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Research Article

Analytical Solution for Finding the Second Zero of the Ahlfors Map for an Annulus Region

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The Ahlfors map is a conformal mapping function that maps a multiply connected region onto a unit disk. It can be written in terms of the Szegö kernel and the Garabedian kernel. In general, a zero of the Ahlfors map can be freely prescribed in a multiply connected region. The remaining zeros are the zeros of the Szegö kernel. For an annulus region, it is known that the second zero of the Ahlfors map can be computed analytically based on the series representation of the Szegö kernel. This paper presents another analytical method for finding the second zero of the Ahlfors map for an annulus region without using the series approach but using a boundary integral equation and knowledge of intersection points.

1. Introduction

A conformal mapping that maps a multiply connected region Ω of connectivity $n > 1$ onto a unit disk $E = \{w : |w| < 1\}$ is known as the Ahlfors map. It generalizes the Riemann map for a simply connected region. The Ahlfors map with a base point $a \in \Omega$ is a n-to-one map. It maps each boundary Γ of Ω corresponding in a one-to-one manner onto the boundary of the unit disk and maps a to the origin [1–3]. The Ahlfors map of a multiply connected region has several applications in modern function theory. For examples, the Bergman kernel and Green's function of a multiply connected region can be written in terms of finitely many Ahlfors map [4, 5]. The Ahlfors map has n zeros in Ω where one of the zeros is $a \in \Omega$ which can be freely prescribed, and the remaining $n-1$ zeros in Ω are unknown [1–3]. It is known that the Ahlfors map can be written in terms of the Szegö kernel and the Garabedian kernel. In [1–3], the zeros of the Ahlfors map with a base point a occur at the pole of the Garabedian kernel at a and the remaining $n-1$ zeros are those of the Szegö kernel.

Kerzman and Stein [6] and Kerzman and Trummer [7] have shown that the Szegö kernel of a simply connected region satisfies an integral equation which is valid also for a multiply connected region [8]. The integral equations with the Kerzman–Stein kernel, Neumann kernel, and Szegö kernel related to the Ahlfors map for a doubly connected region have been derived in [9]. The integral equation with the generalized Neumann kernel for computing the Ahlfors map has been constructed in [10]. The method in [10] is useful provided the zeros are known. Several integral equations involving the Kerzman–Stein kernel, the Neumann kernel, the Szegö kernel, the Neumann-type kernel, and the Kerzman-Stein type kernel have also been derived in [11, 12].

In [2], the formula of the second zero of the Ahlfors map for a doubly connected region has been derived in terms of an integral involving the Szegö kernel and the derivative of the Szegö kernel, but the second zero has not been computed numerically. In [3], the second zero of the Ahlfors map for an annulus region has been obtained analytically from the series representation of the Szegö kernel. The problem of finding the zeros of the Ahlfors map has not been studied in [9–12], but discussed in [13]. The second zero of the Ahlfors map for any doubly connected region has been computed numerically in [13] based on the integral equation and Newton iterative method.

In this paper, we derive a boundary integral equation related to the Ahlfors map for a doubly connected region different from [12]. This integral equation together with knowledge of intersection points provides another analytical method for finding the second zero of the Ahlfors map for an annulus region. Unlike [3], the method does not depend on the series representation of the Szegö kernel.

The organization of this paper is as follows: Section 2 presents some auxiliary materials related to the Ahlfors map and the zeros of the Ahlfors map. In Section 3, we give a boundary integral equation related to the Ahlfors map for a doubly connected region. In Section 4, we present another analytical method for finding the second zero of the Ahlfors map for an annulus region. In Section 5, we give a conclusion.

2. Auxiliary Materials

Let Ω be a bounded doubly connected region with the boundary Γ consisting of two smoothly closed Jordan curves Γ_j, $j = 0, 1$, i.e., $\Gamma = \Gamma_0 \cup \Gamma_1$. The curve Γ_1 lies in the interior of the boundary Γ_0. The outer curve Γ_0 has a counterclockwise orientation, and the inner curve Γ_1 has a clockwise orientation. The curves Γ_j are parameterized by 2π–twice continuously differentiable complex-valued functions $z_j(t)$ with the first derivatives $z_j'(t) \neq 0$, $t \in J_j = [0, 2\pi]$. The total parameter domain J is defined as the disjoint union of two intervals of J_j. The parameterization $z(t)$ of the whole boundary Γ on J is defined as $z(t) = z_j(t)$. The unit tangent to the boundary Γ_j at $z_j(t)$ is given by

$$T\left(z_j(t)\right) = \frac{z_j'(t)}{\left|z_j'(t)\right|}. \tag{1}$$

It is known that the Szegö kernel satisfies the Kerzman–Stein integral equation [1, 6]:

$$S(z, a_0) + \int_\Gamma A(z, w) S(w, a_0) |dw| = g(z), \quad z \in \Gamma,\ a_0 \in \Omega, \tag{2}$$

where

$$A(z, w) = \begin{cases} \dfrac{1}{2\pi i}\left(\dfrac{T(w)}{z - w} - \dfrac{\overline{T(z)}}{\overline{z} - \overline{w}}\right), & z \neq w \in \Gamma, \\ 0, & z = w \in \Gamma, \end{cases} \tag{3}$$

$$g(z) = -\frac{1}{2\pi i}\frac{\overline{T(z)}}{\overline{z} - \overline{a_0}}, \quad z \in \Gamma.$$

The function $A(z, w)$ is continuous on the smooth boundary of Ω, and it is known as a Kerzman–Stein kernel [6, 7]. If z and w are on a circle, then $A(z, w) = 0$. Numerical implementations of computing the Szegö kernel based on (2) are discussed in [7]. With $z = z(t)$ and $w = z(s)$, (2) becomes

$$\begin{aligned} &S(z(t), a_0) + \int_J A(z(t), z(s)) S(z(s), a_0) \left|z'(s)\right| ds \\ &\quad = g(z(t)), \quad z(t) \in \Gamma,\ a_0 \in \Omega. \end{aligned} \tag{4}$$

The derivative of the Szegö kernel has been derived in [13] as a solution of the integral equation:

$$\begin{aligned} &S'(z(t), a_0) z'(t) = g'(z(t)) z'(t) \\ &\quad - \int_J \left[\frac{d}{dt} A(z(t), z(s)) z'(t)\right] S(z(s), a_0) \left|z'(s)\right| ds, \end{aligned} \tag{5}$$

where

$$\begin{aligned} g'(z(t)) z'(t) &= -\frac{1}{2\pi i}\left[\frac{\overline{T'(z(t)) z'(t)}}{\overline{z(t) - a_0}} - \frac{\overline{T(z(t)) z'(t)}}{\overline{(z(t) - a_0)^2}}\right], \\ T'(z(t)) z'(t) &= \frac{z''(t)}{2\left|z'(t)\right|} - \frac{\left(z'(t)\right)^2 \overline{z''(t)}}{2\left|z'(t)\right|^3}, \\ \frac{d}{dt} A(z(t), z(s)) &= \begin{cases} \dfrac{1}{2\pi i}\left[\dfrac{-T(z(s)) z'(t)}{(z(t) - z(s))^2} - \dfrac{\overline{T'(z(t)) z'(t)}}{(\overline{z(t)} - \overline{z(s)})} + \dfrac{\overline{T(z(t)) z'(t)}}{(\overline{z(t)} - \overline{z(s)})^2}\right], & z(t) \neq z(s) \in \Gamma, \\ \dfrac{1}{4\pi\left|z'(t)\right|}\left[\dfrac{1}{3}\operatorname{Im}\left(\dfrac{z'''(t)}{z'(t)}\right) - \operatorname{Re}\left(\dfrac{z''(t)}{z'(t)}\right)\operatorname{Im}\left(\dfrac{z''(t)}{z'(t)}\right)\right], & z(t) = z(s) \in \Gamma. \end{cases} \end{aligned} \tag{6}$$

