, where J is the duality mapping on E and

$$\phi(u, v) = \|u\|^2 - 2\langle u, Jv \rangle + \|v\|^2. \quad (10)$$

Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -generalized nonspreading. A mapping T from C into E is said to be skew-generalized nonspreading if there exist $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha\phi(Tx, Ty) + \beta\phi(x, Ty) + \gamma\phi(Tx, y) + \delta\phi(x, y) \\ & \leq \varepsilon(\phi(Ty, Tx) - \phi(y, Tx)) \\ & \quad + \zeta(\phi(Ty, x) - \phi(y, x)) \end{aligned} \quad (11)$$

for any $x, y \in C$. Such a mapping is said to be $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -skew-generalized nonspreading. These classes include the class of generalized hybrid mappings in a Hilbert space; however, it does not include the class of widely more generalized hybrid mappings. Moreover, they introduced some extensions of attractive point and proved some attractive point theorems. $x \in E$ is an attractive point of T if

$$\phi(x, Ty) \leq \phi(x, y) \quad (12)$$

for any $y \in C$; $x \in E$ is a skew-attractive point of T if

$$\phi(Ty, x) \leq \phi(y, x) \quad (13)$$

for any $y \in C$. Let

$$\begin{aligned} A(T) &= \{x \in E \mid \phi(x, Ty) \leq \phi(x, y) \text{ for any } y \in C\}; \\ B(T) &= \{x \in E \mid \phi(Ty, x) \leq \phi(y, x) \text{ for any } y \in C\}. \end{aligned} \quad (14)$$

They also proved the following.

Theorem 1. Let E be a reflexive and smooth Banach space, let C be a nonempty subset of E , and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -generalized nonspreading mapping from C into itself satisfying $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta > 0$. Then T has an attractive point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Additionally, if E is strictly convex and C is closed and convex, then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Theorem 2. Let E be a strictly convex, reflexive, and smooth Banach space, let C be a nonempty subset of E , and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -skew-generalized nonspreading mapping from C into itself satisfying $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma > 0$. Then T has a skew-attractive point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Additionally, if E is strictly convex and $J(C)$ is closed and convex, then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

On the other hand, in [12] Atsushiba, Iemoto, Kubota, and Takeuchi introduced a concept of acute point as an extension of attractive point in a Hilbert space. Let H be a real Hilbert space, let C be a nonempty subset of H , and let T be a mapping from C into H and $k \in [0, 1]$. $x \in H$ is called a k -acute point of T if

$$\|x - Ty\|^2 \leq \|x - y\|^2 + k\|y - Ty\|^2 \quad (15)$$

for any $y \in C$. Let

$$\begin{aligned} \mathcal{A}_k(T) &= \{x \in H \mid \|x - Ty\|^2 \leq \|x - y\|^2 \\ & \quad + k\|y - Ty\|^2 \text{ for any } y \in C\}. \end{aligned} \quad (16)$$

Moreover, using a concept of acute point, they proved convergence theorems without convexity of C .

Motivated by these results, in this paper, we introduce a new class of mappings on Banach space corresponding to the class of all widely more generalized hybrid mappings on Hilbert space. Moreover, we introduce some extensions of acute point and prove some acute point theorems. Moreover, in the next paper [13], we show some mean convergence theorems for the new class of mappings.

2. Preliminaries

We know that the following hold; for instance, see [14].

(T1) Let E be a normed space, let C be a nonempty, closed, and convex subset in E , and let D be a nonempty, compact, and convex subset in E . Suppose that $C \cap D = \emptyset$. Then

$$d(C, D) \stackrel{\text{def}}{=} \inf \{\|x - y\| \mid x \in C, y \in D\} > 0. \quad (17)$$

(T2) Let E be a normed space and let C and D be nonempty and convex subsets with $d(C, D) > 0$. Then there

exists $x^* \in E^*$, where E^* is the topological dual space of E , such that $\|x^*\| = 1$ and

$$\inf \{ \langle x, x^* \rangle \mid x \in C \} = d(C, D) + \sup \{ \langle y, x^* \rangle \mid y \in D \}. \quad (18)$$

(T3) Let E be a Banach space, let E^* be the topological dual space of E , and let J be the duality mapping on E defined by

$$J(x) = \{x^* \in E^* \mid \|x\|^2 = \langle x, x^* \rangle = \|x^*\|^2\} \quad (19)$$

for any $x \in E$. Then E is strictly convex if and only if J is injective; that is, $x \neq y$ implies $J(x) \cap J(y) = \emptyset$.

(T4) Let E be a Banach space, let E^* be the topological dual space of E , and let J be the duality mapping on E . Then E is reflexive if and only if J is surjective; that is, $\bigcup_{x \in E} J(x) = E^*$.

(T5) Let E be a Banach space and let J be the duality mapping on E . Then E is smooth if and only if J is single-valued.

(T6) Let E be a Banach space and let J be the duality mapping on E . If J is single-valued, then J is norm-to-weak* continuous.

(T7) Let E be a Banach space and let J be the duality mapping on E . Then E is strictly convex if and only if

$$1 - \langle x, y^* \rangle > 0 \quad (20)$$

for any $x, y \in E$ with $x \neq y$ and $\|x\| = \|y\| = 1$ and for any $y^* \in J(y)$.

(T8) Let E be a Banach space and let E^* be the topological dual space of E . Then E is reflexive if and only if E^* is reflexive.

(T9) Let E be a Banach space and let E^* be the topological dual space of E . If E^* is strictly convex, then E is smooth. Conversely, E is reflexive and smooth, then E^* is strictly convex.

(T10) Let E be a Banach space and let E^* be the topological dual space of E . If E^* is smooth, then E is strictly convex. Conversely, E is reflexive and strictly convex, then E^* is smooth.

Let E be a smooth Banach space, let J be the duality mapping on E , and let ϕ be the mapping from $E \times E$ into $[0, \infty)$ defined by

$$\phi(x, y) = \|x\|^2 - 2 \langle x, Jy \rangle + \|y\|^2 \quad (21)$$

for any $x, y \in E$. Since by (T5) J is single-valued, ϕ is well-defined. It is obvious that $x = y$ implies $\phi(x, y) = 0$. Conversely, by (T7)

(T11) If E is also strictly convex, then $\phi(x, y) = 0$ implies $x = y$.

Let E be a strictly convex and smooth Banach space. By (T3) an (T5) J is a bijective mapping from E onto $J(E)$. In particular, if E is also reflexive, then by (T4) J is a bijective mapping from E onto E^* . Suppose that E is strictly convex, reflexive, and smooth. Let ϕ_* be the mapping from $E^* \times E^*$ into $[0, \infty)$ defined by

$$\phi_*(x^*, y^*) = \|x^*\|^2 - 2 \langle J^{-1}y^*, x^* \rangle + \|y^*\|^2 \quad (22)$$

for any $x^*, y^* \in E^*$. Then

$$\phi_*(x^*, y^*) = \phi(J^{-1}y^*, J^{-1}x^*) \quad (23)$$

holds. Therefore

(T11)* $\phi_*(x^*, y^*) = 0$ if and only if $x^* = y^*$.

Let ℓ^∞ be the Banach space consists of all bounded sequences and $\mu \in (\ell^\infty)^*$. Sometimes we denote by $\mu_n x_n$ the value $\mu(\{x_n\}_{n=1}^\infty)$. If $\mu \in (\ell^\infty)^*$ satisfies $\mu(e) = \|\mu\| = 1$, where $e = \{1\}_{n=1}^\infty$, then μ is called a mean. If a mean μ satisfies $\mu_n x_{n+1} = \mu_n x_n$, then μ is called a Banach limit. We know that there exists some Banach limits. If $\{x_n\}_{n=1}^\infty \in \ell^\infty$ and μ is a mean, then the following holds:

$$(T12) \inf\{x_n \mid n \in \mathbb{N}\} \leq \mu_n x_n \leq \sup\{x_n \mid n \in \mathbb{N}\}.$$

The following lemma is introduced in [11]. For completeness we write its proof.

Lemma 3. Let E be a Banach space, let E^* be the dual space of E , let $\{x_n \mid n \in \mathbb{N}\}$ be a bounded sequence in E , and let μ be a mean on ℓ^∞ . Then there exists a unique $z_0 \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$ such that $\mu_n \langle x_n, x^* \rangle = \langle z_0, x^* \rangle$ for any $x^* \in E^*$.

