Research Article

On a Theory for Analysing Second-Order Systems of Ordinary Discrete Equations

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Received 30 December 2018; Accepted 21 April 2019; Published 12 May 2019

Academic Editor: Francisco Balibrea

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We present geometric based methods for solving systems of discrete or difference equations and introduce a technique for finding conservation laws for such systems.

1. Introduction

Depending on a model being studied, some physical laws may be described by differential equations (DEs). Lie group theory provides us with powerful tools for obtaining analytical solutions of such equations [1]. Over the last 30 years, a considerable amount of work has been invested into applying Lie's theory to solve and classify difference equations (Δ Es) (see [2–6] and references therein).

The use of symmetry methods for ordinary difference equations (ODEs) has been introduced by Maeda [2]. He showed that the resulting linearized symmetry condition (LSC) amounts to a set of functional equations which is hard to solve in general. Hydon introduced a technique to solve LSC and obtain symmetries in closed form by repetitive differentiations [5]. For example, a full classification of second-order ODEs according to their point symmetries exists in the literature [6]. However, for systems of difference equations (SDEs), most of results are based on induction methods (see [7–9] and references therein). In this paper, we present a method for solving SDEs using their underlying symmetry.

It has been proved in [5] that every second-order linear homogeneous ODE has an eight-dimension Lie algebra isomorphic to $\mathfrak{sl}(3)$. This is not valid for second-order system of difference equations (SDEs). We shall prove this in Section 3.

2. Groundwork

Let us consider an N -th order system of r Δ Es

$$x_{n+N}^i = \omega_i(n, x_n^1, \dots, x_n^r, x_{n+1}^1, \dots, x_{n+1}^r, \dots, x_{n+N-1}^1, \dots, x_{n+N-1}^r), \quad i = 1, \dots, r. \quad (1)$$

We assume that for each ω_i there exists at least one x_n^j ($i, j = 1, \dots, r$) such that $\partial\omega_i/\partial x_n^j \neq 0$.

Consider a point transformation

$$\Gamma_\epsilon : X \mapsto \widehat{X}(X, \epsilon) \quad (2)$$

where $X = (x_n^1, \dots, x_n^r)$ are continuous variables. Γ will be called one-parameter Lie group of transformations if it satisfies the following properties:

- (i) Γ_0 is the identity map, i.e., $\widehat{X} = X$ for $\epsilon = 0$.
- (ii) $\Gamma_\mu \Gamma_\nu = \Gamma_{\mu+\nu}$ for every μ and ν close to 0.
- (iii) Each $\widehat{x}_n^i$ can be expanded as a Taylor series in a neighbourhood of $\epsilon = 0$.

Therefore, we have

$$\widehat{x}_{n+j}^i = x_{n+j}^i + \epsilon \mathcal{S}^j Q_i(n, x_n^1, \dots, x_n^r, x_{n+1}^1, \dots, x_{n+1}^r, \dots, x_{n+N-1}^1, \dots, x_{n+N-1}^r) + O(\epsilon^2) \quad (3)$$

where Q_i are continuous functions which we shall refer to as characteristics, $i = 1, \dots, r$, $j = 1, \dots, N$, and $\mathcal{S}$ is the “shift” operator. It is defined as follows:

$$\begin{aligned} \mathcal{S} : n &\mapsto n + 1, \\ \mathcal{S}^k(x_n^i) &= x_{n+k}^i \end{aligned} \quad (4)$$

We define the discrete differentiation operator as follows:

$$\Delta = \mathcal{S} - Id \quad (5)$$

where Id is the identity operator.

The symmetry condition for the Δ Es (1) is

$$\begin{aligned} \hat{x}_{n+N}^i &= \omega_i(n, \hat{x}_n^1, \dots, \hat{x}_n^r, \hat{x}_{n+1}^1, \dots, \hat{x}_{n+1}^r, \dots, \hat{x}_{n+N-1}^1, \dots, \\ &\quad \hat{x}_{n+N-1}^r), \quad i = 1, \dots, r. \end{aligned} \quad (6)$$

whenever (1) holds.

