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Research Article

A Halpern-Type Iteration Method for Bregman Nonspreading Mapping and Monotone Operators in Reflexive Banach Spaces

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In this paper, we introduce an iterative method for approximating a common solution of monotone inclusion problem and fixed point of Bregman nonspreading mappings in a reflexive Banach space. Using the Bregman distance function, we study the composition of the resolvent of a maximal monotone operator and the antiresolvent of a Bregman inverse strongly monotone operator and introduce a Halpern-type iteration for approximating a common zero of a maximal monotone operator and a Bregman inverse strongly monotone operator which is also a fixed point of a Bregman nonspreading mapping. We further state and prove a strong convergence result using the iterative algorithm introduced. This result extends many works on finding a common solution of the monotone inclusion problem and fixed-point problem for nonlinear mappings in a real Hilbert space to a reflexive Banach space.

1. Introduction

Let E be a real reflexive Banach space with a norm $\|\cdot\|$ and E^* be the dual space of E. We denote the value of $x^* \in E^*$ at $x \in E$ by $\langle x^*, x\rangle$. A mapping A is called a monotone mapping if for any $x, y \in \operatorname{dom} A$, we have

$$\begin{aligned} &\mu \in Ax, \\ &\nu \in Ay \Longrightarrow \langle \mu - \nu, x - y\rangle \geq 0. \end{aligned} \tag{1}$$

A monotone mapping $A : E \longrightarrow 2^{E^*}$ is said to be maximal monotone if its graph, $G(A) := \{(x,u) \in E \times E^* : u \in Ax\}$, is not properly contained in the graph of any other monotone operator. A basic problem that arises in several branches of applied mathematics [1–7] is to find $x \in E$ such that

$$0 \in Ax. \tag{2}$$

One of the methods for solving this problem is the well-known proximal point algorithm (PPA) introduced by Martinet [8]. Let H be a Hilbert space and let I denote the identity operator on H. The PPA generates for any starting point $x_0 = x \in H$, a sequence $\{x_n\}$ in H by

$$x_{n+1} = (I + \lambda_n A)^{-1} x_n, \quad n = 1, 2, \ldots, \tag{3}$$

where A is a maximal monotone mapping and $\{\lambda_n\}$ is a given sequence of positive real numbers. It has been observed that (3) is equivalent to

$$0 \in Ax_{n+1} + \frac{1}{\lambda_n}(x_{n+1} - x_n), \quad n = 1, 2, \ldots. \tag{4}$$

This algorithm was further developed by Rockafellar [5], who proved that the sequence generated by (3) converges weakly to an element of $A^{-1}(0)$ when $A^{-1}(0)$ is nonempty and $\liminf_{n\longrightarrow\infty}\lambda_n > 0$. Furthermore, Rockafellar [5] asked if the sequence generated by (3) converges strongly in general. This question was answered in the negative by Güler [9] who presented an example of a subdifferential for which the sequence generated by (3) converges weakly but not strongly. Also, the works of Bruck and Reich [10] and

Bauschke et al. [11] are very important in this direction. For more recent results on PPA, see [12–14].

The problem of finding the zeros of the sum of two monotone mappings A and B, is to find a point $x^* \in E$ such that

$$0 \in (A+B)x^*, \tag{5}$$

has recently received attention due to its significant importance in many physical problems. One classical method for solving problem (5) is the forward-backward splitting method [15], which is as follows: for $x_1 \in E$,

$$x_{n+1} = (I + rB)^{-1}(x_n - rAx_n), \quad n \geq 1, \tag{6}$$

where $r > 0$. This method combines the proximal point algorithm and the gradient projection algorithm. In [16], Lions and Mercier introduced the following splitting iterative methods in a real Hilbert space H:

$$\begin{aligned} x_{n+1} &= \left(2J_r^A - I\right)\left(2J_r^B - I\right)x_n, \quad n \geq 1, \\ x_{n+1} &= J_r^A\left(2J_r^B - I\right)x_n + \left(I - J_r^B\right)x_n, \quad n \geq 1, \end{aligned} \tag{7}$$

where $J_r^T = (I + rT)^{-1}$. The first one is called Peaceman–Rachford algorithm and the second one is called Douglas–Rachford algorithm [15]. It was noted that both algorithms converge weakly in general [16, 17].

Many authors have studied the approximation of zero of the sum of two monotone operators (in Hilbert space) and accretive operators (in Banach spaces), but the approximation of the sum of two monotone operators in more general Banach spaces other the Hilbert spaces has not enjoyed such popularity.

Throughout this paper, $f : E \longrightarrow (-\infty, +\infty]$ is a proper lower semicontinuous and convex function, and the Fenchel conjugate of f is the function $f^* : E^* \longrightarrow (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}. \tag{8}$$

We denote by dom$f$ the domain of f, that is, the set $\{x \in E : f(x) < +\infty\}$. For any $x \in \operatorname{intdom} f$ and $y \in E$, the right-hand derivative of f at x in the direction of t is defined by

$$f^o(x, y) \coloneqq \lim_{t \longrightarrow 0^+} \frac{f(x + ty) - f(x)}{t}. \tag{9}$$

The function f is said to be Gâteaux differentiable at x if the limit as $t \longrightarrow 0^+$ in (9) exists for any y. In this case, $f^o(x, y)$ coincides with $\nabla f(x)$, the value of the gradient ∇f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $x \in \operatorname{intdom} f$. The function f is Fréchet differentiable at x if the limit is attained with $\|y\| = 1$ and uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $x \in C$ and $\|y\| = 1$.

The function f is said to be Legendre if it satisfies the following two conditions:

(L1) $\operatorname{intdom} f \neq \emptyset$ and the subdifferential ∂f is single-valued in its domain

(L2) $\operatorname{tdom} f^* \neq \emptyset$ and ∂f^* is single-valued on its domain

The class of Legendre functions in infinite dimensional Banach spaces was first introduced and studied by Bauschke et al. in [18]. Their definition is equivalent to conditions (L1) and (L2) because the space E is assumed to be reflexive (see [18], Theorems 5.4 and 5.6, p. 634). It is well known that in reflexive Banach spaces, $\nabla f = (\nabla f^*)^{-1}$ (see [19], p. 83). When this fact is combined with conditions (L1) and (L2), we obtain

$$\begin{aligned} \operatorname{ran}\nabla f &= \operatorname{dom}\nabla f^* = \operatorname{int}(\operatorname{dom} f)^*, \\ \operatorname{ran}\nabla f^* &= \operatorname{dom}\nabla f = \operatorname{int}(\operatorname{dom} f). \end{aligned} \tag{10}$$

It also follows that f is Legendre if and only if f^* is Legendre (see [18], Corollary 5.5, p. 634) and that the functions f and f^* are Gâteaux differentiable and strictly convex in the interior of their respective domains.