Let $f(z)$ be the Ahlfors function which maps Ω conformally onto the unit disk $E = \{w : |w| < 1\}$ which satisfies the conditions $f(a_j) = 0$, $j = 0, 1$, and $f'(a_0) > 0$, where $a_j \in \Omega$ are the zeros of the Ahlfors map and $a_0 \in \Omega$ can be freely chosen. The boundary values of $f(z)$ is represented by

$$f\left(z_j(t)\right) = e^{i\theta_j(t)}, \quad \Gamma_j : z = z_j(t), \quad 0 \le t \le 2\pi \in J_j, \tag{7}$$

where $\theta_j(t)$, $j = 0, 1$, are the boundary correspondence functions of the Ahlfors map on Γ_j.

Differentiating both sides of (7), we have

$$f'\left(z_j(t)\right)z_j'(t) = f\left(z_j(t)\right)i\theta_j'(t). \tag{8}$$

Taking modulus on both sides of (8) gives

$$\left|f'\left(z_j(t)\right)z_j'(t)\right| = \left|f\left(z_j(t)\right)i\theta_j'(t)\right|. \tag{9}$$

Dividing (8) by (9) and using (1), we get

$$f\left(z_j(t)\right) = \frac{T\left(z_j(t)\right)}{i}\frac{\left|\theta_j'(t)\right|f'\left(z_j(t)\right)}{\theta_j'(t)\left|f'\left(z_j(t)\right)\right|}. \tag{10}$$

The image of Γ_0 remains in a counterclockwise orientation, so $\theta_0'(t) > 0$, and the image of Γ_1 is in reversed orientation, so $\theta_1'(t) > 0$.

Since $|\theta_j'(t)|/\theta_j'(t) = 1$, thus from (10), we get

$$f\left(z_j(t)\right) = \frac{T\left(z_j(t)\right)f'\left(z_j(t)\right)}{i\left|f'\left(z_j(t)\right)\right|}. \tag{11}$$

In general,

$$f(z) = \frac{T(z)f'(z)}{i\left|f'(z)\right|}, \quad z \in \Gamma. \tag{12}$$

The interior values of the Ahlfors map $f(z)$ which is analytic on Ω can be obtained by using the Cauchy integral formula:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w - z}dw, \quad z \in \Omega. \tag{13}$$

For points z closed to Γ, the method of Helsing and Ojala [14] can be used for the efficient numerical computation of (13).

The Ahlfors map can be represented in terms of the Szegö kernel $S(z, a_0)$ and the Garabedian kernel $L(z, a_0)$. It is given as follows [1]:

$$f(z) = \frac{S(z, a_0)}{L(z, a_0)}, \quad z \in \Omega \cup \Gamma, a_0 \in \Omega. \tag{14}$$

Since $L(z, a_0) = i\overline{S(z, a_0)T(z)}$ [1], (14) can be written as

$$f(z(t)) = \frac{S(z(t), a_0)T(z(t))}{i\overline{S(z(t), a_0)}}, \quad z(t) \in \Gamma,\ a_0 \in \Omega. \tag{15}$$

It is shown that the boundary values of the Ahlfors map can be determined from the boundary values of the Szegö kernel since the boundary values of the Garabedian kernel can be determined from the boundary values of the Szegö kernel.

Differentiating both sides of (15), we get

$$f'(z(t))z'(t) = \frac{1}{i}\left[\frac{S(z(t), a_0)T'(z(t))z'(t) + S'(z(t), a_0)z'(t)T(z(t))}{\overline{S(z(t), a_0)}} - \frac{\overline{S'(z(t), a_0)z'(t)}S(z(t), a_0)T(z(t))}{\left(\overline{S(z(t), a_0)}\right)^2}\right]. \tag{16}$$

In [13], an alternative formula for the derivative of the boundary correspondence function of the Ahlfors map has been derived by using (8), (15), and (16), i.e.,

$$\theta'(t) = 2\,\mathrm{Im}\left[\frac{S'(z(t), a_0)z'(t)}{S(z(t), a_0)}\right] + \mathrm{Im}\left[\frac{z''(t)}{z'(t)}\right], \quad z(t) \in \Gamma,\ a_0 \in \Omega. \tag{17}$$

In general, the zeros of the Ahlfors map are unknown except for an annulus region. In [2], the second zero of the Ahlfors map for a doubly connected region is represented by

$$g_1 \equiv a_1 = \frac{1}{2\pi i}\int_\Gamma z\frac{S'(z, a_0)}{S(z, a_0)}dz, \quad z \in \Gamma,\ a_0 \in \Omega. \tag{18}$$

However, numerical computation of a_1 based on (18) is not given in [2] where the boundary values of $S'(z, a_0)$ can be computed by solving the integral equation (5).

The second zero of the Ahlfors map for an annulus region $\{z : \rho \le |z| \le 1\}$ is shown to be $a_1 = -(\rho/\overline{a_0})$, and it has been obtained from the series representation of the Szegö kernel [3]:

$$S(z, a_0) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(z\overline{a}_0)^n}{1 + \rho^{2n+1}}, \quad z \in \Gamma,\ a_0 \in \Omega. \tag{19}$$

Another series representation of the Szegö kernel for an annulus region has been derived in [15] by solving the Kerzman–Stein integral equation (2) using the Adomian decomposition method, and it is given by

$$S(z, a_0) = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{(-1)^{n+1}\rho^{n+1}}{\rho^{2n+2} - z\overline{a}_0}, \quad z \in \Gamma,\ a_0 \in \Omega. \tag{20}$$

The two series (19) and (20) are shown to be equivalent. However, the series in (20) gives a faster convergence compared to the series in (19) [15].

3. A Boundary Integral Equation Related to the Ahlfors Map for an Annulus Region

In this section, a new boundary integral equation related to the Ahlfors map for a doubly connected region different from [12] is derived and applied in Section 4 to determine analytically the second zero of the Ahlfors map. It is given in the following theorem.

Theorem 1. *For all $z \in \Gamma$, the function $\theta'(t)$ of the Ahlfors map for the doubly connected region satisfies*

$$i\theta'(t) + \frac{1}{\pi}\mathrm{PV}\int_J \frac{z'(t)}{z(t)-z(s)}\theta'(s)ds = 2z'(t)\left[\frac{1}{z(t)-a_0} + \frac{1}{z(t)-a_1}\right], \tag{21}$$

where a_0 and a_1 are the zeros of the Ahlfors map.