Lie symmetries are obtained by linearizing the symmetry condition (6) about the identity. We have the following system of linearized symmetry condition (SLSC):

$$\mathcal{S}^N(Q_i) - X\omega_i = 0, \quad i = 1, \dots, r. \quad (7)$$

where the symmetry generator X is given by

$$X = \sum_{j=0}^{N-1} \left(\sum_{i=1}^r \mathcal{S}^j(Q_i) \frac{\partial}{\partial x_{n+j}^i} \right) \quad (8)$$

Definition 1. A function w_n is invariant function under the Lie group of transformations Γ if

$$X(w_n) = 0, \quad (9)$$

where w_n can be found by solving the characteristic equation

$$\begin{aligned} \frac{dx_n^1}{Q_1} &= \dots = \frac{dx_n^r}{Q_r} = \frac{dx_{n+1}^1}{\mathcal{S}(Q_1)} = \dots = \frac{dx_{n+1}^r}{\mathcal{S}(Q_r)} = \dots \\ &= \frac{dx_{n+N-1}^1}{\mathcal{S}^{N-1}(Q_1)} = \dots = \frac{dx_{n+N-1}^r}{\mathcal{S}^{N-1}(Q_r)} = \frac{w_n}{0} \end{aligned} \quad (10)$$

Theorem 2. The discrete differential operator Δ in (5) and the generator of symmetry X in (8) commute.

Proof. We prove the theorem for $N = 1$ any generalisation is straightforward.

$$\begin{aligned} [X, \Delta] F(n, X_n) &= X \{ \Delta(F(n, X_n)) \} \\ &\quad - \Delta \{ X(F(n, X_n)) \}, \\ X_n &= (x_1, x_2, \dots, x_r) \\ &= X \{ F(n+1, X_{n+1}) - F(n, X_n) \} \\ &\quad - \Delta \left(\sum_{i=1}^r Q_i \frac{\partial}{\partial x_n^i} F(n, X_n) \right) \\ &= \sum_{i=1}^r \left\{ \mathcal{S}(Q_i) \frac{\partial}{\partial x_{n+1}^i} F(n+1, X_{n+1}) - Q_i \frac{\partial}{\partial x_n^i} F(n, X_n) \right\} \\ &\quad - \sum_{i=1}^r \left\{ \mathcal{S} \left(Q_i \frac{\partial}{\partial x_n^i} F(n, X_n) \right) - Q_i \frac{\partial}{\partial x_n^i} F(n, X_n) \right\} \\ &= 0 \end{aligned} \quad (11)$$

□

Corollary 3. For each invariant w_n , $\mathcal{S}w_n$ is also an invariant.

Proof. We have $X\{\Delta(w_n)\} = \Delta\{Xw_n\} = 0$.

Equivalently, $X\{(\mathcal{S} - Id)w_n\} = X\{\mathcal{S}(w_n)\} - Xw_n = 0$ or $X(\mathcal{S}w_n) = 0$. □

We shall use this corollary for reductions in Section 3.

A first integral for the system (1) is a quantity $\phi(n, x_n, y_n, x_{n+1}, y_{n+1})$ such that

$$\Delta\phi(n, x_n, y_n, x_{n+1}, y_{n+1}) = 0 \quad (12)$$

whenever (1) holds.

In Section 5, we shall use the condition (12) to develop a constructive technique for obtaining first integrals.

Remark 4. In this paper, we shall consider Lie point symmetry; i.e, the characteristics are given by $Q_i(n, x_n^1, \dots, x_n^r)$.

We refer the reader to [1] for more information on symmetry methods for differential equations.

3. Symmetries and Reductions

3.1. Finding Characteristics. Consider a second-order system of 2 Δ Es

$$\begin{aligned} x_{n+2} &= \omega_1(n, x_n, y_n, x_{n+1}, y_{n+1}), \\ y_{n+2} &= \omega_2(n, x_n, y_n, x_{n+1}, y_{n+1}). \end{aligned} \quad (13)$$

We assume that $\partial\omega_1/\partial x_n \neq 0$ or $\partial\omega_1/\partial y_n \neq 0$ and $\partial\omega_2/\partial x_n \neq 0$ or $\partial\omega_2/\partial y_n \neq 0$, so the system is of second order.