Several interesting examples of the Legendre functions are presented in [18, 20, 21]. A very important example of Legendre function is the function $1/s\|\cdot\|^s$ with $s \in (1, \infty)$, where the Banach space E is smooth and strictly convex, and in particular, a Hilbert space. Throughout this article, we assume that the convex function $f : E \longrightarrow (-\infty, +\infty]$ is Legendre.

Definition 1. Let $f : E \longrightarrow (-\infty, +\infty]$ be a convex and Gâteaux differentiable function, the function $D_f : \operatorname{dom} f \times \operatorname{intdom} f \longrightarrow [0, \infty)$ which is defined by

$$D_f(y, x) \coloneqq f(y) - f(x) - \langle \nabla f(x), y - x\rangle, \tag{11}$$

is called the Bregman distance [22–24].

The Bregman distance does not satisfy the well-known metric properties, but it does have the following important property, which is called the three-point identity: for any $x \in \operatorname{dom} f$ and $y, z \in \operatorname{intdom} f$,

$$D_f(x, y) + D_f(y, z) - D_f(x, z) = \langle \nabla f(z) - \nabla f(y), x - y\rangle. \tag{12}$$

Let C be a nonempty subset of a Banach space E and $T : C \longrightarrow C$ be a mapping, then a point x is called fixed point of T if $Tx = x$. The set of fixed point of T is denoted by $F(T)$. Also, a point $x^* \in C$ is said to be an asymptotic fixed point of T if C contains a sequence $\{x_n\}_{n=1}^\infty$ which converges weakly to x^* and $\lim_{n\longrightarrow\infty}\|x_n - Tx_n\| = 0$ [25]. The set of asymptotic fixed points of T is denoted by $\widehat{F}(T)$.

Definition 2 [26, 27]. Let C be a nonempty, closed, and convex subset of E. A mapping $T : C \longrightarrow \operatorname{int}(\operatorname{dom} f)$ is called

(i) Bregman firmly nonexpansive (BFNE for short) if

$$\begin{aligned} &\langle \nabla f(Tx) - \nabla f(Ty), Tx - Ty\rangle \\ &\leq \langle \nabla f(x) - \nabla f(y), Tx - Ty\rangle, \quad \forall x, y \in C. \end{aligned} \tag{13}$$

(ii) Bregman strongly nonexpansive (BSNE) with respect to a nonempty $\widehat{F}(T)$ if

$$D_f(p, Tx) \le D_f(p, x), \quad (14)$$

for all $p \in \widehat{F}(T)$ and $x \in C$ and if whenever $\{x_n\}_{n=1}^{\infty} \subset C$ is bounded, $p \in \widehat{F}(T)$ and

$$\lim_{n \longrightarrow \infty} \left(D_f(p, x_n) - D_f(p, Tx_n) \right) = 0, \quad (15)$$

it follows that

$$\lim_{n \longrightarrow \infty} D_f(Tx_n, x_n) = 0. \quad (16)$$

(iii) Bregman quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \le D_f(p, x), \quad \forall x \in C \text{ and } p \in F(T). \quad (17)$$

(iv) Bregman skew quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$D_f(Tx, p) \le D_f(x, p), \quad \forall x \in C \text{ and } p \in F(T). \quad (18)$$

(v) Bregman nonspreading if

$$\begin{aligned} D_f(Tx, Ty) + D_f(Ty, Tx) &\le D_f(Tx, y) \\ &+ D_f(Ty, x), \quad \forall x, y \in C. \end{aligned} \quad (19)$$

It is easy to see that every Bregman nonspreading mapping T with $F(T) \neq \emptyset$ is Bregman quasi-nonexpansive. Also Bregman nonspreading mappings include, in particular, the class of nonspreading functions studied by Takahashi et al. in [28, 29]. For more information on Bregman nonspreading mappings, see [30].

In a real Hilbert space H, the nonlinear mapping $T : C \longrightarrow C$ is said to be

(i) Nonexpansive if

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in C. \quad (20)$$

(ii) Quasi-nonexpansive if $F(T) \neq \emptyset$ and

$$\|Tx - p\| \le \|x - p\|, \quad \forall x \in C \text{ and } p \in F(T). \quad (21)$$

(iii) Nonspreading if

$$2\|Tx - Ty\|^2 \le \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C. \quad (22)$$

Clearly, every nonspreading mapping T with $F(T) \neq \emptyset$ is also quasi-nonexpansive mapping. The class of nonspreading mappings is very important due to its relation with maximal monotone operators (see, e.g., [28]).

Let $B : E \longrightarrow 2^{E^*}$ be a maximal monotone operator. The resolvent of B, $\mathrm{Res}_B^f : E \longrightarrow 2^E$, is defined by (see [26])

$$\mathrm{Res}_B^f := (\nabla f + B)^{-1} \circ \nabla f. \quad (23)$$

It is known that Res_B^f is a BFNE operator, single-valued, and $F(\mathrm{Res}_B^f) = B^{-1}(0^*)$ (see [26]). If $f : E \longrightarrow \mathbb{R}$ is a Legendre function which is bounded, uniformly Fréchet differentiable on bounded subsets of E, then Res_B^f is BSNE and $\widehat{F}(\mathrm{Res}_B^f) = F(\mathrm{Res}_B^f)$ (see [31]).

Assume that the Legendre function f satisfies the following range condition:

$$\mathrm{ran}(\nabla f - A) \subseteq \mathrm{ran} \nabla f. \quad (24)$$

An operator $A : E \longrightarrow 2^{E^*}$ is called Bregman inverse strongly monotone (BISM) if $(\mathrm{dom} A)$ $(\mathrm{dom} f)$, and for any $x, y \in \mathrm{int dom} f$ and each $u \in Ax$ and $v \in Ay$, we have

$$\langle u - v, \nabla f^*(\nabla f(x) - u) - \nabla f^*(\nabla f(y) - v) \rangle \ge 0. \quad (25)$$

The class of BISM mappings is a generalization of the class of firmly nonexpansive mappings in Hilbert spaces. Indeed, if $f = 1/2\|\cdot\|^2$, then $\nabla f = \nabla f^* = I$, where I is the identity operator and (25) becomes

$$\langle u - v, x - u - (y - v) \rangle \ge 0, \quad (26)$$

which means

$$\|u - v\|^2 \le \langle x - y, u - v \rangle. \quad (27)$$

Observe that

$$\begin{aligned} \mathrm{dom} A^f &= (\mathrm{dom} A) \cap (\mathrm{int dom} f), \\ \mathrm{ran} A^f &\subset \mathrm{int dom} f. \end{aligned} \quad (28)$$

In other words, T is a (single-valued) firmly nonexpansive operator.