Proof. Define a mapping g from E^* into $\mathbb{R}$ by $g(x^*) = \mu_n \langle x_n, x^* \rangle$ for any $x^* \in E^*$. Then g is linear and bounded clearly and hence $g \in E^{**}$. Since E is reflexive, we can put $z_0 = g$.

Lastly, we show that $z_0 \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$. Assume that $z_0 \notin \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$. Since $\overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$ is nonempty, close, and convex and $\{z_0\}$ is nonempty, compact, and convex, by (T1) and (T2) there exists $x^* \in E^*$ such that $\|x^*\| = 1$ and

$$\begin{aligned} & \inf \{ \langle x, x^* \rangle \mid x \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\} \} \\ &= d(\overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}, \{z_0\}) \\ &+ \sup \{ \langle y, x^* \rangle \mid y \in \{z_0\} \} > \langle z_0, x^* \rangle. \end{aligned} \quad (24)$$

On the other hand, by (T12) we obtain

$$\begin{aligned} & \inf \{ \langle x, x^* \rangle \mid x \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\} \} \\ & \leq \inf \{ \langle x_n, x^* \rangle \mid n \in \mathbb{N} \} \leq \mu_n \langle x_n, x^* \rangle \\ &= \langle z_0, x^* \rangle. \end{aligned} \quad (25)$$

It is a contradiction. Therefore $z_0 \in \overline{\text{co}}\{x_n \mid n \in \mathbb{N}\}$. $\square$

3. Fixed Point Theorems

Let E be a smooth Banach space and let C be a nonempty subset of E . A mapping T from C into E is called a generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned} & \alpha_1 \phi(Tx, Ty) + \alpha_2 \phi(Ty, Tx) + \beta_1 \phi(x, Ty) \\ & + \beta_2 \phi(Ty, x) + \gamma_1 \phi(Tx, y) + \gamma_2 \phi(y, Tx) \\ & + \delta_1 \phi(x, y) + \delta_2 \phi(y, x) + \varepsilon_1 \phi(Tx, x) \\ & + \varepsilon_2 \phi(x, Tx) + \zeta_1 \phi(y, Ty) + \zeta_2 \phi(Ty, y) \leq 0 \end{aligned} \quad (26)$$

for any $x, y \in C$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -generalized pseudocontraction.

If all parameters are negative, of course, all mappings satisfy the above inequality. However, in the following argument, such cases are properly excluded.

Theorem 4. *Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty, closed, and convex subset of E and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -generalized pseudocontraction from C into itself. Then the following hold.*

- (1) Suppose that T satisfies $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0, \alpha_2 + \beta_2 \geq 0, \gamma_2 + \delta_2 \geq 0, \varepsilon_1 \geq 0, \varepsilon_2 \geq 0$, and one of the following

(1-1) $\alpha_1 + \beta_1 + \zeta_1 > 0$ and $\zeta_2 \geq 0$;

(1-2) $\alpha_1 + \beta_1 + \zeta_1 \geq 0$ and $\zeta_2 > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (2) Suppose that T satisfies $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0, \alpha_1 + \gamma_1 \geq 0, \beta_1 + \delta_1 \geq 0, \zeta_1 \geq 0, \zeta_2 \geq 0$, and one of the following:

(2-1) $\alpha_2 + \gamma_2 + \varepsilon_2 > 0$ and $\varepsilon_1 \geq 0$;

(2-2) $\alpha_2 + \gamma_2 + \varepsilon_2 \geq 0$ and $\varepsilon_1 > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (3) Suppose that T satisfies $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 \geq 0, \beta_1 + \gamma_2 + \delta_1 + \delta_2 \geq 0, \varepsilon_2 + \zeta_1 \geq 0$, and one of the following:

(3-1) $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 + \varepsilon_2 + \zeta_1 > 0$ and $\varepsilon_1 + \zeta_2 \geq 0$;

(3-2) $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 + \varepsilon_2 + \zeta_1 \geq 0$ and $\varepsilon_1 + \zeta_2 > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 > 0$, then the fixed point is unique.

Proof. If T has a fixed point z , then $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Moreover, we suppose that (1) holds. Since

$$\phi(u, v) = \phi(u, w) + \phi(w, v) + 2 \langle u - w, Jw - Jv \rangle \quad (27)$$

for any $u, v, w \in E$, we obtain

$$\begin{aligned} & \alpha_1 \phi(Tx, Ty) + \alpha_2 \phi(Ty, Tx) + \beta_1 \phi(x, Ty) \\ & + \beta_2 \phi(Ty, x) + \gamma_1 \phi(Tx, y) + \gamma_2 \phi(y, Tx) \\ & + \delta_1 \phi(x, y) + \delta_2 \phi(y, x) + \varepsilon_1 \phi(Tx, x) \\ & + \varepsilon_2 \phi(x, Tx) + \zeta_1 \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\ & = \alpha_1 \phi(Tx, Ty) + \alpha_2 \phi(Ty, Tx) - \alpha_1 \phi(x, Ty) \\ & + (\alpha_1 + \beta_1) \\ & \cdot (\phi(x, y) + \phi(y, Ty) + 2 \langle x - y, Jy - JTy \rangle) \\ & + \beta_2 \phi(Ty, x) + \gamma_1 \phi(Tx, y) + \gamma_2 \phi(y, Tx) \\ & + \delta_1 \phi(x, y) + \delta_2 \phi(y, x) + \varepsilon_1 \phi(Tx, x) \\ & + \varepsilon_2 \phi(x, Tx) + \zeta_1 \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\ & = \alpha_1 \phi(Tx, Ty) + \alpha_2 \phi(Ty, Tx) - \alpha_1 \phi(x, Ty) \\ & + \beta_2 \phi(Ty, x) + \gamma_1 \phi(Tx, y) + \gamma_2 \phi(y, Tx) \\ & + (\alpha_1 + \beta_1 + \delta_1) \phi(x, y) + \delta_2 \phi(y, x) + \varepsilon_1 \phi(Tx, x) \\ & + \varepsilon_2 \phi(x, Tx) + (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) \\ & + \zeta_2 \phi(Ty, y) + 2(\alpha_1 + \beta_1) \langle x - y, Jy - JTy \rangle. \end{aligned} \quad (28)$$

Since $\alpha_1 + \beta_1 + \delta_1 \geq -\gamma_1, \beta_2 \geq -\alpha_2, \delta_2 \geq -\gamma_2, \varepsilon_1 \geq 0$, and $\varepsilon_2 \geq 0$, we obtain

$$\begin{aligned} & \alpha_1 \phi(Tx, Ty) + \alpha_2 \phi(Ty, Tx) - \alpha_1 \phi(x, Ty) \\ & + \beta_2 \phi(Ty, x) + \gamma_1 \phi(Tx, y) + \gamma_2 \phi(y, Tx) \\ & + (\alpha_1 + \beta_1 + \delta_1) \phi(x, y) + \delta_2 \phi(y, x) \\ & + \varepsilon_1 \phi(Tx, x) + \varepsilon_2 \phi(x, Tx) \\ & + (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\ & + 2(\alpha_1 + \beta_1) \langle x - y, Jy - JTy \rangle \\ & \geq \alpha_1 (\phi(Tx, Ty) - \phi(x, Ty)) \\ & + \alpha_2 (\phi(Ty, Tx) - \phi(Ty, x)) \\ & + \gamma_1 (\phi(Tx, y) - \phi(x, y)) \\ & + \gamma_2 (\phi(y, Tx) - \phi(y, x)) \\ & + (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\ & + 2(\alpha_1 + \beta_1) \langle x - y, Jy - JTy \rangle. \end{aligned} \quad (29)$$

Therefore

$$\begin{aligned}
 & \alpha_1 (\phi(Tx, Ty) - \phi(x, Ty)) \\
 & + \alpha_2 (\phi(Ty, Tx) - \phi(Ty, x)) \\
 & + \gamma_1 (\phi(Tx, y) - \phi(x, y)) \\
 & + \gamma_2 (\phi(y, Tx) - \phi(y, x)) \\
 & + (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\
 & + 2(\alpha_1 + \beta_1) \langle x - y, Jy - JTy \rangle \leq 0.
 \end{aligned} \tag{30}$$