The SLCS (7) reduces to

$$\begin{aligned} \mathcal{S}^2(Q_1) - Q_1\omega_{1,x_n} - Q_2\omega_{1,y_n} - \mathcal{S}(Q_1)\omega_{1,x_{n+1}} \\ - \mathcal{S}(Q_2)\omega_{1,y_{n+1}} = 0, \end{aligned} \quad (14)$$

$$\begin{aligned} \mathcal{S}^2(Q_2) - Q_1\omega_{2,x_n} - Q_2\omega_{2,y_n} - \mathcal{S}(Q_1)\omega_{2,x_{n+1}} \\ - \mathcal{S}(Q_2)\omega_{2,y_{n+1}} = 0. \end{aligned} \quad (15)$$

where $g_{,x} = \partial g / \partial x$, $Q_1 = Q_1(n, x_n, y_n)$, and $Q_2 = Q_2(n, x_n, y_n)$.

The functional equations (14) and (15) contain functions Q_1 and Q_2 with different pairs of arguments making them difficult to solve. For concreteness, if, for instance, the discrete variable n stands for “state” in physics, $Q_1(n, x_n, y_n)$ and $\mathcal{S}(Q_1) \equiv Q_1(n+1, x_{n+1}, y_{n+1})$ belong to two different states.

To overcome this, we proceed as follows.

Step 1 (elimination of $\mathcal{S}^2(Q_1)$ and $\mathcal{S}^2(Q_2)$). We differentiate (total differentiation) (14) and (15) with respect to x_n and y_n , respectively, keeping ω_1 and ω_2 fixed. Here, we take x_{n+1} as function of $x_n, y_n, y_{n+1}, \omega_1, \omega_2$ and y_{n+1} as function of $x_n, y_n, x_{n+1}, \omega_1, \omega_2$.

The total derivative operators are given by

$$\frac{d}{dx_n} = \frac{\partial}{\partial x_n} + \frac{\partial x_{n+1}}{\partial x_n} \frac{\partial}{\partial x_{n+1}} + \frac{\partial y_{n+1}}{\partial x_n} \frac{\partial}{\partial y_{n+1}} + \dots \quad (16)$$

$$\frac{d}{dy_n} = \frac{\partial}{\partial y_n} + \frac{\partial x_{n+1}}{\partial y_n} \frac{\partial}{\partial x_{n+1}} + \frac{\partial y_{n+1}}{\partial y_n} \frac{\partial}{\partial y_{n+1}} + \dots$$

In this case, this is simplified to

$$\begin{aligned} \frac{d}{dx_n} = \frac{\partial}{\partial x_n} - \left(\frac{\omega_{1,x_n}}{\omega_{1,x_{n+1}}} + \frac{\omega_{2,x_n}}{\omega_{2,x_{n+1}}} \right) \frac{\partial}{\partial x_{n+1}} \\ - \left(\frac{\omega_{1,x_n}}{\omega_{1,y_{n+1}}} + \frac{\omega_{2,x_n}}{\omega_{2,y_{n+1}}} \right) \frac{\partial}{\partial y_{n+1}} \end{aligned} \quad (17)$$

$$\begin{aligned} \frac{d}{dy_n} = \frac{\partial}{\partial y_n} - \left(\frac{\omega_{1,y_n}}{\omega_{1,x_{n+1}}} + \frac{\omega_{2,y_n}}{\omega_{2,x_{n+1}}} \right) \frac{\partial}{\partial x_{n+1}} \\ - \left(\frac{\omega_{1,y_n}}{\omega_{1,y_{n+1}}} + \frac{\omega_{2,y_n}}{\omega_{2,y_{n+1}}} \right) \frac{\partial}{\partial y_{n+1}} \end{aligned} \quad (18)$$

So we apply the operator (17) to (14) and (18) to (15) keeping ω_1 and ω_2 fixed. This leads to the determining system

$$\begin{aligned} [Q_1\omega_{1,x_n} + Q_2\omega_{1,y_n}]_{,y_n} + \mathcal{S}(Q_1)\omega_{1,x_{n+1},x_n} + \mathcal{S}(Q_2) \\ \cdot \omega_{1,y_{n+1},x_n} - \left(\frac{\omega_{1,x_n}}{\omega_{1,x_{n+1}}} + \frac{\omega_{2,x_n}}{\omega_{2,x_{n+1}}} \right) [Q_1\omega_{1,x_n} \\ + Q_2\omega_{1,y_n}]_{,x_{n+1}} - \left(\frac{\omega_{1,x_n}}{\omega_{1,y_{n+1}}} + \frac{\omega_{2,x_n}}{\omega_{2,y_{n+1}}} \right) \\ \cdot [\mathcal{S}(Q_1)\omega_{1,x_{n+1},x_{n+1}} + \mathcal{S}(Q_2)\omega_{1,y_{n+1},x_{n+1}} \\ + [\mathcal{S}(Q_1)]_{,x_{n+1}}\omega_{1,x_{n+1}} + [\mathcal{S}(Q_2)]_{,x_{n+1}}\omega_{1,y_{n+1}}] \end{aligned}$$