For any operator $A : E \longrightarrow 2^{E^*}$, the antiresolvent operator $A^f : E \longrightarrow 2^E$ of A is defined by

$$A^f := \nabla f^* \circ (\nabla f - A). \quad (29)$$

It is known that the operator A is BISM if and only if the antiresolvent A^f is a single-valued BFNE (see [32], Lemma 3.2(c) and (d), p. 2109) and $F(A^f) = A^{-1}(0^*)$. For examples and further information on BISM, see [32].

Since the monotone inclusion problems have very close connections with both the fixed-point problems and the equilibrium problems, finding the common solutions of these problems has drawn many people's attention and has become one of the hot topics in the related fields in the past few years [33, 34]. Furthermore, interest in finding the common solution of these problems has also grown because of the possible application of these problems to mathematical models whose constraints can be present as fixed points of mappings and/or monotone inclusion problems and/or equilibrium problems. Such a problem occurs, in particular, in the practical problems as signal processing, network resource allocation, and image recovery (see [35, 36]).

In this paper, we introduce an iterative method for approximating a common solution of monotone inclusion problem and fixed point of Bregman nonspreading mapping in a reflexive Banach space and prove a strong convergence of the sequence generated by our iterative algorithm. This result extends many works on finding common solution of monotone inclusion problem and fixed problem of nonlinear mapping in a real Hilbert space to a reflexive Banach space.

2. Preliminaries

The Bregman projection [22] of $x \in \operatorname{int}(\operatorname{dom} f)$ onto the nonempty, closed, and convex subset $C \subset \operatorname{int}(\operatorname{dom} f)$ is defined as the necessarily unique vector $\operatorname{Proj}_C^f(x) \in C$ satisfying

$$D_f\left(\operatorname{Proj}_C^f(x), x\right) = \inf\left\{D_f(y, x) : y \in C\right\}. \tag{30}$$

It is known from [37] that $z = \operatorname{Proj}_C^f(x)$ if and only if

$$\langle \nabla f(x) - \nabla f(z), y - z\rangle \leq 0, \quad \text{for all } y \in C. \tag{31}$$

We also have

$$\begin{aligned} &D_f\left(y, \operatorname{Proj}_C^f(x)\right) + D_f\left(\operatorname{Proj}_C^f(x), x\right) \\ &\leq D_f(y, x), \quad \text{for all } x \in E, y \in C. \end{aligned} \tag{32}$$

Note that if E is a Hilbert space and $f(x) = 1/2\|x\|^2$, then the Bregman projection of x onto C, i.e., $\arg\min\{\|y - x\| : y \in C\}$, is the metric projection P_C.

Lemma 1 [37]. *Let f be totally convex on int (domf). Let C be a nonempty, closed, and convex subset of int (domf) and* $x \in \operatorname{int}(\operatorname{dom} f)$; *if* $z \in C$, *then the following conditions are equivalent:*

(i) $z = Proj_C^f(x)$

(ii) $\langle \nabla f(x) - \nabla f(z), z - y\rangle \geq 0$ *for all* $y \in C$

(iii) $D_f(y, z) + D_f(z, x) \leq D_f(y, x)$ *for all* $y \in C$

Let $f : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a convex and Gâteaux differentiable function. The function f is said to be totally convex at $x \in \operatorname{int}\operatorname{dom} f$ if its modulus of totally convexity at x, that is, the function $v_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty)$ defined by

$$v_f(x, t) \coloneqq \inf\left\{D_f(y, x) : y \in \operatorname{dom} f, \|y - x\| = t\right\}, \tag{33}$$

is positive for any $t > 0$. The function f is said to be totally convex when it is totally convex at every point $x \in \operatorname{int}(\operatorname{dom} f)$. In addition, the function f is said to be totally convex on bounded set if $v_f(B, t)$ is positive for any nonempty bounded subset B, where the modulus of total convexity of the function f on the set B is the function $v_f : \operatorname{int}(\operatorname{dom} f) \times [0, +\infty)$ defined by

$$v_f(B, t) \coloneqq \inf\left\{v_f(x, t) : x \in B \cap \operatorname{dom} f\right\}. \tag{34}$$

For further details and examples on totally convex functions, see [37–39].

Let $f : E \longrightarrow \mathbb{R}$ be a convex, Legendre, and Gâteaux differentiable function and let the function $V_f : E \times E^* \longrightarrow [0, \infty)$ associated with f (see [23, 40]) be defined by

$$V_f(x, x^*) = f(x) - \langle x^*, x\rangle + f^*(x^*), \quad \forall x \in E, x^* \in E^*. \tag{35}$$

Then V_f is nonnegative and $V_f(x, x^*) = D_f(x, \nabla f^*(x^*))$, $\forall x \in E, x^* \in E^*$. Furthermore, by the subdifferential inequality, we have (see [41])

$$\begin{aligned} &V_f(x, x^*) + \langle y^*, \nabla f^*(x^*) - x\rangle \\ &\leq V_f(x, x^* + y^*), \quad \forall x \in E, x^*, y^* \in E^*. \end{aligned} \tag{36}$$

In addition, if $f : E \longrightarrow (-\infty, +\infty]$ is a proper lower semicontinuous function, then $f^* : E^* \longrightarrow (-\infty, +\infty]$ is a proper *weak** lower semicontinuous and convex function (see [42]). Hence, V_f is convex in the second variable. Thus, for all $z \in E$,

$$D_f\left(z, \nabla f^* \sum_{i=1}^N t_i \nabla f(x_i)\right) \leq \sum_{i=1}^N t_i D_f(z, x_i), \tag{37}$$

where $\{x_i\}_{i=1}^N \subset E$ and $\{t_i\} \subset (0, 1)$ with $\sum_{i=1}^N t_i = 1$.