Replacing x by $T^n z$, we obtain

$$\begin{aligned}
 & \alpha_1 (\phi(T^{n+1} z, Ty) - \phi(T^n z, Ty)) \\
 & + \alpha_2 (\phi(Ty, T^{n+1} z) - \phi(Ty, T^n z)) \\
 & + \gamma_1 (\phi(T^{n+1} z, y) - \phi(T^n z, y)) \\
 & + \gamma_2 (\phi(y, T^{n+1} z) - \phi(y, T^n z)) \\
 & + (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\
 & + 2(\alpha_1 + \beta_1) \langle T^n z - y, Jy - JTy \rangle \leq 0.
 \end{aligned} \tag{31}$$

Applying a Banach limit μ to both sides of this inequality, we obtain

$$\begin{aligned}
 & (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\
 & + 2(\alpha_1 + \beta_1) \mu_n \langle T^n z - y, Jy - JTy \rangle \leq 0.
 \end{aligned} \tag{32}$$

By Lemma 3 there exists a unique $z_0 \in \overline{\text{co}}\{T^n z \mid n \in \mathbb{N}\} \subset C$ such that

$$\mu_n \langle T^n z - y, Jy - JTy \rangle = \langle z_0 - y, Jy - JTy \rangle. \tag{33}$$

Therefore we obtain

$$\begin{aligned}
 & (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\
 & + 2(\alpha_1 + \beta_1) \langle z_0 - y, Jy - JTy \rangle \leq 0.
 \end{aligned} \tag{34}$$

for any $y \in C$. Putting $y = z_0$, we obtain

$$(\alpha_1 + \beta_1 + \zeta_1) \phi(z_0, Tz_0) + \zeta_2 \phi(Tz_0, z_0) \leq 0. \tag{35}$$

In the case of (1-1), since $\alpha_1 + \beta_1 + \zeta_1 > 0$, $\zeta_2 \geq 0$ and E is strictly convex, by (T11) we obtain $z_0 = Tz_0$. In the case of (1-2), since $\alpha_1 + \beta_1 + \zeta_1 \geq 0$, $\zeta_2 > 0$, and E is strictly convex, by (T11), we obtain $z_0 = Tz_0$.

In the case of (2), changing the variables x and y in (26), we obtain

$$\begin{aligned}
 & \alpha_2 \phi(Tx, Ty) + \alpha_1 \phi(Ty, Tx) + \gamma_2 \phi(x, Ty) \\
 & + \gamma_1 \phi(Ty, x) + \beta_2 \phi(Tx, y) + \beta_1 \phi(y, Tx) \\
 & + \delta_2 \phi(x, y) + \delta_1 \phi(y, x) + \zeta_2 \phi(Tx, x) \\
 & + \zeta_1 \phi(x, Tx) + \varepsilon_2 \phi(y, Ty) + \varepsilon_1 \phi(Ty, y) \leq 0.
 \end{aligned} \tag{36}$$

Suppose that (2) holds. Using the result in the case of (1), we can show that T has a fixed point.

In the case of (3), adding (26) and (36), we obtain

$$\begin{aligned}
 & (\alpha_1 + \alpha_2) \phi(Tx, Ty) + (\alpha_1 + \alpha_2) \phi(Ty, Tx) \\
 & + (\beta_1 + \gamma_2) \phi(x, Ty) + (\beta_2 + \gamma_1) \phi(Ty, x) \\
 & + (\beta_2 + \gamma_1) \phi(Tx, y) + (\beta_1 + \gamma_2) \phi(y, Tx) \\
 & + (\delta_1 + \delta_2) \phi(x, y) + (\delta_1 + \delta_2) \phi(y, x) \\
 & + (\varepsilon_1 + \zeta_2) \phi(Tx, x) + (\varepsilon_2 + \zeta_1) \phi(x, Tx) \\
 & + (\varepsilon_2 + \zeta_1) \phi(y, Ty) + (\varepsilon_1 + \zeta_2) \phi(Ty, y) \leq 0
 \end{aligned} \tag{37}$$

Suppose that (3) holds. Using the result in the case of (1), we can show that T has a fixed point.

Lastly, suppose that $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$ in the case of (1). Let z_1 and z_2 be fixed points of T . Then

$$\begin{aligned}
 0 & \geq \alpha_1 \phi(Tz_1, Tz_2) + \alpha_2 \phi(Tz_2, Tz_1) + \beta_1 \phi(z_1, Tz_2) \\
 & + \beta_2 \phi(Tz_2, z_1) + \gamma_1 \phi(Tz_1, z_2) + \gamma_2 \phi(z_2, Tz_1) \\
 & + \delta_1 \phi(z_1, z_2) + \delta_2 \phi(z_2, z_1) + \varepsilon_1 \phi(Tz_1, z_1) \\
 & + \varepsilon_2 \phi(z_1, Tz_1) + \zeta_1 \phi(z_2, Tz_2) + \zeta_2 \phi(Tz_2, z_2) \\
 & = (\alpha_1 + \beta_1 + \gamma_1 + \delta_1) \phi(z_1, z_2) \\
 & + (\alpha_2 + \beta_2 + \gamma_2 + \delta_2) \phi(z_2, z_1).
 \end{aligned} \tag{38}$$

By (T11) we obtain $z_1 = z_2$ holds. The cases of (2) and (3) are similar. $\square$

Let E^* be the dual space of a strictly convex, reflexive, and smooth Banach space E and let C^* be a nonempty subset of E^* . A mapping T^* from C^* into E^* is called a $*$ -generalized pseudocontraction if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2 \in \mathbb{R}$ such that

$$\begin{aligned}
 & \alpha_1 \phi_*(T^* x^*, T^* y^*) + \alpha_2 \phi_*(T^* y^*, T^* x^*) \\
 & + \beta_1 \phi_*(x^*, T^* y^*) + \beta_2 \phi_*(T^* y^*, x^*) \\
 & + \gamma_1 \phi_*(T^* x^*, y^*) + \gamma_2 \phi_*(y^*, T^* x^*) \\
 & + \delta_1 \phi_*(x^*, y^*) + \delta_2 \phi_*(y^*, x^*) \\
 & + \varepsilon_1 \phi_*(T^* x^*, x^*) + \varepsilon_2 \phi_*(x^*, T^* x^*) \\
 & + \zeta_1 \phi_*(y^*, T^* y^*) + \zeta_2 \phi_*(T^* y^*, y^*) \leq 0
 \end{aligned} \tag{39}$$

for any $x^*, y^* \in C^*$. Such a mapping is called an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ - $*$ -generalized pseudocontraction.

Theorem 5. Let E^* be the dual space of a strictly convex, reflexive, and smooth Banach space E , let C^* be a nonempty, closed, and convex subset of E^* , and let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ - $*$ -generalized pseudocontraction from C^* into itself. Then the following hold.

- (1) Suppose that T^* satisfies $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \beta_2 \geq 0$, $\gamma_2 + \delta_2 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, and one of the following:

$$(1-1) \alpha_1 + \beta_1 + \zeta_1 > 0 \text{ and } \zeta_2 \geq 0;$$

$$(1-2) \alpha_1 + \beta_1 + \zeta_1 \geq 0 \text{ and } \zeta_2 > 0.$$

Then T^* has a fixed point if and only if there exists $z^* \in C^*$ such that $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (2) Suppose that T^* satisfies $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \gamma_1 \geq 0$, $\beta_1 + \delta_1 \geq 0$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, and one of the following:

$$(2-1) \alpha_2 + \gamma_2 + \varepsilon_2 > 0 \text{ and } \varepsilon_1 \geq 0;$$

$$(2-2) \alpha_2 + \gamma_2 + \varepsilon_2 \geq 0 \text{ and } \varepsilon_1 > 0.$$

Then T^* has a fixed point if and only if there exists $z^* \in C^*$ such that $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (3) Suppose that T^* satisfies $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 \geq 0$, $\beta_1 + \gamma_2 + \delta_1 + \delta_2 \geq 0$, $\varepsilon_2 + \zeta_1 \geq 0$, and one of the following:

$$(3-1) \alpha_1 + \alpha_2 + \beta_1 + \gamma_2 + \varepsilon_2 + \zeta_1 > 0 \text{ and } \varepsilon_1 + \zeta_2 \geq 0;$$

$$(3-2) \alpha_1 + \alpha_2 + \beta_1 + \gamma_2 + \varepsilon_2 + \zeta_1 \geq 0 \text{ and } \varepsilon_1 + \zeta_2 > 0.$$

Then T^* has a fixed point if and only if there exists $z^* \in C^*$ such that $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 > 0$, then the fixed point is unique.