Proof. The Ahlfors map for a doubly connected region can be written as

$$f(z) = \prod_{j=0}^{1}\left(z-a_j\right)g(z), \tag{22}$$

where $g(z)$ is analytic in Ω and $g(z) \neq 0$. Applying log on both sides of (22) gives

$$\log f(z) = \sum_{j=0}^{1}\log\left(z-a_j\right) + \log g(z), \quad z \in \Gamma. \tag{23}$$

Differentiating on both sides of (23), we get

$$\frac{f'(z)}{f(z)} = \sum_{j=0}^{1}\frac{1}{z-a_j} + \frac{g'(z)}{g(z)}. \tag{24}$$

Observe that

$$\begin{aligned}\frac{1}{2\pi i}\mathrm{PV}\int_\Gamma \frac{1}{w-z}\frac{f'(w)}{f(w)}dw &= \frac{1}{2\pi i}\mathrm{PV}\int_\Gamma \frac{1}{w-z}\cdot\left[\sum_{j=0}^{1}\frac{1}{w-a_j} + \frac{g'(w)}{g(w)}\right]dw \\ &= \frac{1}{2\pi i}\mathrm{PV}\sum_{j=0}^{1}\int_\Gamma \frac{1}{w-z}\frac{1}{w-a_j}dw + \frac{1}{2\pi i}\mathrm{PV}\int_\Gamma \frac{1}{w-z}\frac{g'(w)}{g(w)}dw.\end{aligned} \tag{25}$$

Applying the residue theory [16], Sokhotskyi formula [17], and (24) to (25), we get

$$\begin{aligned}\frac{1}{2\pi i}\mathrm{PV}\int_\Gamma \frac{1}{w-z}\frac{f'(w)}{f(w)}dw &= \frac{1}{2}\sum_{j=0}^{1}\frac{1}{z-a_j} + \sum_{j=0}^{1}\frac{1}{a_j-z} + \frac{1}{2}\frac{g'(z)}{g(z)} \\ &= \frac{1}{2}\frac{f'(z)}{f(z)} - \sum_{j=0}^{1}\frac{1}{z-a_j}.\end{aligned} \tag{26}$$

Letting $z = z(t)$ and $w = z(s)$ and multiplying (26) by $z'(t)$ gives

$$\frac{1}{2}\frac{f'(z(t))z'(t)}{f(z(t))} + \frac{1}{2\pi i}\mathrm{PV}\int_J \frac{z'(t)}{z(t)-z(s)}\frac{f'(z(s))}{f(z(s))}z'(s)ds = z'(t)\sum_{j=0}^{1}\frac{1}{z(t)-a_j}. \tag{27}$$

Since from (8), $i\theta'(t) = ((f'(z(t))z'(t))/(f(z(t))))$; thus, (27) yields

$$i\theta'(t) + \frac{1}{\pi}\mathrm{PV}\int_J \frac{z'(t)}{z(t)-z(s)}\theta'(s)ds = 2z'(t)\left[\frac{1}{z(t)-a_0} + \frac{1}{z(t)-a_1}\right]. \tag{28}$$

This completes the proof of Theorem 1. □

If we take the imaginary part on both sides of (21), then it reduces the integral equation derived in [12], i.e.,

$$\theta'(t) + \int_J N(t,s)\theta'(s)ds = 2\,\mathrm{Im}\left[\frac{z'(t)}{z(t)-a_0} + \frac{z'(t)}{z(t)-a_1}\right], \tag{29}$$

where

$$N(t,s) = \begin{cases}\dfrac{1}{\pi}\mathrm{Im}\left[\dfrac{z'(t)}{z(t)-z(s)}\right], & t \neq s, \\[2ex] \dfrac{1}{2\pi}\mathrm{Im}\left[\dfrac{z''(t)}{z'(t)}\right], & t = s,\end{cases} \tag{30}$$

and $\theta'(t)$ is defined in (17). The kernel $N(t,s)$ is the classical Neumann kernel. If Ω is a simply connected region, then (29) reduces to the Warschawski's integral equation for the Riemann map as given in [17] (p. 394-395) with $R(0) = 0$, i.e.,

$$\theta'(t) + \int_J N(t,s)\theta'(s)ds = 2\,\mathrm{Im}\left[\frac{z'(t)}{z(t)}\right]. \tag{31}$$

Next we show how to find the second zero a_1 of the Ahlfors map for an annulus region using (21) and knowledge of intersection points.

4. Finding the Second Zero of the Ahlfors Map for an Annulus Region

We represent the left-hand side and the right-hand side of (21), respectively, as

$$h(t) = i\theta'(t) + \frac{1}{\pi}\mathrm{PV}\int_J \frac{z'(t)}{z(t)-z(s)}\theta'(s)ds, \tag{32}$$

$$g(t) = 2z'(t)\left[\frac{1}{z(t)-a_0} + \frac{1}{z(t)-a_1}\right]. \tag{33}$$

We consider an annulus region Ω bounded by

$$\Gamma_0 : z_0(t) = e^{it},$$
$$\Gamma_1 : z_1(t) = \rho e^{-it}, \tag{34}$$

where $0 < \rho < 1$ and $0 \le t \le 2\pi$. Given a_0 is known, and we assume a_1 is unknown. In the next two examples, we show the graphs of $h(t)$ for special values of ρ and a_0. The graph of $h(t)$ depends on $\theta'(t)$ which is computed based on (17). The function $\theta'(t)$ in (17) depends on $S(z(t), a_0)$ and $S'(z(t), a_0)$ which are computed by solving (2) and (5), respectively. Thus, the graph of $h(t)$ does not depend on a_1. If a_1 is known, then the graph of $g(t)$ is the same as the graph of $h(t)$.

Example 1. Given $a_0 = -0.5 - 0.5i$ and $\rho = 0.2$, and the annulus region is shown in Figure 1. The graphs of the functions $h_0(t) = h(z_0(t))$ and $h_1(t) = h(z_1(t))$ are shown in Figure 2.

Example 2. Given $a_0 = 0.2 - 0.4i$ and $\rho = 0.3$, and the annulus region is shown in Figure 3. The graphs of the functions $h_0(t) = h(z_0(t))$ and $h_1(t) = h(z_1(t))$ are shown in Figure 4.

Figures 2 and 4 show some common characteristics:

(i) C1: the graphs of $h_0(t)$ and $h_1(t)$ have the same shape and size

(ii) C2: the graphs of $h_0(t)$ and $h_1(t)$ are symmetrical with respect to the imaginary axis

(iii) C3: both graphs also have intersection points on the imaginary axis

(iv) C4: the graph of $h_0(t)$ is $4i$ units above the graph of $h_1(t)$

These characteristics are unique for the Ahlfors map for the annulus region. Note that the graphs in Figures 2 and 4 can also be generated using (33) provided $a_1 = -(\rho/\overline{a_0})$ is known. In Example 3, we consider plotting (33) using $a_1 \neq -(\rho/\overline{a_0})$.