Lemma 2 (see [43]). *Let* $r > 0$ *be a constant and let* $f : E \longrightarrow \mathbb{R}$ *be a continuous uniformly convex function on bounded subsets of E. Then*

$$f\left(\sum_{k=0}^{\infty} \alpha_k x_k\right) \leq \sum_{k=0}^{\infty} \alpha_k f(x_k) - \alpha_i \alpha_j \rho_r\left(\left\|x_i - x_j\right\|\right), \tag{38}$$

for all $i, j \in \mathbb{N} \cup 0$, $x_k \in B_r$, $\alpha_k \in (0, 1)$, *and* $k \in \mathbb{N} \cup 0$ *with* $\sum_{k=0}^{\infty} \alpha_k = 1$, *where* ρ_r *is the gauge of uniform convexity of f.*

Recall that a function f is said to be sequentially consisted (see [37]) if for any two sequences $\{x_n\}$ and $\{y_n\}$ in E such that the first one is bounded,

$$\lim_{n \longrightarrow \infty} D_f(y_n, x_n) = 0 \Longrightarrow \lim_{n \longrightarrow \infty} \left\|y_n - x_n\right\| = 0. \tag{39}$$

The following lemma follows from [44].

Lemma 3. *If domf contains at least two points, then the function f is totally convex on bounded sets if and only if the function f is sequentially consistent.*

Lemma 4 (see [45]). *Let* $f : E \longrightarrow (-\infty, +\infty]$ *be a Legendre function and let* $A : E \longrightarrow 2^{E^*}$ *be a BISM operator such that* $A^{-1}(0^*) \neq \varnothing$. *Then the following statements hold:*

(i) $A^{-1}(0^*) = F(A^f)$

(ii) For any $w \in A^{-1}(0^*)$ and $x \in \operatorname{dom} A^f$, we have

$$D_f\left(w, A^f x\right) + D_f\left(A^f x, x\right) \leq D_f(w, x). \tag{40}$$

Remark 1. If the Legendre function f is uniformly Fréchet differentiable and bounded on bounded subsets of E, then the antiresolvent A^f is a single-valued BSNE operator which satisfies $F(A^f) = \widehat{F}(A^f)$ (cf. [31]).

Lemma 5 (see [46]). *If $f : E \longrightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.*

Lemma 6 (see [44]). *Let $f : E \longrightarrow \mathbb{R}$ be a Gâteaux differentiable and totally convex function. If $x_1 \in E$ and the sequence $\{D_f(x_n, x_1)\}$ is bounded, then the sequence $\{x_n\}$ is also bounded.*

Lemma 7 (see [45]). *Assume that $f : E \longrightarrow \mathbb{R}$ is a Legendre function which is uniformly Fréchet differentiable and bounded on bounded subset of E. Let C be a nonempty, closed, and convex subset of E. Let $\{T_i : 1 \le i \le N\}$ be BSNE operators which satisfy $\widehat{F}(T_i) = F(T_i)$ for each $1 \le i \le N$ and let $T := w_n T_{N-1} \ldots T_1$. If*

$$\cap \{F(T_i) : 1 \le i \le N\}, \tag{41}$$

and $F(T)$ are nonempty, then T is also BSNE with $F(T) = \widehat{F}(T)$.

Lemma 8 (Demiclosedness principle [30]). *Let C be a nonempty subset of a reflexive Banach space. Let $g : E \longrightarrow \mathbb{R}$ be a strict convex, Gâteaux differentiable, and locally bounded function. Let $T : C \longrightarrow E$ be a Bregman nonspreading mapping. If $x_n \rightharpoonup p$ in C and $\lim_{n\longrightarrow\infty} \|Tx_n - x_n\| = 0$, then $p \in F(T)$.*

Lemma 9 (see [47]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers satisfying*

$$a_{n+1} \le (1 - t_n)a_n + t_n \delta_n \forall n \ge 0, \tag{42}$$

where $\{t_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(i) $\sum_{n=o}^{\infty} t_n = \infty$

(ii) $\limsup_{n\longrightarrow\infty} \delta_n \le 0$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Lemma 10 [48]. *Let $\{a_n\}$ be a sequence of real numbers such that there exists a nondecreasing subsequence $\{n_i\}$ of $\{n\}$, that is, $a_{n_i} \le a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a nondecreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \longrightarrow \infty$, and the following properties are satisfied for all (sufficiently large number $k \in \mathbb{N}$): $a_{m_k} \le a_{m_k+1}$ and $a_k \le a_{m_k+1}$, $m_k = \max\{j \le k : a_j \le a_{j+1}\}$.*

3. Main Results

Theorem 1. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f : E \longrightarrow \mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $A : E \longrightarrow 2^{E^*}$ be a Bregman inverse strongly monotone operator, $B : E \longrightarrow 2^{E^*}$ be a maximal monotone operator, and $T : C \longrightarrow C$ be a Bregman nonspreading mapping. Suppose $\Gamma := F(\mathrm{Res}_A^f \circ A^f) \cap F(T) \ne \emptyset$. Let $\{\gamma_n\} \subset (0,1)$ and $\{\alpha_n\}, \beta_n$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n + \beta_n + \delta_n = 1$. Given $u \in E$ and $x_1 \in C$ arbitrarily, let $\{x_n\}$ and $\{y_n\}$ be sequences in E generated by*

$$\begin{cases} y_n = \nabla f^*(\gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)), \\ x_{n+1} = \mathrm{Proj}_C^f\left(\nabla f^*\left(\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f(y_n)\right)\right)\right), \quad n \ge 1. \end{cases} \tag{43}$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\longrightarrow\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$

(ii) $(1-\alpha_n)a < \delta_n$, $\alpha_n \le b < 1$, $a \in (0, 1/2)$

(iii) $0 \le c < \liminf_{n\longrightarrow\infty} \gamma_n \le \limsup_{n\longrightarrow\infty} \gamma_n < 1$

Then $\{x_n\}$ converges strongly to $\mathrm{Proj}_\Gamma^f u$, where Proj_Γ^f is the Bregman projection of E onto Γ.

Proof. First we observe that $F(\mathrm{Res}_B^f \circ A^f) = (A+B)^{-1}0$ and $F(\mathrm{Res}_B^f \circ A^f) = F(\mathrm{Res}_B^f) \cap F(A^f)$. Thus, since Res_B^f and A^f are BSNE operators and $F(\mathrm{Res}_B^f) \cap F(A^f) = (A+B)^{-1}0 \ne \emptyset$, it then follows from Lemma 7 that $\mathrm{Res}_B^f \circ A^f$ is BSNE and $F(\mathrm{Res}_B^f \circ A^f) = \widehat{F}(\mathrm{Res}_B^f \circ A^f)$.

We next show that $\{x_n\}$ and $\{y_n\}$ are bounded.