$$\begin{aligned} - \left(\frac{\omega_{1,x_n}}{\omega_{1,y_{n+1}}} + \frac{\omega_{2,x_n}}{\omega_{2,y_{n+1}}} \right) [Q_1\omega_{1,x_n} + Q_2\omega_{1,y_n}]_{,y_{n+1}} \\ - \left(\frac{\omega_{1,x_n}}{\omega_{1,y_{n+1}}} + \frac{\omega_{2,x_n}}{\omega_{2,y_{n+1}}} \right) [\mathcal{S}(Q_1)\omega_{1,x_{n+1},y_{n+1}} \\ + \mathcal{S}(Q_2)\omega_{1,y_{n+1},y_{n+1}} + [\mathcal{S}(Q_1)]_{,y_{n+1}}\omega_{1,x_{n+1}} \\ + [\mathcal{S}(Q_2)]_{,y_{n+1}}\omega_{1,y_{n+1}}] = 0, \end{aligned} \quad (19)$$

$$\begin{aligned} [Q_1\omega_{2,x_n} + Q_2\omega_{2,y_n}]_{,y_n} + \mathcal{S}(Q_1)\omega_{2,x_{n+1},y_n} + \mathcal{S}(Q_2) \\ \cdot \omega_{2,y_{n+1},y_n} - \left(\frac{\omega_{1,y_n}}{\omega_{1,x_{n+1}}} + \frac{\omega_{2,y_n}}{\omega_{2,x_{n+1}}} \right) [Q_1\omega_{2,x_n} \\ + Q_2\omega_{2,y_n}]_{,x_{n+1}} - \left(\frac{\omega_{1,y_n}}{\omega_{1,y_{n+1}}} + \frac{\omega_{2,y_n}}{\omega_{2,y_{n+1}}} \right) \\ \cdot [\mathcal{S}(Q_1)\omega_{2,x_{n+1},x_{n+1}} + \mathcal{S}(Q_2)\omega_{2,y_{n+1},x_{n+1}} \\ + [\mathcal{S}(Q_1)]_{,x_{n+1}}\omega_{2,x_{n+1}} + [\mathcal{S}(Q_2)]_{,x_{n+1}}\omega_{2,y_{n+1}}] \\ - \left(\frac{\omega_{1,y_n}}{\omega_{1,y_{n+1}}} + \frac{\omega_{2,y_n}}{\omega_{2,y_{n+1}}} \right) [Q_1\omega_{2,x_n} + Q_2\omega_{2,y_n}]_{,y_{n+1}} \\ - \left(\frac{\omega_{1,y_n}}{\omega_{1,x_{n+1}}} + \frac{\omega_{2,y_n}}{\omega_{2,x_{n+1}}} \right) [\mathcal{S}(Q_1)\omega_{2,x_{n+1},y_{n+1}} \\ + \mathcal{S}(Q_2)\omega_{2,y_{n+1},y_{n+1}} + [\mathcal{S}(Q_1)]_{,y_{n+1}}\omega_{2,x_{n+1}} \\ + [\mathcal{S}(Q_2)]_{,y_{n+1}}\omega_{2,y_{n+1}}] = 0. \end{aligned} \quad (20)$$

Step 2 (elimination of $\mathcal{S}(Q_1)$ and $\mathcal{S}(Q_2)$). We now differentiate (19) and (20) with respect to x_n and y_n , respectively, keeping x_{n+1} and y_{n+1} fixed. This means that we apply the operator $\partial/\partial x_n$ on (19) and $\partial/\partial y_n$ on (20). For a second-order SDEs we need at most to differentiate four times. After separating with respect to x_{n+1} and y_{n+1} the resulting equations, we obtain a system of DEs in Q_1 and Q_2 which is solvable by hand or by using a computer algebra package.