Example 3. Given $a_0 = 0.2 - 0.4i$ and $\rho = 0.3$, and the annulus region is shown in Figure 3. We choose $a_1 = -0.2 + 0.6i \neq -(\rho/\overline{a_0})$. The graphs of the functions $g_0(t) = g(z_0(t))$ and $g_1(t) = g(z_1(t))$ are shown in Figure 5.

In Example 4, we consider plotting (32) involving nonconcentric circles.

Example 4. Consider the region bounded by nonconcentric circles $\Gamma_0 : z_0(t) = 2e^{it}$ and $\Gamma_1 : z_1(t) = -0.4 + 0.4i + \rho e^{-it}$ with $a_0 = 1 + i$ and $\rho = 0.5$. The nonconcentric circles region is shown in Figure 6. The graphs of the functions $h_0(t) = h(z_0(t))$ and $h_1(t) = h(z_1(t))$ are shown in Figure 7.

The graphs in Figures 5 and 7 no longer satisfy the characteristics C1, C2, C3, and C4. Based on these observations, we prove the following theorem to determine the second zero a_1 of the Ahlfors map for an annulus region.

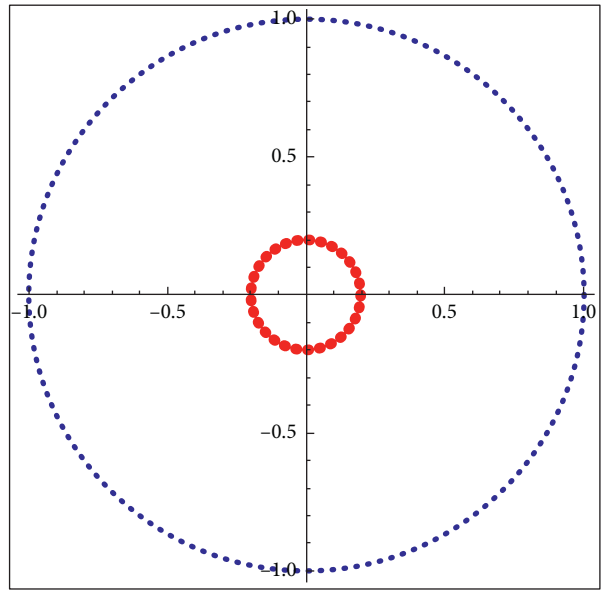

Figure 1: Annulus region Ω for Example 1.

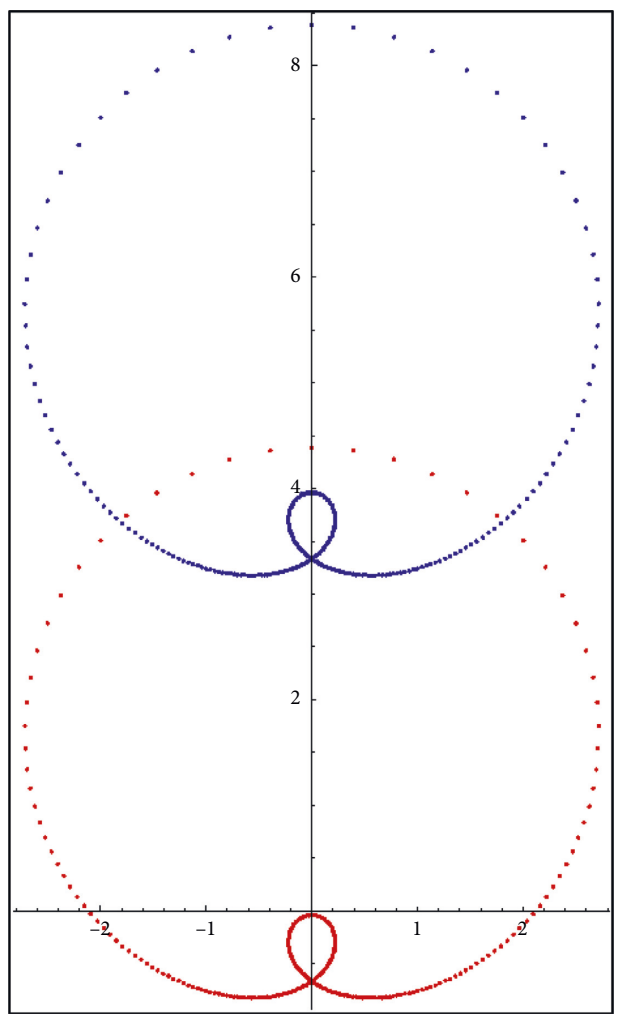

Figure 2: Graphs of $h_0(t)$ (top curve) and $h_1(t)$ (bottom curve) for Example 1.

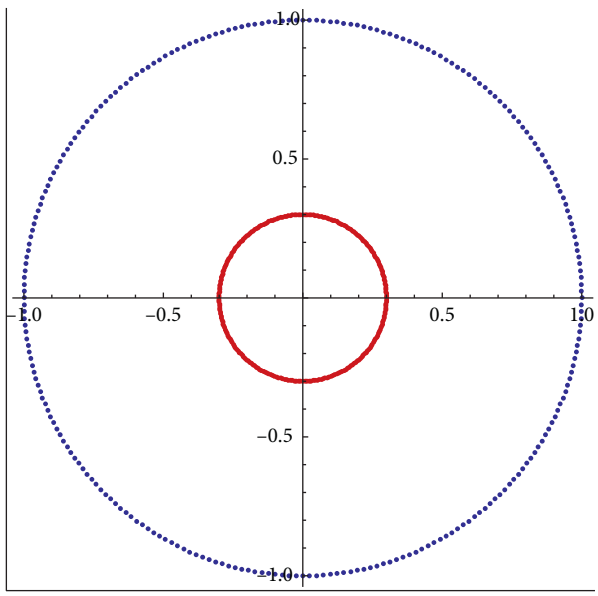

Figure 3: Annulus region Ω for Example 2.

Theorem 2. *Let $g(t)$ be defined in (33), where $g_0(t) = g(z_0(t))$, $g_1(t) = g(z_1(t))$, $z_0(t) = e^{it}$, $t \in J_0$, $z_1(t) = \rho e^{-it}$, $t \in J_1$, and $a_0 = re^{i\phi}$ be given. Then, there exists $t_1, t_2, t_1^*, t_2^* \in [0, 2\pi]$, $t_1 \neq t_2$, and $t_1^* \neq t_2^*$ such that*

$$\begin{aligned} g_0(t_1) &= g_0(t_2), \\ g_1(t_1^*) &= g_1(t_2^*). \end{aligned} \tag{35}$$

Furthermore, $a_1 = -(\rho/\overline{a_0})$.

Proof. From (33),

$$g(t) = 2z'(t)\left[\frac{1}{z(t) - a_0} + \frac{1}{z(t) - a_1}\right]. \tag{36}$$

We consider for two cases which are $z_0(t) = e^{it}$, $t \in J_0$, and $z_1(t) = \rho e^{-it}$, $t \in J_1$.