Let $p \in \Gamma$, then from (43), we have

$$\begin{aligned} D_f(p, y_n) &= D_f(p, \nabla f^*(\gamma_n \nabla f(x_n)) + (1-\gamma_n)\nabla f(Tx_n)) \\ &\le \gamma_n D_f(p, x_n) + (1-\gamma_n) D_f(p, Tx_n) \\ &\le \gamma_n D_f(p, x_n) + (1-\gamma_n) D_f(p, x_n) \\ &= D_f(p, x_n). \end{aligned} \tag{44}$$

Also

$$\begin{aligned} D_f(p, x_{n+1}) &\le D_f\Big(p, \nabla f^*\Big[\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) \\ &\quad + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f(y_n)\right)\Big]\Big) \\ &\le \alpha_n D_f(p, u) + \beta_n D_f(p, y_n) \\ &\quad + \delta_n D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \\ &\le \alpha_n D_f(p, u) + \beta_n D_f(p, y_n) + \delta_n D_f(p, y_n) \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, y_n) \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, x_n) \\ &\le \max\{D_f(p, u), D_f(p, x_n)\} \\ &\quad \vdots \\ &\le \max\{D_f(p, u), D_f(p, x_1)\}. \end{aligned} \tag{45}$$

Hence $\{D_f(p, x_n)\}$ is bounded. Therefore, by Lemma 6, $\{x_n\}$ is also bounded, and consequently, $\{y_n\}$ is also bounded.

We now show that x_n converges strongly to $\overline{x} = \mathrm{Proj}_{\Gamma}^{f}(u)$. To do this, we first show that if there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup q \in C$, then $q \in \Gamma$.

Let $s = \sup\{\|\nabla f(x_n)\|, \|\nabla f(Tx_n)\|\}$ and $\rho_s^* : E^* \longrightarrow \mathbb{R}$ be the gauge of uniform convexity of the conjugate function f^*. From Lemma 2 and (9), we have

$$\begin{aligned}
D_f(p, y_n) &\leq D_f(p, \nabla f^*(\gamma_n \nabla f(x_n)) + (1-\gamma_n)\nabla f(Tx_n)) \\
&= V_f(p, \gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)) \\
&= f(p) - \langle p, \gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)\rangle \\
&\quad + f^*(\gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)) \\
&\leq \gamma_n f(p) - \gamma_n \langle p, \nabla f(x_n)\rangle + \gamma_n f^*(\nabla f(x_n)) \\
&\quad + (1-\gamma_n) f(p) - (1-\gamma_n)\langle p, \nabla f(Tx_n)\rangle \\
&\quad + (1-\gamma_n) f^*(\nabla f(Tx_n)) - \gamma_n(1-\gamma_n)\rho_s^* \\
&\quad \cdot \left(\|\nabla f(x_n) - \nabla f(Tx_n)\|\right) \\
&= \gamma_n D_f(p, x_n) + (1-\gamma_n) D_f(p, Tx_n) \\
&\quad - \gamma_n(1-\gamma_n)\rho_s^*\left(\|\nabla f(x_n) - \nabla f(Tx_n)\|\right) \\
&\leq D_f(p, x_n) - \gamma_n(1-\gamma_n)\rho_s^* \\
&\quad \cdot \left(\|\nabla f(x_n) - \nabla f(Tx_n)\|\right).
\end{aligned} \tag{46}$$

Thus, from (45), we have

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq \alpha_n D_f(p, u) + (1-\alpha_n) \\
&\quad \cdot \left[D_f(p, x_n) - \gamma_n(1-\gamma_n)\rho_s^*\left(\|\nabla f(x_n) - \nabla f(Tx_n)\|\right)\right].
\end{aligned} \tag{47}$$

We consider the following two cases for the rest of the proof. □

Case A. Suppose $\{D_f(p, x_n)\}$ is monotonically nonincreasing. Then, $\{D_f(p, x_n)\}$ converges and $D_f(p, x_n) - D_f(p, x_{n+1}) \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, from (47), we have

$$\begin{aligned}
&(1-\alpha_n)(1-\gamma_n)\gamma_n \rho_s^*\left(\|\nabla f(x_n) - \nabla f(Tx_n)\|\right) \\
&\leq \alpha_n\left(D_f(p, u) - D_f(p, x_n)\right) + D_f(p, x_n) - D_f(p, x_{n+1}).
\end{aligned} \tag{48}$$

Since $\alpha_n \longrightarrow 0$, $n \longrightarrow \infty$, then we have

$$\lim_{n\longrightarrow\infty} \gamma_n(1-\gamma_n)\rho_s^*\left(\|\nabla f(x_n) - \nabla f(Tx_n)\|\right) = 0, \tag{49}$$

and hence, by condition (iii) and the property of ρ_s^*, we have

$$\lim_{n\longrightarrow\infty} \|\nabla f(x_n) - \nabla f(Tx_n)\| = 0. \tag{50}$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^*, we have

$$\lim_{n\longrightarrow\infty} \|x_n - Tx_n\| = 0. \tag{51}$$

Again

$$\begin{aligned}
\|\nabla f(x_n) - \nabla f(y_n)\| &= \|\nabla f(x_n) - (\gamma_n \nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n))\| \\
&= (1-\gamma_n)\|\nabla f(x_n) - \nabla f(Tx_n)\| \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{52}$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we have that

$$\lim_{n\longrightarrow\infty} \|x_n - y_n\| = 0. \tag{53}$$

Now, let $w_n = \nabla f^*(\beta_n/1-\alpha_n \nabla f(y_n) + \delta_n/1-\alpha_n \nabla f(\mathrm{Res}_B^f \circ A^f y_n))$, then

$$\begin{aligned}
D_f(p, w_n) &= D_f\left(p, \nabla f^*\left[\frac{\beta_n}{1-\alpha_n}\nabla f(y_n) + \frac{\delta_n}{1-\alpha_n}\nabla f\left(\mathrm{Res}_B^f \circ A^f y_n\right)\right]\right) \\
&\leq \frac{\beta_n}{1-\alpha_n} D_f(p, y_n) + \frac{\delta_n}{1-\alpha_n} D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \\
&\leq \frac{\beta_n + \delta_n}{1-\alpha_n} D_f(p, y_n) \\
&= D_f(p, y_n).
\end{aligned} \tag{54}$$

Therefore, we have

$$\begin{aligned}
0 &\leq D_f(p, x_n) - D_f(p, w_n) \\
&= D_f(p, x_n) - D_f(p, x_{n+1}) + D_f(p, x_{n+1}) - D_f(p, w_n) \\
&\leq D_f(p, x_n) - D_f(p, x_{n+1}) + \alpha_n D_f(p, u) \\
&\quad + (1-\alpha_n) D_f(p, w_n) - D_f(p, w_n) \\
&= D_f(p, x_n) - D_f(p, x_{n+1}) \\
&\quad + \alpha_n\left[D_f(p, u) - D_f(p, w_n)\right] \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty.
\end{aligned} \tag{55}$$