Step 3 (explicit form of constant of integration). When integrating in Step 2 to obtain the characteristics Q_1 and Q_2 , we have constant of integration which appears to be functions of n . To obtain their explicit form, we need to substitute the results obtained in Step 2 in (19) and (20). If we do not succeed in obtaining all the constant of integration, we need further substitution in the SLSC (14) and (15).

3.2. Reductions. Consider a second-order SDEs

$$\begin{aligned} x_{n+2} &= \omega_1(n, x_n, y_n, x_{n+1}, y_{n+1}), \\ y_{n+2} &= \omega_2(n, x_n, y_n, x_{n+1}, y_{n+1}) \end{aligned} \quad (21)$$

and its symmetry generator

$$X = Q_1 \frac{\partial}{\partial x_n} + Q_2 \frac{\partial}{\partial y_n} + \mathcal{S}Q_1 \frac{\partial}{\partial x_{n+1}} + \mathcal{S}Q_2 \frac{\partial}{\partial y_{n+1}} \quad (22)$$

The method of characteristics for partial differential equations (PDEs)

$$\frac{dx_n}{Q_1} = \frac{dy_n}{Q_2} = \frac{dx_{n+1}}{Q_1} = \frac{dy_{n+1}}{Q_2} = \frac{W_n}{0} \quad (23)$$

leads to three independent constants of integration K_1, K_2, K_3 . Each invariant under X is function of those constant. $W_n = f(K_1, K_2, K_3)$.

For second-order systems, two invariants suffice to do reduction of the systems.

Let

$$\begin{aligned} u_n &= f_1(n, x_n, y_n, x_{n+1}, y_{n+1}), \\ v_n &= f_2(n, x_n, y_n, x_{n+1}, y_{n+1}) \end{aligned} \quad (24)$$

be the invariants functions under X . We choose them in a way that the Jacobian is nonzero.

$$\left| \frac{\partial(f_1, f_2)}{\partial(x_{n+1}, y_{n+1})} \right| \neq 0 \quad (25)$$

That is, (24) can be inverted as follows:

$$\begin{aligned} x_{n+1} &= g_1(n, x_n, y_n, u_n, v_n), \\ y_{n+1} &= g_2(n, x_n, y_n, u_n, v_n) \end{aligned} \quad (26)$$

By Corollary 3, $\mathcal{S}v_n$ and $\mathcal{S}u_n$ are also invariant functions. Therefore, the solution of (21) satisfies

$$\begin{aligned} u_{n+1} &= \Omega_1(u_n, v_n), \\ v_{n+1} &= \Omega_2(u_n, v_n) \end{aligned} \quad (27)$$

(27) is a first-order SΔEs which can be solved by further reductions or by using computer algebra software (maple, Mathematica, ...) for linear systems. Note that there exist some first-order systems which cannot be solved analytically.

The general solution is

$$\begin{aligned} u_n &= u(n, C_1, C_2), \\ v_n &= v(n, C_1, C_2) \end{aligned} \quad (28)$$

for some constant C_1, C_2 .

So the second-order system (21) is equivalent to the first-order system obtained by substituting (28) in (26):

$$\begin{aligned} x_{n+1} &= g_1(n, x_n, y_n, u(n, C_1, C_2), v(n, C_1, C_2)), \\ y_{n+1} &= g_2(n, x_n, y_n, u(n, C_1, C_2), v(n, C_1, C_2)) \end{aligned} \quad (29)$$

(29) also admits the symmetries generated by X . The best way to integrate any first-order analytic ΔE is to use its canonical coordinates [10],

$$T_n = T(n, x_n, y_n), \quad (30)$$

which satisfy

$$XT_n = 1 \quad (31)$$

The obvious choice of canonical coordinates is (see [10])

$$\begin{aligned} t_n &= \int \frac{dx_n}{Q_1(n, x_n, y_n, g_1(n, x_n, y_n, u(n, C_1, C_2), v(n, C_1, C_2)), g_2(n, x_n, y_n, u(n, C_1, C_2), v(n, C_1, C_2)))}, \\ s_n &= \int \frac{dy_n}{Q_2(n, x_n, y_n, g_1(n, x_n, y_n, u(n, C_1, C_2), v(n, C_1, C_2)), g_2(n, x_n, y_n, u(n, C_1, C_2), v(n, C_1, C_2)))} \end{aligned} \quad (32)$$