Case 1. $z_0(t) = e^{it}$, $t \in J_0$.

Let $z_0(t) = e^{it}$, $t \in J_0$, and $a_0 = re^{i\phi}$ be given, and $a_1 = se^{i\theta}$ is unknown. Since $g_0(t)$ is periodic, there exists $t_1, t_2 \in [0, 2\pi]$ with $t_1 \neq t_2$ such that

$$g_0(t_1) = g_0(t_2). \tag{37}$$

This gives

$$\operatorname{Re} g_0(t_1) = \operatorname{Re} g_0(t_2), \tag{38}$$

$$\operatorname{Im} g_0(t_1) = \operatorname{Im} g_0(t_2). \tag{39}$$

From (38) and (39), respectively, we have

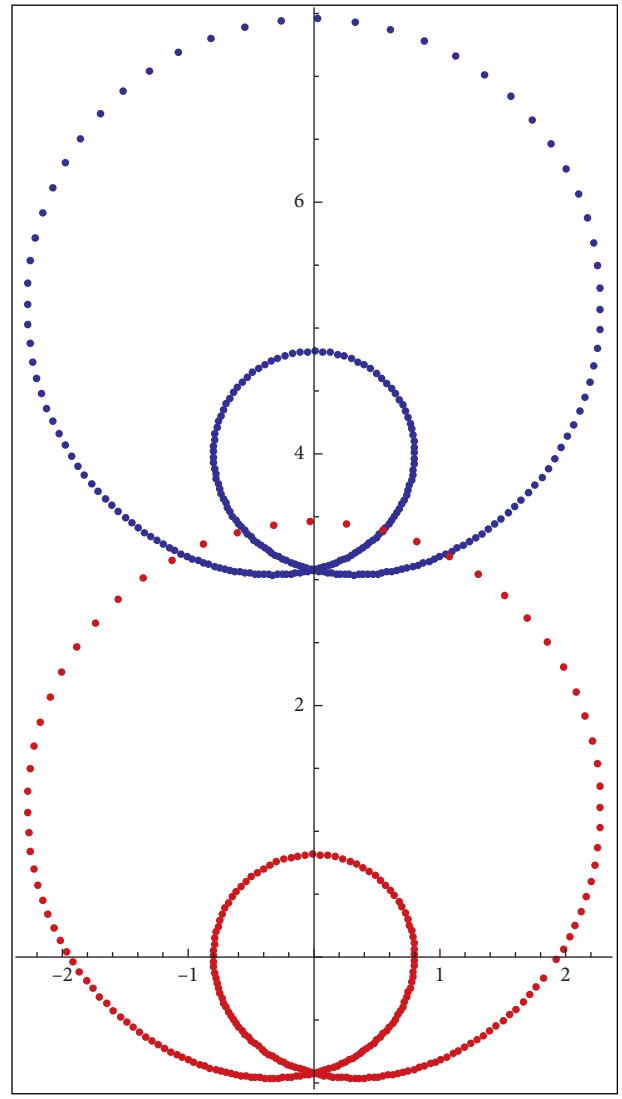

Figure 4: Graphs $h_0(t)$ (top curve) and $h_1(t)$ (bottom curve) for Example 2.

$$\begin{aligned} &-2\left[\frac{s\sin(\theta - t_1)}{1 - 2s\cos(\theta - t_1) + s^2} + \frac{r\sin(\phi - t_1)}{1 - 2r\cos(\phi - t_1) + r^2}\right] \\ &\quad = -2\left[\frac{s\sin(\theta - t_2)}{1 - 2s\cos(\theta - t_2) + s^2} + \frac{r\sin(\phi - t_2)}{1 - 2r\cos(\phi - t_2) + r^2}\right], \end{aligned} \tag{40}$$

$$\begin{aligned} &2\left[\frac{1 - s\cos(\theta - t_1)}{1 - 2s\cos(\theta - t_1) + s^2} + \frac{1 - r\cos(\phi - t_1)}{1 - 2r\cos(\phi - t_1) + r^2}\right] \\ &\quad = 2\left[\frac{1 - s\cos(\theta - t_2)}{1 - 2s\cos(\theta - t_2) + s^2} + \frac{1 - r\cos(\phi - t_2)}{1 - 2r\cos(\phi - t_2) + r^2}\right]. \end{aligned} \tag{41}$$

From (40) and (41), we seek t_1 and t_2 such that

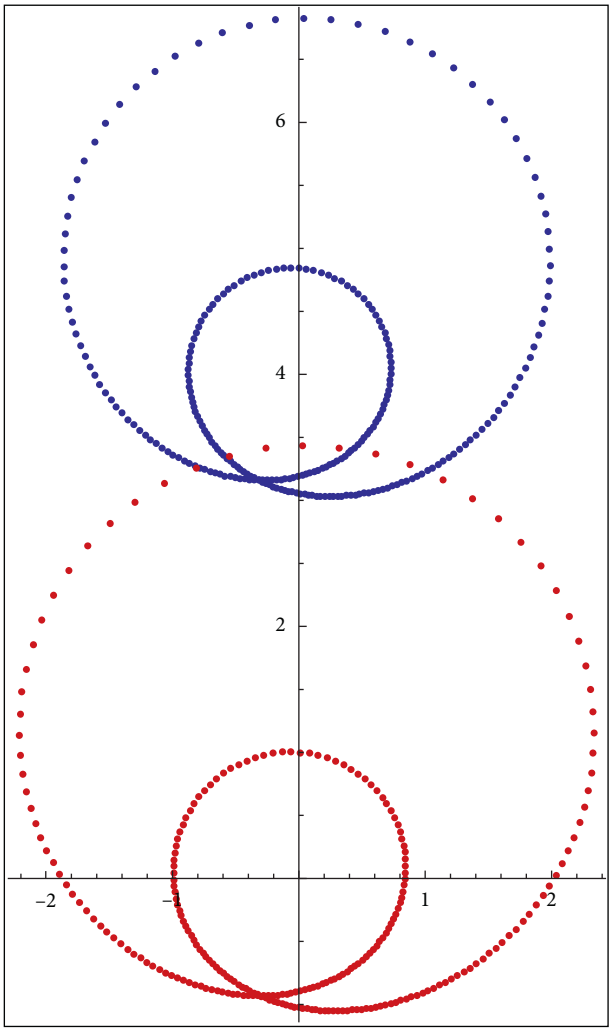

Figure 5: Graphs of $g_0(t)$ (top curve) and $g_1(t)$ (bottom curve) for Example 3.

$$\cos(\phi - t_1) = \cos(\phi - t_2), \tag{42}$$

that is,

$$\begin{aligned}&\cos(\phi - t_1) - \cos(\phi - t_2)\\ &\quad = 2\sin\left(\frac{\phi - t_1 + \phi - t_2}{2}\right)\sin\left(\frac{\phi - t_2 - (\phi - t_1)}{2}\right).\end{aligned} \tag{43}$$