More so

$$\begin{aligned}
D_f(p, w_n) &\leq \frac{\beta_n}{1-\alpha_n} D_f(p, y_n) + \frac{\delta_n}{1-\alpha_n} D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \\
&= D_f(p, y_n) - \left(1 - \frac{\beta_n}{1-\alpha_n}\right) D_f(p, y_n) \\
&\quad + \frac{\delta_n}{1-\alpha_n} D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \\
&\leq D_f(p, x_n) + \frac{\delta_n}{1-\alpha_n}\left[D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right)\right. \\
&\quad \left. - D_f(p, y_n)\right].
\end{aligned} \tag{56}$$

Since $(1-\alpha_n)a < \delta_n$ and $\alpha_n \leq b < 1$, we have

$$a\left(D_f(p, y_n) - D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right)\right) < \frac{\delta_n}{1-\alpha_n}\left[D_f(p, y_n) - D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right)\right] \le D_f(p, x_n) - D_f(p, w_n) \longrightarrow 0, \quad \text{as}\, n \longrightarrow \infty. \tag{57}$$

Thus,

$$D_f(p, y_n) - D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{58}$$

Therefore, since $\mathrm{Res}_B^f \circ A^f$ is BSNE, we have that $\lim_{n\longrightarrow\infty} D_f(y_n, \mathrm{Res}_B^f \circ A^f y_n) = 0$, which implies that

$$\lim_{n\longrightarrow\infty}\left\|y_n - \mathrm{Res}_B^f \circ A^f y_n\right\| = 0. \tag{59}$$

Setting $u_n = \nabla f^*[\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f(\mathrm{Res}_B^f \circ A^f y_n)]$, for each $n \ge 1$, we have

$$\begin{aligned} D_f(y_n, u_n) &= D_f\left(y_n, \nabla f^*\left[\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f y_n\right)\right]\right) \\ &\le \alpha_n D_f(y_n, u) + \beta_n D_f(y_n, y_n) + \delta_n D_f \cdot \left(y_n, \mathrm{Res}_B^f \circ A^f y_n\right) \longrightarrow 0. \end{aligned} \tag{60}$$

Thus,

$$\lim_{n\longrightarrow\infty}\left\|y_n - u_n\right\| = 0. \tag{61}$$

Therefore, from (47), we have

$$\left\|u_n - x_n\right\| \le \left\|u_n - y_n\right\| + \left\|y_n - x_n\right\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{62}$$

Moreover, since $x_{n+1} = \mathrm{Proj}_C^f u_n$, then

$$D_f(p, x_{n+1}) + D_f(x_{n+1}, u_n) \le D_f(p, u_n), \tag{63}$$

and therefore, we have that

$$\begin{aligned} D_f(x_{n+1}, u_n) &\le D_f(p, u_n) - D_f(p, x_{n+1}) \\ &\le \alpha_n D_f(p, u) + \beta_n D_f(p, y_n) + \delta_n D_f\left(p, \mathrm{Res}_B^f \circ A^f y_n\right) - D_f(p, x_{n+1}) \\ &= \alpha_n D_f(p, u) + (1-\alpha_n) D_f(p, y_n) - D_f(p, x_{n+1}) \\ &\le \alpha_n\left(D_f(p, u) - D_f(p, x_n)\right) + D_f(p, x_n) - D_f(p, x_{n+1}) \longrightarrow 0, \quad n \longrightarrow \infty, \end{aligned} \tag{64}$$

which implies

$$\left\|x_{n+1} - u_n\right\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{65}$$

Hence,

$$\left\|x_{n+1} - x_n\right\| \le \left\|x_{n+1} - u_n\right\| + \left\|u_n - x_n\right\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{66}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $q \in C$ as $n \longrightarrow \infty$. Since $\lim_{n\longrightarrow\infty}\|x_{n_i} - Tx_{n_i}\| = 0$, it follows from Lemma 8 that $q \in F(T)$. Also, since $\|x_{n_i} - y_{n_i}\| \longrightarrow 0$, it implies that y_{n_i} also converges weakly to $q \in E$. Therefore, from (59), we have that $q \in F(\mathrm{Res}_B^f \circ A^f)$, and hence, $q \in \Gamma = F(T) \cap F(\mathrm{Res}_B^f \circ A^f)$.

Next, we show that $\{x_n\}$ converges strongly to $\overline{x} = \mathrm{Proj}_\Gamma^f(u)$.

Now from (43), we have

$$\begin{aligned} D_f(\overline{x}, x_{n+1}) &\le D_f\left(\overline{x}, \nabla f^*\left[\alpha_n \nabla f(u) + \beta_n \nabla f y_n + \delta_n \nabla f \mathrm{Res}_B^f \circ A^f y_n\right]\right) \\ &= V_f\left(\overline{x}, \alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f\right) y_n\right) \\ &\le V_f\left(\overline{x}, \alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f\right) y_n - \alpha_n(\nabla f(u) - \nabla f(\overline{x}))\right) \\ &\quad - \langle -\alpha_n(\nabla f(u) - \nabla f(\overline{x})), \nabla f^* \cdot \left[\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f\right) y_n\right] - \overline{x}\rangle \\ &= V_f\left(\overline{x}, \alpha_n \nabla f(\overline{x}) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f\right) y_n\right) + \alpha_n\langle \nabla f(u) - \nabla f(w), u_n - \overline{x}\rangle \\ &= D_f\left(\overline{x}, \nabla f^*\left[\alpha_n \nabla f(\overline{x}) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Res}_B^f \circ A^f\right) y_n\right]\right) + \alpha_n\langle \nabla f(u) - \nabla f(\overline{x}), u_n - \overline{x}\rangle \\ &= \alpha_n D_f(\overline{x}, \overline{x}) + \beta_n D_f(\overline{x}, y_n) + \delta_n D_f\left(\overline{x}, \mathrm{Res}_B^f \circ A^f y_n\right) + \alpha_n\langle \nabla f(u) - \nabla f(\overline{x}), u_n - \overline{x}\rangle \\ &\le \beta_n D_f(\overline{x}, y_n) + \delta_n D_f(\overline{x}, y_n) + \alpha_n\langle \nabla f(u) - \nabla f(\overline{x}), u_n - \overline{x}\rangle \\ &= (1-\alpha_n) D_f(\overline{x}, y_n) + \alpha_n\langle \nabla f(u) - \nabla f(\overline{x}), u_n - \overline{x}\rangle \\ &\le (1-\alpha_n) D_f(\overline{x}, x_n) + \alpha_n\langle \nabla f(u) - \nabla f(\overline{x}), u_n - \overline{x}\rangle. \end{aligned} \tag{67}$$

Choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle \nabla f(u) - \nabla f(\overline{x}), x_n - \overline{x}\rangle = \lim_{j\longrightarrow\infty}\langle \nabla f(u) - \nabla f(\overline{x}), x_{n_j} - \overline{x}\rangle. \tag{68}$$

Since $x_{n_j} \rightharpoonup q$, it follows from Lemma 1(ii) that

$$\limsup_{n\longrightarrow\infty}\langle\nabla f(u)-\nabla f(\overline{x}), x_n-\overline{x}\rangle = \lim_{j\longrightarrow\infty}\langle\nabla f(u) - \nabla f(\overline{x}), x_{n_j}-\overline{x}\rangle = \langle\nabla f(u)-\nabla f(\overline{x}), q-\overline{x}\rangle \leq 0. \quad (69)$$

Since $\|u_n - x_n\| \longrightarrow 0$, $n\longrightarrow\infty$, then,

$$\limsup_{n\longrightarrow\infty}\langle\nabla f(u)-\nabla f(\overline{x}), u_n-\overline{x}\rangle \leq 0. \quad (70)$$

Hence, by Lemma 9 and (67), we conclude that $D_f(\overline{x}, x_n)\longrightarrow 0$, $n\longrightarrow\infty$. Therefore, $\{x_n\}$ converges strongly to $\overline{x} = \text{Proj}_\Gamma^f(u)$.

Case B. Suppose that there exists a subsequence $\{n_j\}$ of $\{n\}$ such that

$$D_f\left(x_{n_j}, w\right) < D_f\left(x_{n_j+1}, w\right), \quad (71)$$

for all $j\in\mathbb{N}$. Then, by Lemma 10, there exists a nondecreasing sequence $\{m_k\}\subset\mathbb{N}$ with $m_k\longrightarrow\infty$ as $n\longrightarrow\infty$ such that

$$D_f\left(p, x_{m_k}\right) \leq D_f\left(p, x_{m_k+1}\right), \quad D_f(p, x_k) \leq D_f\left(p, x_{m_k+1}\right), \quad (72)$$

for all $k\in\mathbb{N}$. Following the same line of arguments as in Case I, we have that

$$\begin{aligned}
&\lim_{k\longrightarrow\infty}\left\|Tx_{m_k}-x_{m_k}\right\| = 0,\\
&\lim_{k\longrightarrow\infty}\left\|\text{Res}_B^f A^f y_{m_k} - y_{m_k}\right\| = 0,\\
&\lim_{k\longrightarrow\infty}\left\|w_{m_k}-x_{m_k}\right\| = 0,\\
&\limsup_{k\longrightarrow\infty}\langle\nabla f(u)-\nabla f(p), w_{m_k}-p\rangle \leq 0.
\end{aligned} \quad (73)$$

From (67), we have

$$D_f\left(p, x_{m_k+1}\right) \leq \left(1-\alpha_{m_k}\right)D_f\left(p, x_{m_k}\right) + \alpha_{m_k}\langle\nabla f(u)-\nabla f(p), w_{m_k}-p\rangle. \quad (74)$$

Since $D_f(p, x_{m_k}) \leq D_f(p, x_{m_k+1})$, it follows from (74) that

$$\begin{aligned}
\alpha_{m_k}D_f\left(p, x_{m_k}\right) &\leq D_f\left(p, x_{m_k}\right) - D_f\left(p, x_{m_k+1}\right) + \alpha_{m_k}\langle\nabla f(u)-\nabla f(x^*), w_{m_k}-p\rangle\\
&\leq \alpha_{m_k}\langle\nabla f(u)-\nabla f(p), w_{m_k}-p\rangle.
\end{aligned} \quad (75)$$

Since $\alpha_{m_k} > 0$, we obtain

$$D_f\left(p, x_{m_k}\right) \leq \langle\nabla f(u)-\nabla f(p), w_{m_k}-p\rangle. \quad (76)$$

Then from (73), it follows that $D_f(p, x_{m_k})\longrightarrow 0$ as $k\longrightarrow\infty$. Combining $D_f(p, x_{m_k})\longrightarrow 0$ with (74), we obtain $D_f(p, x_{m_k+1})\longrightarrow 0$ as $k\longrightarrow\infty$. Since $D_f(p, x_k) \leq D_f(p, x_{m_k+1})$ for all $k\in\mathbb{N}$, we have $x_k\longrightarrow p$ as $k\longrightarrow\infty$, which implies that $x_n\longrightarrow p$ as $n\longrightarrow\infty$.

Therefore, from the above two cases, we conclude that $\{x_n\}$ converges strongly to $\overline{x} = \text{Proj}_\Gamma^f u$.

This completes the proof.

Corollary 1. *Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E and $f: E\longrightarrow\mathbb{R}$ a Legendre function which is bounded, uniformly Fréchet differentiable, and totally convex on bounded subsets of E. Let $A: E\longrightarrow 2^{E^*}$ be a Bregman inverse strongly monotone operator, $B: E\longrightarrow 2^{E^*}$ be a maximal monotone operator, and $T: C\longrightarrow C$ be a Bregman firmly nonexpansive mapping. Suppose $\Gamma := F(Res_A^f\circ A^f)\cap F(T)\neq\varnothing$. Let $\{\gamma_n\}\subset(0,1)$ and $\{\alpha_n\}, \beta_n$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\delta_n = 1$. Given $u\in E$ and $x_1\in C$ arbitrarily, let $\{x_n\}$ and $\{y_n\}$ be sequences in E generated by*

$$\begin{cases}
y_n = \nabla f^*(\gamma_n\nabla f(x_n) + (1-\gamma_n)\nabla f(Tx_n)),\\
x_{n+1} = \text{Proj}_C^f\left(\nabla f^*\left(\alpha_n\nabla f(u) + \beta_n\nabla f(y_n) + \delta_n\nabla f\left(\text{Res}_B^f\circ A^f(y_n)\right)\right)\right), \quad n\geq 1.
\end{cases} \quad (77)$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$

(ii) $(1-\alpha_n)a < \delta_n$, $\alpha_n\leq b < 1$, $a\in(0,1/2)$

(iii) $0\leq c < \liminf_{n\longrightarrow\infty}\gamma_n \leq \limsup_{n\longrightarrow\infty}\gamma_n < 1$

Then, $\{x_n\}$ converges strongly to $\text{Proj}_\Gamma^f u$, where Proj_Γ^f is the Bregman projection of E onto Γ.