Proof. If T^* has a fixed point z^* , then $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z^* \in C^*$ such that $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Moreover, we suppose that (1) holds. From (23) and (27) we obtain

$$\begin{aligned} \phi_*(u^*, v^*) &= \phi_*(u^*, w^*) + \phi_*(w^*, v^*) \\ &\quad + 2 \langle J^{-1}w^* - J^{-1}v^*, u^* - w^* \rangle \end{aligned} \quad (40)$$

for any $u^*, v^*, w^* \in E^*$. Similarly to the proof of Theorem 4 we obtain

$$\begin{aligned} &\alpha_1 (\phi_*(T^* x^*, T^* y^*) - \phi_*(x^*, T^* y^*)) \\ &\quad + \alpha_2 (\phi_*(T^* y^*, T^* x^*) - \phi_*(T^* y^*, x^*)) \\ &\quad + \gamma_1 (\phi_*(T^* x^*, y^*) - \phi_*(x^*, y^*)) \\ &\quad + \gamma_2 (\phi_*(y^*, T^* x^*) - \phi_*(y^*, x^*)) \\ &\quad + (\alpha_1 + \beta_1 + \zeta_1) \phi_*(y^*, T^* y^*) \\ &\quad + \zeta_2 \phi_*(T^* y^*, y^*) \\ &\quad + 2(\alpha_1 + \beta_1) \langle J^{-1}y^* - J^{-1}T^* y^*, x^* - y^* \rangle \leq 0. \end{aligned} \quad (41)$$

Replacing x^* by $(T^*)^n z^*$, we obtain

$$\begin{aligned} &\alpha_1 (\phi_*((T^*)^{n+1} z^*, T^* y^*) - \phi_*((T^*)^n z^*, T^* y^*)) \\ &\quad + \alpha_2 (\phi_*(T^* y^*, (T^*)^{n+1} z^*) \\ &\quad - \phi_*(T^* y^*, (T^*)^n z^*)) + \gamma_1 (\phi_*((T^*)^{n+1} z^*, y^*) \\ &\quad - \phi_*((T^*)^n z^*, y^*)) + \gamma_2 (\phi_*(y^*, (T^*)^{n+1} z^*) \\ &\quad - \phi_*(y^*, (T^*)^n z^*)) + (\alpha_1 + \beta_1 + \zeta_1) \\ &\quad \cdot \phi_*(y^*, T^* y^*) + \zeta_2 \phi_*(T^* y^*, y^*) + 2(\alpha_1 + \beta_1) \\ &\quad \cdot \langle J^{-1}y^* - J^{-1}T^* y^*, (T^*)^n z^* - y^* \rangle \leq 0. \end{aligned} \quad (42)$$

Applying a Banach limit μ to both sides of this inequality, we obtain

$$\begin{aligned} &(\alpha_1 + \beta_1 + \zeta_1) \phi_*(y^*, T^* y^*) + \zeta_2 \phi_*(T^* y^*, y^*) \\ &\quad + 2(\alpha_1 + \beta_1) \mu_n \langle J^{-1}y^* - J^{-1}T^* y^*, (T^*)^n z^* - y^* \rangle \\ &\leq 0. \end{aligned} \quad (43)$$

By Lemma 3 there exists a unique $z_0^* \in \overline{\text{co}}\{(T^*)^n z^* \mid n \in \mathbb{N}\} \subset C^*$ such that

$$\begin{aligned} &\mu_n \langle J^{-1}y^* - J^{-1}T^* y^*, (T^*)^n z^* - y^* \rangle \\ &= \langle J^{-1}y^* - J^{-1}T^* y^*, z_0^* - y^* \rangle. \end{aligned} \quad (44)$$

Therefore we obtain

$$\begin{aligned} &(\alpha_1 + \beta_1 + \zeta_1) \phi_*(y^*, T^* y^*) + \zeta_2 \phi_*(T^* y^*, y^*) \\ &\quad + 2(\alpha_1 + \beta_1) \langle J^{-1}y^* - J^{-1}T^* y^*, z_0^* - y^* \rangle \leq 0. \end{aligned} \quad (45)$$

for any $y^* \in C^*$. Putting $y^* = z_0^*$, we obtain

$$(\alpha_1 + \beta_1 + \zeta_1) \phi_*(z_0^*, T^* z_0^*) + \zeta_2 \phi_*(T^* z_0^*, z_0^*) \leq 0. \quad (46)$$

In the case of (1-1), since $\alpha_1 + \beta_1 + \zeta_1 > 0$, $\zeta_2 \geq 0$, and E^* is strictly convex, by $(T11)^*$ we obtain $z_0^* = T^* z_0^*$. In the case of (1-2), since $\alpha_1 + \beta_1 + \zeta_1 \geq 0$, $\zeta_2 > 0$, and E^* is strictly convex, by $(T11)^*$ we obtain $z_0^* = T^* z_0^*$.

In the cases of (2) and (3) we can show similarly. $\square$

By Theorem 5 we obtain the following.

Theorem 6. Let E be a strictly convex, reflexive, and smooth Banach space, let C be a nonempty subset of E satisfying $J(C)$ is closed and convex, and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -generalized pseudocontraction from C into itself. Then the following hold.

- (1) Suppose that T satisfies $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \beta_1 \geq 0$, $\gamma_1 + \delta_1 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, and one of the following:

$$(1-1) \alpha_2 + \beta_2 + \zeta_2 > 0 \text{ and } \zeta_1 \geq 0;$$

$$(1-2) \alpha_2 + \beta_2 + \zeta_2 \geq 0 \text{ and } \zeta_1 > 0.$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (2) Suppose that T satisfies $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \gamma_2 \geq 0$, $\beta_2 + \delta_2 \geq 0$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, and one of the following:

(2-1) $\alpha_1 + \gamma_1 + \varepsilon_1 > 0$ and $\varepsilon_2 \geq 0$;

(2-2) $\alpha_1 + \gamma_1 + \varepsilon_1 \geq 0$ and $\varepsilon_2 > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (3) Suppose that T satisfies $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 \geq 0$, $\beta_2 + \gamma_1 + \delta_1 + \delta_2 \geq 0$, $\varepsilon_1 + \zeta_2 \geq 0$, and one of the following:

(3-1) $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 + \varepsilon_1 + \zeta_2 > 0$ and $\varepsilon_2 + \zeta_1 \geq 0$;

(3-2) $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 + \varepsilon_1 + \zeta_2 \geq 0$ and $\varepsilon_2 + \zeta_1 > 0$.

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 > 0$, then the fixed point is unique.

Proof. If T has a fixed point z , then $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Since E is strictly convex, reflexive, and smooth, by (T8), (T9), and (T10) E^* is also so. By assumption $J(C)$ is nonempty, close, and convex. Let $T^* = J T J^{-1}$. Then T^* is a mapping from $J(C)$ into itself. Putting $z^* = Jz$, since $(T^*)^n z^* = J T^n z$, $\|(T^*)^n z^*\| = \|J T^n z\| = \|T^n z\|$, and hence $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Putting $x^* = Jx$ and $y^* = Jy$, (26) is equivalent to

$$\begin{aligned} & \alpha_2 \phi_*(T^* x^*, T^* y^*) + \alpha_1 \phi_*(T^* y^*, T^* x^*) \\ & + \beta_2 \phi_*(x^*, T^* y^*) + \beta_1 \phi_*(T^* y^*, x^*) \\ & + \gamma_2 \phi_*(T^* x^*, y^*) + \gamma_1 \phi_*(y^*, T^* x^*) \\ & + \delta_2 \phi_*(x^*, y^*) + \delta_1 \phi_*(y^*, x^*) \\ & + \varepsilon_2 \phi_*(T^* x^*, x^*) + \varepsilon_1 \phi_*(x^*, T^* x^*) \\ & + \zeta_2 \phi_*(y^*, T^* y^*) + \zeta_1 \phi_*(T^* y^*, y^*) \leq 0 \end{aligned} \quad (47)$$

from (23). Therefore T^* is an $(\alpha_2, \alpha_1, \beta_2, \beta_1, \gamma_2, \gamma_1, \delta_2, \delta_1, \varepsilon_2, \varepsilon_1, \zeta_2, \zeta_1)$ -*-generalized pseudocontraction from $J(C)$ into itself. By Theorem 5 we have that T^* has a fixed point z_0^* . Then $J^{-1} z_0^*$ is a fixed point of T . $\square$