3.3. Applications

3.3.1. Example 1. Consider the most general homogeneous second-order linear system of difference equations

$$\begin{aligned} x_{n+2} &= a_1(n)x_n + a_2(n)y_n + a_3(n)x_{n+1} \\ &\quad + a_4(n)y_{n+1}, \\ y_{n+2} &= b_1(n)x_n + b_2(n)y_n + b_3(n)x_{n+1} + b_4(n)y_{n+1} \end{aligned} \quad (33)$$

where $a_i(n), b_i(n)$, $i = 1, \dots, 4$ are arbitrary functions.

One can readily verify that the determining system (19) and (20) amounts to

$$\begin{aligned} Q_{1,x_n x_n} &= Q_{2,x_n x_n} = 0, \\ Q_{1,y_n y_n} &= Q_{2,y_n y_n} = 0 \end{aligned} \quad (34)$$

Therefore

$$\begin{aligned} Q_1(n, x_n, y_n) &= C_1 x_n + C_2 y_n + F_1(n), \\ Q_2(n, x_n, y_n) &= C_3 x_n + C_4 y_n + F_2(n) \end{aligned} \quad (35)$$

where C_i , $i = 1, \dots, 4$ are constants.

The characteristics in (35) must satisfy the SLSC (14) and (15). Hence, we have

$$\begin{aligned} F_1(n+2) - [a_1(n)F_1(n) + a_2(n)F_2(n) \\ + a_3(n)F_1(n+1) + a_4(n)F_2(n+1)] &= 0, \\ F_2(n+2) - [b_1(n)F_1(n) + b_2(n)F_2(n) \\ + b_3(n)F_1(n+1) + b_4(n)F_2(n+1)] &= 0. \end{aligned} \quad (36)$$

and

$$\begin{aligned} C_1 &= C_4, \\ C_2 &= C_3 = 0 \end{aligned} \quad (37)$$

So, (35) is simplified to

$$\begin{aligned} Q_1(n, x_n, y_n) &= C_1 x_n + F_1(n), \\ Q_2(n, x_n, y_n) &= C_1 y_n + F_2(n) \end{aligned} \quad (38)$$

The first generator of symmetry for a second-order homogeneous linear system (36) is the scaling symmetry given by

$$X = x_n \partial_{x_n} + y_n \partial_{y_n} \quad (39)$$

The system (36), which governs the remaining generators of the Lie point symmetry for the system (33), is of second order in F_1 and F_2 . Its general solution is

$$\begin{aligned} F_1(n) &= g_1(n, K_1, K_2, K_3, K_4), \\ F_2(n) &= g_2(n, K_1, K_2, K_3, K_4) \end{aligned} \quad (40)$$

where K_i , $i = 1, \dots, 4$ are constants.

So, the most large Lie algebra of symmetry generators which can be obtained from a homogeneous second-order system of 2 difference equations has dimension five.

For clarification let us consider

$a_1(n) = a_3(n) = a_4(n) = 0$, $a_2(n) = 1$ and $b_2(n) = b_3(n) = b_4(n) = 0$, $b_1(n) = 1$. The system (33) becomes

$$\begin{aligned} x_{n+2} &= y_n, \\ y_{n+2} &= x_n \end{aligned} \quad (41)$$

The system which governs the remaining generators of the Lie point symmetry in this case is given by

$$\begin{aligned} F_1(n+2) - F_2(n) &= 0, \\ F_2(n+2) - F_1(n) &= 0 \end{aligned} \quad (42)$$

The general solutions for this system will be

$$\begin{aligned} F_1(n) &= \frac{C_1 [1 + (-1)^n + i^n + (-i)^n]}{4} \\ &+ \frac{C_2 [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\ &+ \frac{C_3 [1 + (-1)^n - i^n - (-i)^n]}{4} \\ &+ \frac{C_4 [1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \\ F_2(n) &= \frac{C_3 [1 + (-1)^n + i^n + (-i)^n]}{4} \\ &+ \frac{C_4 [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\ &+ \frac{C_1 [1 + (-1)^n - i^n - (-i)^n]}{4} \\ &+ \frac{C_2 [1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \end{aligned} \quad (43)$$