Corollary 2. *Let C be a nonempty, closed, and convex subset of a real Hilbert space H. Let $A: H\longrightarrow H$ be a single-valued 1-inverse strongly monotone operator, $B: E\longrightarrow 2^{E^*}$ be a maximal monotone operator, and $T: C\longrightarrow C$ be a firmly nonexpansive mapping. Suppose $\Gamma := F((I+B)^{-1}(I-A))\cap F(T)\neq\varnothing$. Let $\{\gamma_n\}\subset(0,1)$ and $\{\alpha_n\}, \beta_n$, and $\{\delta_n\}$ be sequences in $(0,1)$ such that $\alpha_n+\beta_n+\delta_n = 1$. Given $u\in E$ and $x_1\in C$ arbitrarily, let $\{x_n\}$ and $\{y_n\}$ be sequences in E generated by*

$$\begin{cases}
y_n = \gamma_n x_n + (1-\gamma_n)Tx_n,\\
x_{n+1} = P_C\left(\alpha_n u + \beta_n y_n + \delta_n(I+B)^{-1}(I-A)y_n\right), \quad n\geq 1.
\end{cases} \quad (78)$$

Suppose the following conditions are satisfied:

(i) $\lim_{n\longrightarrow\infty}\alpha_n = 0$ and $\sum_{n=1}^{\infty}\alpha_n = \infty$

(ii) $(1-\alpha_n)a < \delta_n$, $\alpha_n\leq b < 1$, $a\in(0,1/2)$

(iii) $0\leq c < \liminf_{n\longrightarrow\infty}\gamma_n \leq \limsup_{n\longrightarrow\infty}\gamma_n < 1$

Then, $\{x_n\}$ converges strongly to $P_\Gamma u$, where P_Γ is the metric projection of H onto Γ.

4. Application

In this section, we apply our result to obtain a common solution of variational inequality problem (VIP) and equilibrium problem (EP) in real reflexive Banach spaces.

Let C be a nonempty, closed, and convex subset of a real reflexive Banach space E. Suppose $g : C \times C \longrightarrow \mathbb{R}$ is a bifunction that satisfies the following conditions:

A1 $g(x, x) = 0, \quad \forall x \in C$

A2 $g(x, y) + g(y, x) \leq 0, \quad \forall x, y \in C$

A3 $\limsup_{t \downarrow 0} g(tz + (1-t)x, y) \leq g(x, y), \quad \forall x, y, z \in C$

A4 $g(x, .)$ is convex and lower semicontinuous, for each $x \in C$.

The equilibrium problem with respect to g is to find $\overline{x} \in C$ such that

$$g(\overline{x}, y) \geq 0, \quad \forall y \in C. \tag{79}$$

We denote the set of solutions of (79) by EP(g). The resolvent of a bifunction $g : C \times C \longrightarrow \mathbb{R}$ that satisfies $A1 - A4$ (see [49]) is the operator $T_g^f : E \longrightarrow 2^C$ defined by

$$T_g^f(x) := \{z \in C : g(z, y) + \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0, \quad \forall y \in C\}. \tag{80}$$

Lemma 11 ([27], Lemma 1, 2). *Let $f : E \longrightarrow (-\infty, \infty)$ be a coercive Legendre function and let C be a nonempty, closed, and convex subset of E. Suppose the bifunction $g : C \times C \longrightarrow \mathbb{R}$ satisfies $A1 - A4$, then*

(1) $\operatorname{dom}(T_g^f) = E$.

(2) T_g^f is single valued

(3) T_g^f is Bregman firmly nonexpansive

(4) $F(T_g^f) = \mathrm{EP}(g)$

(5) $\mathrm{EP}(g)$ is a closed and convex subset of C

(6) $D_f(u, T_g^f(x)) + D_f(T_g^f(x), x) \leq D_f(u, x)$, for all $x \in E$ and for all $u \in F(T_g^f)$

Let $A : E \longrightarrow E^*$ be a Bregman inverse strongly monotone mapping and let C be a nonempty, closed, and convex subset of dom$A$. The variational inequality problem corresponding to A is to find $x \in C$, such that

$$\langle Ax^*, y - x^* \rangle \geq 0, \quad \forall y \in C. \tag{81}$$

The set of solutions of (81) is denoted by $\mathrm{VI}(C, A)$.

Lemma 12 (see [25, 46]). *Let $A : E \longrightarrow E^*$ be a Bregman inverse strongly monotone mapping and $f : E \longrightarrow (-\infty, \infty]$ be a Legendre and totally convex function that satisfies the range condition. If C is a nonempty, closed, and convex subset of $domA \cap \operatorname{int}(dom f)$, then*

(1) $P_C^f \circ A^f$ is Bregman relatively nonexpansive mapping

(2) $F(P_C^f \circ A^f) = \mathrm{VI}(C, A)$

Now let i_C be the indicator function of a closed convex subset C of E, defined by

$$i_C(x) = \begin{cases} 0, & x \in C, \\ +\infty, & \text{otherwise.} \end{cases} \tag{82}$$

The subdifferential of the indicator function $\partial i_C(\overline{x}) = N_C(\overline{x})$, where C is a closed subset of a Banach space E and $N_C \subset E^*$ is the normal cone defined by

$$N_C(\overline{x}) = \begin{cases} \{v \in E^* : \langle v, x - \overline{x} \rangle \leq 0, & \text{for all } x \in C\}, \ \overline{x} \in C, \\ \varnothing, & x \notin C. \end{cases} \tag{83}$$

The normal cone N_C is maximal monotone and the resolvent of the normal cone corresponds to the Bregman projection (see [50], Example 4.4) that is $\mathrm{Res}_{N_C}^f = \mathrm{Proj}_C^f$.

Therefore, if we let $B = N_C$ and $T = T_g^f$, then the iterative algorithm (77) becomes

$$\begin{cases} y_n = \nabla f^*\left(\gamma_n \nabla f(x_n) + (1 - \gamma_n) \nabla f\left(T_g^f x_n\right)\right), \\ x_{n+1} = \mathrm{Proj}_C^f\left(\nabla f^*\left(\alpha_n \nabla f(u) + \beta_n \nabla f(y_n) + \delta_n \nabla f\left(\mathrm{Proj}_C^f \circ A^f(y_n)\right)\right)\right), \quad n \geq 1. \end{cases} \tag{84}$$

Thus, from Corollary 1, we obtain a strong convergence result for approximating a point $x \in \mathrm{VI}(C, A) \cap \mathrm{EP}(g)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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