4. Acute Point and Skew-Acute Point

Let E be a smooth Banach space, let C be a nonempty subset of E , let T be a mapping from C into E , and let $k, \ell \in \mathbb{R}$. $x \in E$ is called a (k, ℓ) -acute point of T if

$$\phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) + \ell\phi(Ty, y) \quad (48)$$

for any $y \in C$. $x \in E$ is called a (k, ℓ) -skew-acute point of T if

$$\phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) + \ell\phi(Ty, y) \quad (49)$$

for any $y \in C$. Let

$$\begin{aligned} \mathcal{A}_{k,\ell}(T) &= \{x \in E \mid \phi(x, Ty) \leq \phi(x, y) + k\phi(y, Ty) \\ &+ \ell\phi(Ty, y) \text{ for any } y \in C\}; \\ \mathcal{B}_{k,\ell}(T) &= \{x \in E \mid \phi(Ty, x) \leq \phi(y, x) + k\phi(y, Ty) \\ &+ \ell\phi(Ty, y) \text{ for any } y \in C\}. \end{aligned} \quad (50)$$

It is obvious that

$$\begin{aligned} \mathcal{A}_{k_1, \ell_1}(T) &\subset \mathcal{A}_{k_2, \ell_2}(T), \\ \mathcal{B}_{k_1, \ell_1}(T) &\subset \mathcal{B}_{k_2, \ell_2}(T) \end{aligned} \quad (51)$$

for any $k_1, k_2, \ell_1, \ell_2 \in \mathbb{R}$ with $k_1 \leq k_2$ and $\ell_1 \leq \ell_2$.

The following lemmas are important property characterizing them.

Lemma 7. Let E be a smooth Banach space, let C be a nonempty subset of E , let T be a mapping from C into E , and let $k, \ell \in \mathbb{R}$. Then $\mathcal{A}_{k,\ell}(T)$ is closed and convex.

Proof. Since (48) is equivalent to

$$\begin{aligned} 2 \langle x, Jy - JTy \rangle &\leq (k-1)\phi(y, Ty) + \ell\phi(Ty, y) \\ &+ 2 \langle y, Jy - JTy \rangle \end{aligned} \quad (52)$$

from (27), $\mathcal{A}_{k,\ell}(T)$ is closed and convex. $\square$

Lemma 8. Let E be a smooth Banach space, let C be a nonempty subset of E , let T be a mapping from C into E , and let $k, \ell \in \mathbb{R}$. Then $\mathcal{B}_{k,\ell}(T)$ is closed.

Proof. Inequality (49) is equivalent to

$$\begin{aligned} 2 \langle y - Ty, Jx \rangle &\leq k\phi(y, Ty) + (\ell-1)\phi(Ty, y) \\ &+ 2 \langle y - Ty, Jy \rangle \end{aligned} \quad (53)$$

from (27). Moreover, by (T6) J is norm-to-weak* continuous. Therefore $\mathcal{B}_{k,\ell}(T)$ is closed. $\square$

Let E^* be the dual space of a strictly convex, reflexive, and smooth Banach space E , let C^* be a nonempty subset of E^* , let T^* be a mapping from C^* into E^* , and let $k, \ell \in \mathbb{R}$. $x^* \in E^*$ is called a (k, ℓ) -*-acute point of T^* if

$$\begin{aligned} \phi_*(x^*, T^* y^*) &\leq \phi_*(x^*, y^*) + k\phi_*(y^*, T^* y^*) \\ &+ \ell\phi_*(T^* y^*, y^*) \end{aligned} \quad (54)$$

for any $y^* \in C^*$. $x^* \in E^*$ is called a (k, ℓ) -*-skew-acute point of T^* if

$$\begin{aligned} \phi_*(T^* y^*, x^*) &\leq \phi_*(y^*, x^*) + k\phi_*(y^*, T^* y^*) \\ &+ \ell\phi_*(T^* y^*, y^*) \end{aligned} \quad (55)$$

for any $y^* \in C^*$. Let

$$\begin{aligned} \mathcal{A}_{k,\ell}^*(T^*) &= \{x^* \in E^* \mid \phi_*(x^*, T^* y^*) \leq \phi_*(x^*, y^*) \\ &\quad + k\phi_*(y^*, T^* y^*) + \ell\phi_*(T^* y^*, y^*) \text{ for any } y^* \\ &\quad \in C^*\}; \\ \mathcal{B}_{k,\ell}^*(T^*) &= \{x^* \in E^* \mid \phi_*(T^* y^*, x^*) \leq \phi_*(y^*, x^*) \\ &\quad + k\phi_*(y^*, T^* y^*) + \ell\phi_*(T^* y^*, y^*) \text{ for any } y^* \\ &\quad \in C^*\}. \end{aligned} \quad (56)$$

Lemma 9. Let E^* be the dual space of a strictly convex, reflective, and smooth Banach space E , let C^* be a nonempty subset of E^* , let T^* be a mapping from C^* into E^* , and let $k, \ell \in \mathbb{R}$. Then $\mathcal{A}_{k,\ell}^*(T^*)$ is closed and convex.

Proof. Since

$$\begin{aligned} \phi_*(x^*, T^* y^*) &\leq \phi_*(x^*, y^*) + k\phi_*(y^*, T^* y^*) \\ &\quad + \ell\phi_*(T^* y^*, y^*) \end{aligned} \quad (57)$$

is equivalent to

$$\begin{aligned} 2 \langle J^{-1} y^* - J^{-1} T^* y^*, x^* \rangle \\ \leq (k-1) \phi_*(y^*, T^* y^*) + \ell\phi_*(T^* y^*, y^*) \\ + 2 \langle J^{-1} y^* - J^{-1} T^* y^*, y^* \rangle \end{aligned} \quad (58)$$

from (27) and (23), $\mathcal{A}_{k,\ell}^*(T^*)$ is closed and convex. $\square$

Lemma 10. Let E^* be the dual space of a strictly convex, reflexive, and smooth Banach space E , let C^* be a nonempty subset of E^* , let T^* be a mapping from C^* into E^* , and let $k, \ell \in \mathbb{R}$. Then $\mathcal{B}_{k,\ell}^*(T^*)$ is closed.

Proof. Since

$$\begin{aligned} \phi_*(T^* y^*, x^*) &\leq \phi_*(y^*, x^*) + k\phi_*(y^*, T^* y^*) \\ &\quad + \ell\phi_*(T^* y^*, y^*) \end{aligned} \quad (59)$$

is equivalent to

$$\begin{aligned} 2 \langle J^{-1} x^*, y^* - T^* y^* \rangle \\ \leq k\phi_*(y^*, T^* y^*) + (\ell-1) \phi_*(T^* y^*, y^*) \\ + 2 \langle J^{-1} y^*, y^* - T^* y^* \rangle \end{aligned} \quad (60)$$

from (27) and (23), moreover, by (T6) J^{-1} is norm-to-weak* continuous. Therefore $\mathcal{B}_{k,\ell}^*(T^*)$ is closed. $\square$

Lemma 11. Let E be a strictly convex, reflective, and smooth Banach space, let C be a nonempty subset of E , let T be a mapping from C into E , let $T^* = JTJ^{-1}$, and let $k, \ell \in \mathbb{R}$. Then

$$\begin{aligned} \mathcal{A}_{k,\ell}^*(T^*) &= J(\mathcal{B}_{\ell,k}(T)), \\ \mathcal{B}_{k,\ell}^*(T^*) &= J(\mathcal{A}_{\ell,k}(T)). \end{aligned} \quad (61)$$

In particular, $J(\mathcal{B}_{k,\ell}(T))$ is closed and convex and $J(\mathcal{A}_{k,\ell}(T))$ is closed.