Therefore we have 5 generators of the Lie point symmetry spanned by

$$\begin{aligned} \mathcal{X}_0 &= x_n \partial_{x_n} + y_n \partial_{y_n} \\ \mathcal{X}_1 &= \frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \partial_{x_n} \\ &+ \frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \partial_{y_n} \\ \mathcal{X}_2 &= \frac{[1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \partial_{x_n} \\ &+ \frac{[1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \partial_{y_n} \\ \mathcal{X}_3 &= \frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \partial_{x_n} \\ &+ \frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \partial_{y_n} \\ \mathcal{X}_4 &= \frac{[1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \partial_{x_n} \\ &+ \frac{[1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \partial_{y_n} \end{aligned} \quad (44)$$

3.3.2. Example 2. Consider the system

$$\begin{aligned} x_{n+2} &= \frac{x_n y_{n+1} + 1}{x_n + y_{n+1}} \\ y_{n+2} &= \frac{y_n x_{n+1} + 1}{y_n + x_{n+1}} \end{aligned} \quad (45)$$

(45) is a special case of systems investigated in [11], where the author looked at the stability of the systems.

We choose the ansatz $Q_1(n, x_n)$, $Q_2(n, y_n)$.

The determining system (19) and (20) amounts to

$$\begin{aligned} & -Q_{2,y_n} x_{n+1}^2 y_n^2 + SQ_{1,x_{n+1}} x_{n+1}^2 y_n^2 + 2Q_{2,x_{n+1}}^2 y_n^2 \\ & - 2SQ_{1,y_n}^2 x_{n+1} + Q_{2,y_n} x_{n+1}^2 + Q_{2,y_n} y_n^2 \\ & - SQ_{1,x_{n+1}} x_{n+1}^2 - SQ_{1,x_{n+1}} y_n^2 - 2Q_{2,y_n} \end{aligned} \quad (46)$$

$$\begin{aligned} & + 2SQ_{1,x_{n+1}} - Q_{2,y_n} + SQ_{1,x_{n+1}} = 0 \\ & - Q_{1,x_n} x_n^2 y_{n+1}^2 + SQ_{2,y_{n+1}} x_n^2 y_{n+1}^2 + 2Q_{1,y_{n+1}}^2 x_n^2 \\ & - 2SQ_{2,x_n}^2 y_{n+1} + Q_{1,x_n} x_n^2 + Q_{1,x_n} y_{n+1}^2 \\ & - SQ_{2,y_{n+1}} x_n^2 - SQ_{2,y_{n+1}} y_{n+1}^2 - 2Q_{1,x_n} \\ & + 2SQ_{2,y_{n+1}} - Q_{1,x_n} + SQ_{2,y_{n+1}} = 0 \end{aligned} \quad (47)$$

Differentiating twice (46) with respect to x_n and twice (47) with respect to y_n keeping x_{n+1} and y_{n+1} fixed we obtain, after separating with respect to x_{n+1} and y_{n+1} , the following system of Des:

$$\begin{aligned} Q_1'' + x_n Q_1''' + 2 \frac{Q_1}{x_n^2} - 2 \frac{Q_1'}{x_n} + \frac{Q_1''}{x_n^2} - \frac{Q_1'''}{x_n} &= 0 \\ Q_2'' + y_n Q_2''' + 2 \frac{Q_2}{y_n^2} - 2 \frac{Q_2'}{y_n} + \frac{Q_2''}{y_n^2} - \frac{Q_2'''}{y_n} &= 0 \end{aligned} \quad (48)$$

whose most general solutions are

$$\begin{aligned} Q_1(n, x_n) &= F_1(n) x_n + F_2(n) (x_n^2 - 1) \ln \frac{x_n + 1}{x_n - 1} \\ &+ F_3(n) (x_n^2 - 1) \\ Q_2(n, y_n) &= F_4(n) y_n + F_5(n) (y_n^2 - 1) \ln \frac{y_n + 1}{y_n - 1} \\ &+ F_6(n) (y_n^2 - 1) \end{aligned} \quad (49)$$