Since $t_1 \neq t_2$, thus this implies

$$\begin{aligned}\frac{\phi - t_1 + \phi - t_2}{2} &= \pi k,\\ t_2 &= 2\phi - t_1 - 2\pi k,\\ k &= 0, \pm 1, \ldots.\end{aligned} \tag{44}$$

Choosing $k = 0$ gives

$$t_2 = 2\phi - t_1. \tag{45}$$

Then,

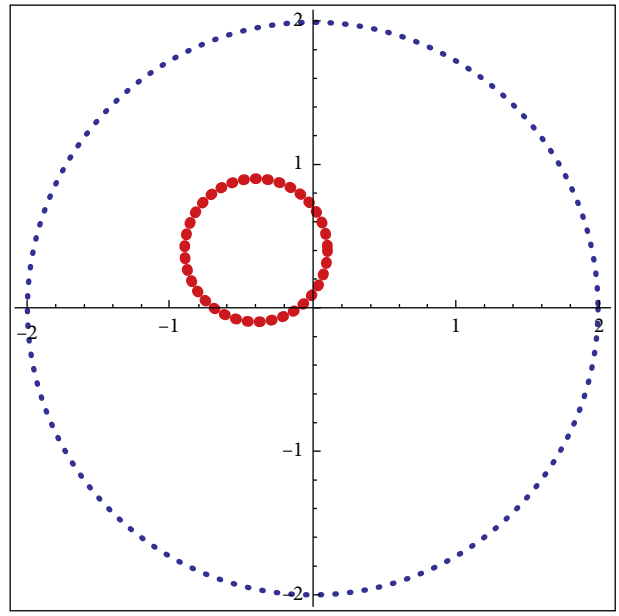

Figure 6: Nonconcentric circles region for Example 4.

$$\sin(\phi - t_2) = \sin(\phi - (2\phi - t_1)) = -\sin(\phi - t_1). \tag{46}$$

Using (42) and (46) into (40) and (41) gives

$$\begin{aligned}&\frac{s\sin(\theta - t_1)}{1 - 2s\cos(\theta - t_1) + s^2} - \frac{s\sin(\theta - t_2)}{1 - 2s\cos(\theta - t_2) + s^2}\\ &\quad = -\frac{2r\sin(\phi - t_1)}{1 - 2r\cos(\phi - t_1) + r^2},\end{aligned} \tag{47}$$

$$\frac{1 - s\cos(\theta - t_1)}{1 - 2s\cos(\theta - t_1) + s^2} = \frac{1 - s\cos(\theta - t_2)}{1 - 2s\cos(\theta - t_2) + s^2}. \tag{48}$$

Simplifying (48) to solve θ, we get

$$\begin{aligned}&[1 - s\cos(\theta - t_1)]\left[1 - 2s\cos(\theta - t_2) + s^2\right]\\ &\quad = [1 - s\cos(\theta - t_2)]\left[1 - 2s\cos(\theta - t_1) + s^2\right],\\ &s\left(s^2 - 1\right)[\cos(\theta - t_1) - \cos(\theta - t_2)] = 0.\end{aligned} \tag{49}$$

Since $s \neq 0$ and $s^2 - 1 \neq 0$, we have

$$\cos(\theta - t_1) - \cos(\theta - t_2) = 0. \tag{50}$$

Solving (50) and applying (45) gives

$$\begin{aligned}\frac{\theta - t_1 + \theta - t_2}{2} &= \pi k,\\ 2\theta - t_1 - (2\phi - t_1) &= 2\pi k,\\ \theta &= \phi + \pi k,\\ k &= 0, \pm 1, \ldots.\end{aligned} \tag{51}$$

Choosing $k = \pm 1$ gives

$$\theta = \phi \pm \pi. \tag{52}$$

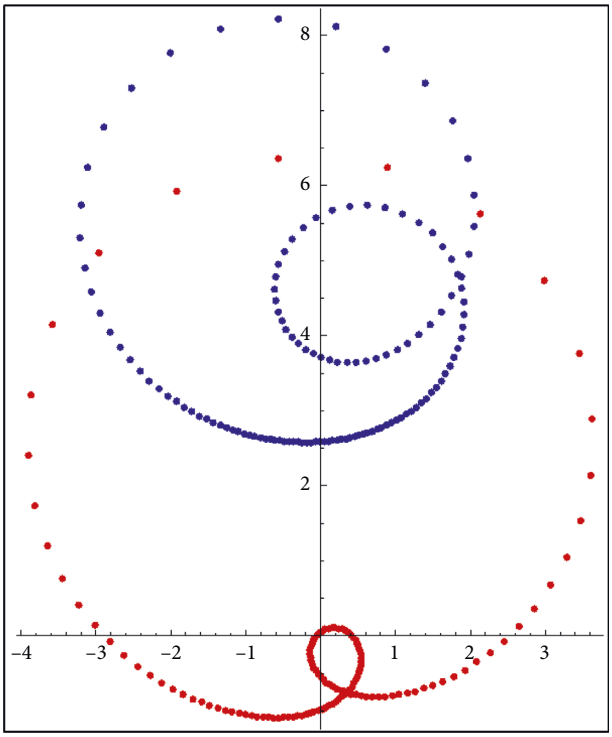

Figure 7: Graphs $h_0(t)$ (top curve) and $h_1(t)$ (bottom curve) for Example 4.

Then, for $p = 1, 2$, we have

$$\begin{aligned}\sin(\theta - t_p) &= \sin[(\phi \pm \pi) - t_p] \\ &= \sin(\phi \pm \pi)\cos t_p - \cos(\phi \pm \pi)\sin t_p \\ &= -\sin(\phi - t_p),\end{aligned} \tag{53}$$

$$\begin{aligned}\cos(\theta - t_p) &= \cos[(\phi \pm \pi) - t_p] \\ &= \cos(\phi \pm \pi)\cos t_p + \sin(\phi \pm \pi)\sin t_p \\ &= -\cos(\phi - t_p).\end{aligned} \tag{54}$$

Applying (46), (50), (53), and (54) into (47), we get

$$\begin{aligned}&-\frac{s\sin(\phi - t_1)}{1 + 2s\cos(\phi - t_1) + s^2} + \frac{s\sin(\phi - t_2)}{1 + 2s\cos(\phi - t_2) + s^2} \\ &\quad = -\frac{2r\sin(\phi - t_1)}{1 - 2r\cos(\phi - t_1) + r^2}, \\ &-\frac{s\sin(\phi - t_1)}{1 + 2s\cos(\phi - t_1) + s^2} - \frac{s\sin(\phi - t_1)}{1 + 2s\cos(\phi - t_1) + s^2} \\ &\quad = -\frac{2r\sin(\phi - t_1)}{1 - 2r\cos(\phi - t_1) + r^2}, \\ &\frac{s}{1 + 2s\cos(\phi - t_1) + s^2} = \frac{r}{1 - 2r\cos(\phi - t_1) + r^2}.\end{aligned} \tag{55}$$

It shows that $s \neq r$. This implies

$$\cos(\phi - t_1) = \frac{(s - r)(1 - rs)}{4rs} \equiv A, \tag{56}$$

or

$$\frac{s - r}{4rs} = \frac{\cos(\phi - t_1)}{1 - rs}. \tag{57}$$

Case 2. $z_1(t) = \rho e^{-it}$, $t \in J_1$.