Proof. Let $x^* \in \mathcal{A}_{k,\ell}^*(T^*)$. Then

$$\begin{aligned} \phi_*(x^*, T^* y^*) &\leq \phi_*(x^*, y^*) + k\phi_*(y^*, T^* y^*) \\ &\quad + \ell\phi_*(T^* y^*, y^*) \end{aligned} \quad (62)$$

for any $y^* \in J(C)$. From (23)

$$\begin{aligned} \phi(J^{-1} T^* y^*, J^{-1} x^*) &\leq \phi(J^{-1} y^*, J^{-1} x^*) \\ &\quad + k\phi(J^{-1} T^* y^*, J^{-1} y^*) \\ &\quad + \ell\phi(J^{-1} y^*, J^{-1} T^* y^*) \end{aligned} \quad (63)$$

for any $y^* \in J(C)$. Since $J^{-1} T^* = TJ^{-1}$, putting $y = J^{-1} y^*$, we obtain

$$\begin{aligned} \phi(Ty, J^{-1} x^*) &\leq \phi(y, J^{-1} x^*) + \ell\phi(y, Ty) \\ &\quad + k\phi(Ty, y). \end{aligned} \quad (64)$$

Therefore $J^{-1} x^* \in \mathcal{B}_{\ell,k}(T)$ and hence $\mathcal{A}_{k,\ell}^*(T^*) = J(\mathcal{B}_{\ell,k}(T))$.

$\mathcal{B}_{k,\ell}^*(T^*) = J(\mathcal{A}_{\ell,k}(T))$ can be shown similarly.

Moreover, by Lemma 9 $J(\mathcal{B}_{k,\ell}(T))$ is closed and convex and by Lemma 10 $J(\mathcal{A}_{k,\ell}(T))$ is closed. $\square$

Lemma 12. Let E be a strictly convex and smooth Banach space, let C be a nonempty subset of E , let T be a mapping from C into E , and let $k, \ell \in \mathbb{R}$. Then the following hold.

- (1) If $(k, \ell) \in (-\infty, 1] \times (-\infty, 0] \setminus \{(1, 0)\}$, then $C \cap \mathcal{A}_{k,\ell}(T)$ is included in the set of all fixed points of T .
- (2) If $(k, \ell) \in (-\infty, 0] \times (-\infty, 1] \setminus \{(0, 1)\}$, then $C \cap \mathcal{B}_{k,\ell}(T)$ is included in the set of all fixed points of T .

Proof. Let $x \in C \cap \mathcal{A}_{k,\ell}(T)$. Then (48) holds for any $y \in C$. Putting $y = x$, by (T11) we obtain $(1-k)\phi(x, Tx) - \ell\phi(Tx, x) \leq 0$. If $(k, \ell) \in (-\infty, 1] \times (-\infty, 0] \setminus \{(1, 0)\}$, then by (T11) we obtain $x = Tx$.

Let $x \in C \cap \mathcal{B}_{k,\ell}(T)$. Then (49) holds for any $y \in C$. Putting $y = x$, by (T11) we obtain $-k\phi(x, Tx) + (1-\ell)\phi(Tx, x) \leq 0$. If $(k, \ell) \in (-\infty, 0] \times (-\infty, 1] \setminus \{(0, 1)\}$, then by (T11) we obtain $x = Tx$. $\square$

5. Acute Point and Skew-Acute Point Theorems

Theorem 13. Let E be a reflexive and smooth Banach space, let C be a nonempty subset of E , and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -generalized pseudocontraction from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Then the following hold.

- (1) If $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \beta_2 \geq 0$, $\gamma_2 + \delta_2 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, and $\alpha_1 + \beta_1 > 0$, then there exists a $(-\zeta_1/(\alpha_1 + \beta_1), -\zeta_2/(\alpha_1 + \beta_1))$ -acute point.

- (2) If $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \gamma_1 \geq 0$, $\beta_1 + \delta_1 \geq 0$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, and $\alpha_2 + \gamma_2 > 0$, then there exists a $(-\varepsilon_2/(\alpha_2 + \gamma_2), -\varepsilon_1/(\alpha_2 + \gamma_2))$ -acute point.
- (3) If $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 \geq 0$, $\beta_1 + \gamma_2 + \delta_1 + \delta_2 \geq 0$, $\varepsilon_1 + \zeta_2 \geq 0$, $\varepsilon_2 + \zeta_1 \geq 0$, and $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 > 0$, then there exists a $(-\varepsilon_2 + \zeta_1)/(\alpha_1 + \alpha_2 + \beta_1 + \gamma_2), -(\varepsilon_1 + \zeta_2)/(\alpha_1 + \alpha_2 + \beta_1 + \gamma_2))$ -acute point.

Proof. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. In the case of (1), similarly to the proof of Theorem 4, we have that there exists a unique $z_0 \in \overline{\text{co}}\{T^n z \mid n \in \mathbb{N}\}$ such that

$$\begin{aligned} & (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\ & + 2(\alpha_1 + \beta_1) \langle z_0 - y, Jy - JTy \rangle \leq 0. \end{aligned} \quad (65)$$

for any $y \in C$. Since

$$\begin{aligned} 2 \langle p - q, Jr - Js \rangle &= \phi(p, s) + \phi(q, r) - \phi(p, r) \\ &\quad - \phi(q, s) \end{aligned} \quad (66)$$

for any $p, q, r, s \in E$, we obtain

$$\begin{aligned} & (\alpha_1 + \beta_1 + \zeta_1) \phi(y, Ty) + \zeta_2 \phi(Ty, y) \\ & + (\alpha_1 + \beta_1) (\phi(z_0, Ty) - \phi(z_0, y) - \phi(y, Ty)) \\ & = (\alpha_1 + \beta_1) \phi(z_0, Ty) - (\alpha_1 + \beta_1) \phi(z_0, y) \\ & + \zeta_1 \phi(y, Ty) + \zeta_2 \phi(Ty, y) \leq 0. \end{aligned} \quad (67)$$

Since $\alpha_1 + \beta_1 > 0$, we obtain

$$\begin{aligned} \phi(z_0, Ty) &\leq \phi(z_0, y) - \frac{\zeta_1}{\alpha_1 + \beta_1} \phi(y, Ty) \\ &\quad - \frac{\zeta_2}{\alpha_1 + \beta_1} \phi(Ty, y). \end{aligned} \quad (68)$$

Therefore z_0 is a $(-\zeta_1/(\alpha_1 + \beta_1), -\zeta_2/(\alpha_1 + \beta_1))$ -acute point.

In the cases of (2), changing the variables x and y in (26), we have that there exists a unique $z_0 \in \overline{\text{co}}\{T^n z \mid n \in \mathbb{N}\}$ such that

$$\begin{aligned} & (\alpha_2 + \gamma_2) \phi(z_0, Ty) - (\alpha_2 + \gamma_2) \phi(z_0, y) \\ & + \varepsilon_2 \phi(y, Ty) + \varepsilon_1 \phi(Ty, y) \leq 0. \end{aligned} \quad (69)$$

Since $\alpha_2 + \gamma_2 > 0$, we obtain

$$\begin{aligned} \phi(z_0, Ty) &\leq \phi(z_0, y) - \frac{\varepsilon_2}{\alpha_2 + \gamma_2} \phi(y, Ty) \\ &\quad - \frac{\varepsilon_1}{\alpha_2 + \gamma_2} \phi(Ty, y). \end{aligned} \quad (70)$$

Therefore z_0 is a $(-\varepsilon_2/(\alpha_2 + \gamma_2), -\varepsilon_1/(\alpha_2 + \gamma_2))$ -acute point.