Let $z_1(t) = \rho e^{-it}$, $t \in J_1$. If $g_1(t)$ is periodic, then there exists $t_1^*, t_2^* \in [0, 2\pi]$, $t_1^* \neq t_2^*$, such that

$$g_1(t_1^*) = g_1(t_2^*). \tag{58}$$

This gives

$$\operatorname{Re} g_1(t_1^*) = \operatorname{Re} g_1(t_2^*), \tag{59}$$

$$\operatorname{Im} g_1(t_1^*) = \operatorname{Im} g_1(t_2^*). \tag{60}$$

From (59) and (60), respectively, we have

$$\begin{aligned}&2\rho\left[\frac{s\sin(\theta + t_1^*)}{\rho^2 - 2\rho s\cos(\theta + t_1^*) + s^2} + \frac{r\sin(\phi + t_1^*)}{\rho^2 - 2\rho r\cos(\phi + t_1^*) + r^2}\right] \\ &\quad = 2\rho\left[\frac{s\sin(\theta + t_2^*)}{\rho^2 - 2\rho s\cos(\theta + t_2^*) + s^2} + \frac{r\sin(\phi + t_2^*)}{\rho^2 - 2\rho r\cos(\phi + t_2^*) + r^2}\right],\end{aligned} \tag{61}$$

$$\begin{aligned}&-2\rho\left[\frac{\rho - s\cos(\theta + t_1^*)}{\rho^2 - 2\rho s\cos(\theta + t_1^*) + s^2} + \frac{\rho - r\cos(\phi + t_1^*)}{\rho^2 - 2\rho r\cos(\phi + t_1^*) + r^2}\right] \\ &\quad = -2\rho\left[\frac{\rho - s\cos(\theta + t_2^*)}{\rho^2 - 2\rho s\cos(\theta + t_2^*) + s^2}\right. \\ &\qquad \left. + \frac{\rho - r\cos(\phi + t_2^*)}{\rho^2 - 2\rho r\cos(\phi + t_2^*) + r^2}\right].\end{aligned} \tag{62}$$

From (61) and (62), we seek t_1^* and t_2^* such that

$$\cos(\phi + t_1^*) = \cos(\phi + t_2^*), \tag{63}$$

that is,

$$\begin{aligned}\cos(\phi + t_1^*) - \cos(\phi + t_2^*) &= 2\sin\left(\frac{\phi + t_1^* + \phi + t_2^*}{2}\right) \\ &\quad \cdot \sin\left(\frac{\phi + t_2^* - (\phi + t_1^*)}{2}\right).\end{aligned} \tag{64}$$

This implies

$$\begin{aligned}\frac{\phi + t_1^* + \phi + t_2^*}{2} &= \pi k, \\ t_2^* &= -2\phi - t_1^* + 2\pi k, \\ k &= 0, \pm 1, \ldots.\end{aligned} \tag{65}$$

Choosing $k = 0$ gives

$$t_2^* = -2\phi - t_1^*. \tag{66}$$

Then,

$$\sin\left(\phi + t_2^*\right) = \sin\left(\phi + \left(-2\phi - t_1^*\right)\right) = -\sin\left(\phi + t_1^*\right). \tag{67}$$

Since $\theta = \phi \pm \pi$, then for $p = 1, 2$, we have

$$\begin{aligned}\sin\left(\theta + t_p^*\right) &= \sin\left[(\phi \pm \pi) + t_p^*\right] \\ &= \sin(\phi \pm \pi)\cos t_p^* + \cos(\phi \pm \pi)\sin t_p^* \\ &= -\sin\left(\phi + t_p^*\right),\end{aligned} \tag{68}$$

$$\begin{aligned}\cos\left(\theta + t_p^*\right) &= \cos\left[(\phi \pm \pi) + t_p^*\right] \\ &= \cos(\phi \pm \pi)\cos t_p^* - \sin(\phi \pm \pi)\sin t_p^* \\ &= -\cos\left(\phi + t_p^*\right).\end{aligned} \tag{69}$$

Applying (63), (67), (68), and (69) into (61), we get

$$\begin{aligned}&\frac{s\sin\left(\theta + t_1^*\right)}{\rho^2 - 2\rho s\cos\left(\theta + t_1^*\right) + s^2} + \frac{r\sin\left(\phi + t_1^*\right)}{\rho^2 - 2\rho r\cos\left(\phi + t_1^*\right) + r^2} \\ &\quad = \frac{s\sin\left(\theta + t_2^*\right)}{\rho^2 - 2\rho s\cos\left(\theta + t_2^*\right) + s^2} + \frac{r\sin\left(\phi + t_2^*\right)}{\rho^2 - 2\rho r\cos\left(\phi + t_2^*\right) + r^2}, \\ &-\frac{s\sin\left(\phi + t_1^*\right)}{\rho^2 + 2\rho s\cos\left(\phi + t_1^*\right) + s^2} + \frac{r\sin\left(\phi + t_1^*\right)}{\rho^2 - 2\rho r\cos\left(\phi + t_1^*\right) + r^2} \\ &\quad = -\frac{s\sin\left(\phi + t_2^*\right)}{\rho^2 + 2\rho s\cos\left(\phi + t_2^*\right) + s^2} + \frac{r\sin\left(\phi + t_2^*\right)}{\rho^2 - 2\rho r\cos\left(\phi + t_2^*\right) + r^2}, \\ &-\frac{s\sin\left(\phi + t_1^*\right)}{\rho^2 + 2\rho s\cos\left(\phi + t_1^*\right) + s^2} + \frac{r\sin\left(\phi + t_1^*\right)}{\rho^2 - 2\rho r\cos\left(\phi + t_1^*\right) + r^2} \\ &\quad = \frac{s\sin\left(\phi + t_1^*\right)}{\rho^2 + 2\rho s\cos\left(\phi + t_1^*\right) + s^2} - \frac{r\sin\left(\phi + t_1^*\right)}{\rho^2 - 2\rho r\cos\left(\phi + t_1^*\right) + r^2}, \\ &\frac{s}{\rho^2 + 2\rho s\cos\left(\phi + t_1^*\right) + s^2} = \frac{r}{\rho^2 - 2\rho r\cos\left(\phi + t_1^*\right) + r^2}.\end{aligned} \tag{70}$$

It shows that $s \neq r$. This implies

$$\cos\left(\phi + t_1^*\right) = \frac{(s - r)\left(\rho^2 - rs\right)}{4\rho rs} \equiv B, \tag{71}$$

or

$$\frac{s - r}{4rs} = \frac{\rho\cos\left(\phi + t_1^*\right)}{\rho^2 - rs}. \tag{72}$$

Equating (56) and (71) and solving for s, we get

$$s = \frac{\rho(\rho A - B)}{r(A - \rho B)}. \tag{73}$$

Adding both sides of (56) and (71) gives

$$A + B = \frac{s - r}{4rs}\left(1 - rs + \frac{\rho^2 - rs}{\rho}\right). \tag{74}$$