In the case of (3), adding (26) and (36), we have that there exists a unique $z_0 \in \overline{\text{co}}\{T^n z \mid n \in \mathbb{N}\}$ such that

$$\begin{aligned} & (\alpha_1 + \alpha_2 + \beta_1 + \gamma_2) \phi(z_0, Ty) \\ & - (\alpha_1 + \alpha_2 + \beta_1 + \gamma_2) \phi(z_0, y) \\ & + (\varepsilon_2 + \zeta_1) \phi(y, Ty) + (\varepsilon_1 + \zeta_2) \phi(Ty, y) \leq 0. \end{aligned} \quad (71)$$

Since $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 > 0$, we obtain

$$\begin{aligned} \phi(z_0, Ty) &\leq \phi(z_0, y) - \frac{\varepsilon_2 + \zeta_1}{\alpha_1 + \alpha_2 + \beta_1 + \gamma_2} \phi(y, Ty) \\ &\quad - \frac{\varepsilon_1 + \zeta_2}{\alpha_1 + \alpha_2 + \beta_1 + \gamma_2} \phi(Ty, y). \end{aligned} \quad (72)$$

Therefore z_0 is a $(-\varepsilon_2 + \zeta_1)/(\alpha_1 + \alpha_2 + \beta_1 + \gamma_2), -(\varepsilon_1 + \zeta_2)/(\alpha_1 + \alpha_2 + \beta_1 + \gamma_2))$ -acute point. $\square$

Theorem 14. Let E^* be the dual space of a strictly convex, reflexive, and smooth Banach space E , let C^* be a nonempty subset of E^* , and let T^* be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -*-generalized pseudocontraction from C^* into itself. Suppose that there exists $z^* \in C^*$ such that $\{(T^*)^n z^* \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Then the following hold.

- (1) If $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \beta_2 \geq 0$, $\gamma_2 + \delta_2 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, and $\alpha_1 + \beta_1 > 0$, then there exists a $(-\zeta_1/(\alpha_1 + \beta_1), -\zeta_2/(\alpha_1 + \beta_1))$ -*-acute point.
- (2) If $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \gamma_1 \geq 0$, $\beta_1 + \delta_1 \geq 0$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, and $\alpha_2 + \gamma_2 > 0$, then there exists a $(-\varepsilon_2/(\alpha_2 + \gamma_2), -\varepsilon_1/(\alpha_2 + \gamma_2))$ -*-acute point.
- (3) If $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 \geq 0$, $\beta_1 + \gamma_2 + \delta_1 + \delta_2 \geq 0$, $\varepsilon_1 + \zeta_2 \geq 0$, $\varepsilon_2 + \zeta_1 \geq 0$, and $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 > 0$, then there exists a $(-\varepsilon_2 + \zeta_1)/(\alpha_1 + \alpha_2 + \beta_1 + \gamma_2), -(\varepsilon_1 + \zeta_2)/(\alpha_1 + \alpha_2 + \beta_1 + \gamma_2))$ -*-acute point.

Proof. Similar to the proof of Theorem 5 we obtain

$$\begin{aligned} & (\alpha_1 + \beta_1 + \zeta_1) \phi_*(y^*, T^* y^*) + \zeta_2 \phi_*(T^* y^*, y^*) \\ & + 2(\alpha_1 + \beta_1) \langle J^{-1} y^* - J^{-1} T^* y^*, z_0^* - y^* \rangle \leq 0. \end{aligned} \quad (73)$$

for any $y^* \in C^*$. Since

$$\begin{aligned} & 2 \langle J^{-1} r^* - J^{-1} s^*, p^* - q^* \rangle \\ & = \phi_*(p^*, s^*) + \phi_*(q^*, r^*) - \phi_*(p^*, r^*) \\ & \quad - \phi_*(q^*, s^*) \end{aligned} \quad (74)$$

for any $p^*, q^*, r^*, s^* \in E^*$, we obtain

$$\begin{aligned} & (\alpha_1 + \beta_1 + \zeta_1) \phi_*(y^*, T^* y^*) + \zeta_2 \phi_*(T^* y^*, y^*) \\ & + 2(\alpha_1 + \beta_1) \langle J^{-1} y^* - J^{-1} T^* y^*, z_0^* - y^* \rangle \\ & = (\alpha_1 + \beta_1) \phi_*(z_0^*, T^* y^*) - (\alpha_1 + \beta_1) \phi_*(z_0^*, y^*) \\ & + \zeta_1 \phi_*(y^*, T^* y^*) + \zeta_2 \phi_*(T^* y^*, y^*) \leq 0. \end{aligned} \quad (75)$$

Since $\alpha_1 + \beta_1 > 0$, we obtain

$$\begin{aligned} \phi_*(z_0^*, T^* y^*) &\leq \phi_*(z_0^*, y^*) - \frac{\zeta_1}{\alpha_1 + \beta_1} \phi_*(y^*, T^* y^*) \\ &\quad - \frac{\zeta_2}{\alpha_1 + \beta_1} \phi_*(T^* y^*, y^*). \end{aligned} \quad (76)$$

Therefore z_0^* is a $(-\zeta_1/(\alpha_1 + \beta_1), -\zeta_2/(\alpha_1 + \beta_1))$ -*-acute point. In the cases of (2) and (3) we can show similarly. $\square$

By Theorem 14 we obtain the following.

Theorem 15. Let E be a strictly convex, reflexive, and smooth Banach space, let C be a nonempty subset of E , and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -generalized pseudocontraction from C into itself. Suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. Then the following hold.

- (1) If $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \beta_1 \geq 0$, $\gamma_1 + \delta_1 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, and $\alpha_2 + \beta_2 > 0$, then there exists a $(-\zeta_2/(\alpha_2 + \beta_2), -\zeta_1/(\alpha_2 + \beta_2))$ -skew-acute point.
- (2) If $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \gamma_2 \geq 0$, $\beta_2 + \delta_2 \geq 0$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, and $\alpha_1 + \gamma_1 > 0$, then there exists a $(-\varepsilon_1/(\alpha_1 + \gamma_1), -\varepsilon_2/(\alpha_1 + \gamma_1))$ -skew-acute point.
- (3) If $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 \geq 0$, $\beta_2 + \gamma_1 + \delta_1 + \delta_2 \geq 0$, $\varepsilon_1 + \zeta_2 \geq 0$, $\varepsilon_2 + \zeta_1 \geq 0$, and $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 > 0$, then there exists a $(-\varepsilon_1 + \zeta_2)/(\alpha_1 + \alpha_2 + \beta_2 + \gamma_1), -(\varepsilon_2 + \zeta_1)/(\alpha_1 + \alpha_2 + \beta_2 + \gamma_1))$ -skew-acute point.

Proof. By Theorem 14, $T^* = JTJ^{-1}$ has a $*$ -acute point $z_0^* \in \overline{\text{co}}\{(T^*)^n Jz \mid n \in \mathbb{N}\}$. By Lemma 11 $J^{-1}z_0^*$ is a skew-acute point of T . $\square$

6. Applications

By Theorem 13 we can show easily the first part of Theorem 1. Note that $A(T) = \mathcal{A}_{0,0}(T)$.

Proof of the First Part of Theorem 1. Let E be a reflexive and smooth Banach space, let C be a nonempty subset of E , and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -generalized nonspreading mapping from C into itself satisfying $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta > 0$.

Suppose that T has an attractive point z . Then it is clear that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. T is an $(\alpha, -\varepsilon, \beta, \varepsilon, \gamma, -\zeta, \delta, \zeta, 0, 0, 0, 0)$ -generalized pseudocontraction. These parameters satisfy the conditions of Theorem 13 (1). Therefore there exists a $(0, 0)$ -acute point, that is, an attractive point. $\square$

By Theorem 15 we can show easily the first part of Theorem 2. Note that $B(T) = \mathcal{B}_{0,0}(T)$.

Proof of the First Part of Theorem 2. Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E , and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -skew-generalized nonspreading mapping from C into itself satisfying $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma > 0$.

Suppose that T has a skew-attractive point z . Then it is clear that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. T is an $(\alpha, -\varepsilon, \beta, -\zeta, \gamma, \varepsilon, \delta, \zeta, 0, 0, 0, 0)$ -generalized pseudocontraction. These parameters satisfy the conditions of Theorem 15 (2). Therefore there exists a $(0, 0)$ -skew-acute point, that is, a skew-attractive point. $\square$

By Theorem 13 and Lemma 12 (1) we obtain the following.

Theorem 16. Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty, closed, and convex subset of E , and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -generalized pseudocontraction from C into itself. Then the following hold.