Using (73) into (74), we obtain

$$\begin{aligned}&A + B = \frac{\left[\rho(\rho A - B) - r^2(A - B)\right](A + B)\left(1 - \rho^2\right)}{4\rho r(\rho A - B)(A - B)}, \\ &(A + B)\left[1 - \frac{\left(1 - \rho^2\right)}{4}\left(\frac{1}{r(A - \rho B)} - \frac{r}{\rho(\rho A - B)}\right)\right] = 0, \\ &(A + B)\left[1 - \frac{\left(1 - \rho^2\right)}{4\rho r}\left(\frac{A\left(\rho^2 - r^2\right) + \rho B\left(r^2 - 1\right)}{\rho\left(A^2 + B^2\right) - AB\left(1 + \rho^2\right)}\right)\right] = 0.\end{aligned} \tag{75}$$

Since $1 - (((1 - \rho^2)/4\rho r)\ (A(\rho^2 - r^2) + \rho B(r^2 - 1)/(\rho(A^2 + B^2) - AB(1 + \rho^2)))) \neq 0$, this implies

$$A + B = 0. \tag{76}$$

Applying (56) and (71) into (76) and solving for s, we get

$$\begin{aligned}&\frac{s\left(1 + r^2\right) - r\left(1 + s^2\right)}{4rs} + \frac{s\left(\rho^2 + r^2\right) - r\left(\rho^2 + s^2\right)}{4\rho rs} = 0, \\ &rs^2 - s\left(\rho + r^2\right) + \rho r = 0, \\ &s = \frac{\rho}{r}, r.\end{aligned} \tag{77}$$

Since $s \neq r$, we must have

$$s = \frac{\rho}{r}. \tag{78}$$

Thus $a_1 = se^{i\theta} = (\rho/r)e^{i(\phi\pm\pi)} = -(\rho/r)e^{i\phi} = -(\rho/\overline{a_0})$. This completes the proof of Theorem 2. □

Remark. Using $s = (\rho/r)$, (53), (54), and (56), it can be shown that $\operatorname{Re} g_0(t_1) = \operatorname{Re} g_0(t_2) = 0$, and similarly by using $s = (\rho/r)$, (68), (69), and (71), it can be shown that $\operatorname{Re} g_1(t_1^*) = \operatorname{Re} g_1(t_2^*) = 0$.

Now we are in the position to prove the characteristic C4. The following theorem shows that there exists q such that $h_0(t) - h_1(t + q) = 4i$.

Theorem 3. *Let $g(t)$ be defined in (33), where $g_0(t) = g(z_0(t)), g_1(s) = g(z_1(s)), z_0(t) = e^{it}, t \in J_0, z_1(s) = \rho e^{-is}$, $s \in J_1$, and $a_0 = re^{i\phi}$ be given. Then, there exists a constant q such that*

$$g_0(t) - g_1(t + q) = 4i, \tag{79}$$

which is equivalent to $h_0(t) - h_1(t + q) = 4i$.

Proof. Since $z_0(t) = e^{it}$, $t \in J_0$, (33) becomes

$$\begin{aligned}g_0(t) &= 2ie^{it}\left[\frac{1}{e^{it} - a_0} + \frac{1}{e^{it} - a_1}\right] \\ &= 2i\left[\frac{2e^{2it} - (a_0 + a_1)e^{it}}{e^{2it} - (a_0 + a_1)e^{it} + a_0 a_1}\right].\end{aligned} \tag{80}$$

Since $z_1(s) = \rho e^{-is}$, $s \in J_1$, (33) becomes

$$g_1(s) = -2i\rho e^{-is}\left[\frac{1}{\rho e^{-is} - a_0} + \frac{1}{\rho e^{-is} - a_1}\right]. \tag{81}$$

Letting $s = t + q$, we get

$$\begin{aligned} g_1(t+q) &= -2i\rho e^{-i(t+q)}\left[\frac{1}{\rho e^{-i(t+q)} - a_0} + \frac{1}{\rho e^{-i(t+q)} - a_1}\right] \\ &= -2i\rho\left[\frac{2\rho - (a_0 + a_1)e^{i(t+q)}}{\rho^2 - \rho(a_0 + a_1)e^{i(t+q)} + a_0 a_1 e^{2i(t+q)}}\right] \\ &= \frac{-2i\rho}{a_0 a_1 e^{2iq}} \\ &\quad \cdot \left[\frac{2\rho - (a_0 + a_1)e^{i(t+q)}}{(\rho^2/a_0 a_1 e^{2iq}) - (\rho(a_0 + a_1)e^{i(t-q)}/a_0 a_1) + e^{2it}}\right]. \end{aligned} \tag{82}$$

Comparing the denominators (80) and (82), we seek q such that

$$1 = \frac{\rho e^{-iq}}{a_0 a_1}, \tag{83}$$

$$a_0 a_1 = \frac{\rho^2}{a_0 a_1 e^{2iq}}. \tag{84}$$

Observe that (84) is just the square of (83). Note that Theorem 2 gives $a_1 = -(\rho/\overline{a_0})$.

Hence, (83) becomes

$$e^{iq} = \frac{\rho}{a_0(-(\rho/\overline{a_0}))} = -\frac{\overline{a_0}}{a_0} = -\frac{re^{-i\phi}}{re^{i\phi}} = -e^{-2i\phi} = e^{-2i\phi}e^{\pm i\pi}. \tag{85}$$

This yields

$$q = -2\phi \pm \pi. \tag{86}$$

Thus, (82) simplifies to

$$g_1(t+q) = -2i\left[\frac{2a_0 a_1 - (a_0 + a_1)e^{it}}{e^{2it} - (a_0 + a_1)e^{it} + a_0 a_1}\right]. \tag{87}$$

Hence,

$$\begin{aligned} g_0(t) - g_1(t+q) &= 2i\left[\frac{2e^{2it} - (a_0 + a_1)e^{it}}{e^{2it} - (a_0 + a_1)e^{it} + a_0 a_1}\right] \\ &\quad + 2i\left[\frac{2a_0 a_1 - (a_0 + a_1)e^{it}}{e^{2it} - (a_0 + a_1)e^{it} + a_0 a_1}\right] \\ &= \frac{4i}{e^{2it} - (a_0 + a_1)e^{it} + a_0 a_1} \\ &\quad \cdot \left[e^{2it} - (a_0 + a_1)e^{it} + a_0 a_1\right] \\ &= 4i. \end{aligned} \tag{88}$$

Since $h(t) = g(t)$, $h_0(t) - h_1(t+q) = 4i$. This completes the proof of Theorem 3. □

5. Conclusion

In this paper, we have presented a boundary integral equation related to the Ahlfors map for a doubly connected region and an analytical method for finding the second zero of the Ahlfors map for an annulus region. The method does not depend on the series representation of the Szegö kernel. The method is based on some geometrical properties and knowledge of intersection points. Probably the method can be extended to some other doubly connected regions.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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