- (1) Suppose that T satisfies $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \beta_2 \geq 0$, $\gamma_2 + \delta_2 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, $\alpha_1 + \beta_1 > 0$, and one of the following:

$$(1-1) \alpha_1 + \beta_1 + \zeta_1 > 0 \text{ and } \zeta_2 \geq 0;$$

$$(1-2) \alpha_1 + \beta_1 + \zeta_1 \geq 0 \text{ and } \zeta_2 > 0.$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (2) Suppose that T satisfies $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \gamma_1 \geq 0$, $\beta_1 + \delta_1 \geq 0$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, $\alpha_2 + \gamma_2 > 0$, and one of the following:

$$(2-1) \alpha_2 + \gamma_2 + \varepsilon_2 > 0 \text{ and } \varepsilon_1 \geq 0;$$

$$(2-2) \alpha_2 + \gamma_2 + \varepsilon_2 \geq 0 \text{ and } \varepsilon_1 > 0.$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (3) Suppose that T satisfies $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 \geq 0$, $\beta_1 + \gamma_2 + \delta_1 + \delta_2 \geq 0$, $\varepsilon_2 + \zeta_1 \geq 0$, $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 > 0$, and one of the following:

$$(3-1) \alpha_1 + \alpha_2 + \beta_1 + \gamma_2 + \varepsilon_2 + \zeta_1 > 0 \text{ and } \varepsilon_1 + \zeta_2 \geq 0;$$

$$(3-2) \alpha_1 + \alpha_2 + \beta_1 + \gamma_2 + \varepsilon_2 + \zeta_1 \geq 0 \text{ and } \varepsilon_1 + \zeta_2 > 0.$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 > 0$, then the fixed point is unique.

Proof. If T has a fixed point z , then $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. If T satisfies $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \beta_2 \geq 0$, $\gamma_2 + \delta_2 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, and $\alpha_1 + \beta_1 > 0$, then by Theorem 13 there exists $z_0 \in \overline{\text{co}}\{T^n z \mid n \in \mathbb{N}\} \subset C$ such that it is a $(-\zeta_1/(\alpha_1 + \beta_1), -\zeta_2/(\alpha_1 + \beta_1))$ -acute point. Moreover,

If (1-1) holds, then $-\zeta_1/(\alpha_1 + \beta_1) < 1$ and $-\zeta_2/(\alpha_1 + \beta_1) \leq 0$;

If (1-2) holds, then $-\zeta_1/(\alpha_1 + \beta_1) \leq 1$ and $-\zeta_2/(\alpha_1 + \beta_1) < 0$.

By Lemma 12 (1) z_0 is a fixed point of T .

In the cases of (2) and (3) we can show similarly. $\square$

Remark 17. Theorem 16 is the same as Theorem 4. The only difference is that Theorem 16 needs more assumptions than Theorem 4.

By Theorem 16 we can show easily the second part of Theorem 1.

Proof of the Second Part of Theorem 1. Let E be a strictly convex, reflexive, and smooth Banach space, let C be a nonempty, closed, and convex subset of E , and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -generalized nonspreading mapping from C into itself satisfying $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \beta > 0$.

Suppose that T has a fixed point z . Then it is clear that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. T is an $(\alpha, -\varepsilon, \beta, \varepsilon, \gamma, -\zeta, \delta, \zeta, 0, 0, 0, 0)$ -generalized pseudocontraction. These parameters satisfy the conditions of Theorem 16 (1). Therefore there exists a fixed point. $\square$

By Theorem 15 and Lemma 12 (2) we obtain the following.

Theorem 18. *Let E be a strictly convex, reflexive, and smooth Banach space, let C be a nonempty subset of E satisfying $J(C)$ is closed and convex, and let T be an $(\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, \delta_1, \delta_2, \varepsilon_1, \varepsilon_2, \zeta_1, \zeta_2)$ -generalized pseudocontraction from C into itself. Then the following hold.*

- (1) *Suppose that T satisfies $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \beta_1 \geq 0$, $\gamma_1 + \delta_1 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, $\alpha_2 + \beta_2 > 0$, and one of the following:*

$$(1-1) \alpha_2 + \beta_2 + \zeta_2 > 0 \text{ and } \zeta_1 \geq 0;$$

$$(1-2) \alpha_2 + \beta_2 + \zeta_2 \geq 0 \text{ and } \zeta_1 > 0.$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (2) *Suppose that T satisfies $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 \geq 0$, $\alpha_2 + \gamma_2 \geq 0$, $\beta_2 + \delta_2 \geq 0$, $\zeta_1 \geq 0$, $\zeta_2 \geq 0$, $\alpha_1 + \gamma_1 > 0$, and one of the following:*

$$(2-1) \alpha_1 + \gamma_1 + \varepsilon_1 > 0 \text{ and } \varepsilon_2 \geq 0;$$

$$(2-2) \alpha_1 + \gamma_1 + \varepsilon_1 \geq 0 \text{ and } \varepsilon_2 > 0.$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \beta_1 + \gamma_1 + \delta_1 > 0$ or $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 > 0$, then the fixed point is unique.

- (3) *Suppose that T satisfies $\alpha_1 + \alpha_2 + \beta_1 + \gamma_2 \geq 0$, $\beta_2 + \gamma_1 + \delta_1 + \delta_2 \geq 0$, $\varepsilon_1 + \zeta_2 \geq 0$, $\alpha_1 + \alpha_2 + \beta_2 + \gamma_1 > 0$, and one of the following:*

$$(3-1) \alpha_1 + \alpha_2 + \beta_2 + \gamma_1 + \varepsilon_1 + \zeta_2 > 0 \text{ and } \varepsilon_2 + \zeta_1 \geq 0;$$

$$(3-2) \alpha_1 + \alpha_2 + \beta_2 + \gamma_1 + \varepsilon_1 + \zeta_2 \geq 0 \text{ and } \varepsilon_2 + \zeta_1 > 0.$$

Then T has a fixed point if and only if there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Moreover, if $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 + \gamma_1 + \gamma_2 + \delta_1 + \delta_2 > 0$, then the fixed point is unique.

Proof. If T has a fixed point z , then $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. If T satisfies $\alpha_2 + \beta_2 + \gamma_2 + \delta_2 \geq 0$, $\alpha_1 + \beta_1 \geq 0$, $\gamma_1 + \delta_1 \geq 0$, $\varepsilon_1 \geq 0$, $\varepsilon_2 \geq 0$, and $\alpha_2 + \beta_2 > 0$, then by Theorem 15 there exists $z_0 \in J^{-1}(\overline{\text{co}}\{(T^*)^n Jz \mid n \in \mathbb{N}\}) \subset C$ such that it is a $(-\zeta_1/(\alpha_2 + \beta_2), -\zeta_2/(\alpha_2 + \beta_2))$ -skew-acute point. Moreover,

$$\text{If (1-1) holds, then } -\zeta_1/(\alpha_2 + \beta_2) \leq 0 \text{ and } -\zeta_2/(\alpha_2 + \beta_2) < 1;$$

$$\text{If (1-2) holds, then } -\zeta_1/(\alpha_2 + \beta_2) < 0 \text{ and } -\zeta_2/(\alpha_2 + \beta_2) \leq 1.$$

By Lemma 12 (2) z_0 is a fixed point of T .

In the cases of (2) and (3) we can show similarly. $\square$

Remark 19. Theorem 18 is the same as Theorem 6. The only difference is that Theorem 18 needs more assumptions than Theorem 6.

By Theorem 18 we can show easily the second part of Theorem 2.

Proof of the Second Part of Theorem 2. Let E be a strictly convex, reflexive and smooth Banach space, let C be a nonempty subset of E satisfying $J(C)$ is closed and convex, and let T be an $(\alpha, \beta, \gamma, \delta, \varepsilon, \zeta)$ -skew-generalized nonspreading mapping from C into itself satisfying $\alpha + \beta + \gamma + \delta \geq 0$ and $\alpha + \gamma > 0$.

Suppose that T has a fixed point z . Then it is clear that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded.

Conversely, suppose that there exists $z \in C$ such that $\{T^n z \mid n \in \mathbb{N} \cup \{0\}\}$ is bounded. T is an $(\alpha, -\varepsilon, \beta, -\zeta, \gamma, \varepsilon, \delta, \zeta, 0, 0, 0, 0)$ -generalized pseudocontraction. These parameters satisfy the conditions of Theorem 18 (2). Therefore there exists a fixed point. $\square$

Data Availability

No data were used to support this study.

Disclosure

An earlier version of this study was presented as an oral presentation in The 6th Asian Conference on Nonlinear Analysis and Optimization 5–9 November 2018.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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