To obtain the nature of functions $F_1, \dots, F_6$ we substitute (49) in (46) and (47). After separating with respect to x_n, x_{n+1}, y_n and y_{n+1} we get the following SΔEs:

$$\begin{aligned} -4F_2(n) - 2F_4(n+1) + 4F_5(n+1) - 2F_1(n) &= 0 \\ 4F_2(n) - 2F_4(n+1) - 4F_5(n+1) + 2F_1(n) &= 0 \\ -4F_5(n) - 2F_1(n+1) + 4F_2(n+1) - 2F_4(n) &= 0 \\ 4F_5(n) - 2F_1(n+1) - 4F_2(n+1) + 2F_4(n) &= 0 \end{aligned} \quad (50)$$

whose solutions are

$$\begin{aligned} F_1(n) &= F_4(n) = 0 \\ F_2(n) &= C_1 + (-1)^n C_2 \\ F_5(n) &= C_1 - (-1)^n C_2 \end{aligned} \quad (51)$$

The remaining unknown functions $F_3(n)$ and $F_6(n)$ are determined by substituting (51) and (49) into the SLSC (14) and (15). This leads to the SΔEs

$$\begin{aligned} F_3(n) - F_3(n+2) + F_6(n+1) &= 0 \\ F_6(n) + F_3(n+1) - F_6(n+2) &= 0 \end{aligned} \quad (52)$$

The general solutions to (52) are given by

$$\begin{aligned} F_3(n) &= \frac{1 + (-1)^n}{\sqrt{5}} \left\{ \frac{[(-1 + \sqrt{5})^{n-1} + (1 + \sqrt{5})^{n-1}] C_3}{2^n} \right. \\ &\quad \left. - \frac{[(-1 + \sqrt{5})^n - (1 + \sqrt{5})^n] C_6}{2^{n+1}} \right\} \\ &\quad \cdot \frac{-1 + (-1)^n}{\sqrt{5}} \left\{ \frac{[(-1 + \sqrt{5})^{n-1} - (1 + \sqrt{5})^{n-1}] C_5}{2^n} \right. \\ &\quad \left. - \frac{[(-1 + \sqrt{5})^n + (1 + \sqrt{5})^n] C_4}{2^{n+1}} \right\} \end{aligned} \quad (53)$$

$F_6(n)$

$$\begin{aligned} &= \frac{1 + (-1)^n}{\sqrt{5}} \left\{ \frac{[(-1 + \sqrt{5})^{n-1} + (1 + \sqrt{5})^{n-1}] C_5}{2^n} \right. \\ &\quad \left. - \frac{[(-1 + \sqrt{5})^n - (1 + \sqrt{5})^n] C_4}{2^{n+1}} \right\} \\ &\quad \cdot \frac{-1 + (-1)^n}{\sqrt{5}} \left\{ \frac{[(-1 + \sqrt{5})^{n-1} - (1 + \sqrt{5})^{n-1}] C_3}{2^n} \right. \\ &\quad \left. - \frac{[(-1 + \sqrt{5})^n + (1 + \sqrt{5})^n] C_6}{2^{n+1}} \right\} \end{aligned}$$

where $C_1, \dots, C_6$ are arbitrary constants. It follows that the characteristics are given by

$$\begin{aligned} Q_1 &= (C_1 + (-1)^n C_2) (x_n^2 - 1) \ln \frac{x_n + 1}{x_n - 1} \\ &+ \left\{ \frac{1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} + (1 + \sqrt{5})^{n-1}] C_3}{2^n} \right. \right. \\ &\quad \left. \left. - \frac{[(-1 + \sqrt{5})^n - (1 + \sqrt{5})^n] C_6}{2^{n+1}} \right] \right\} \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{-1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} - (1 + \sqrt{5})^{n-1}] C_5}{2^n} \right. \\
& \left. - \frac{[(-1 + \sqrt{5})^n + (1 + \sqrt{5})^n] C_4}{2^{n+1}} \right] \Bigg\} (x_n^2 - 1) \\
Q_2 = & (C_1 - (-1)^n C_2) (y_n^2 - 1) \ln \frac{y_n + 1}{y_n - 1} \\
& + \left\{ \frac{1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} + (1 + \sqrt{5})^{n-1}] C_5}{2^n} \right. \right. \\
& \left. \left. - \frac{[(-1 + \sqrt{5})^n - (1 + \sqrt{5})^n] C_4}{2^{n+1}} \right] \right. \\
& \cdot \frac{-1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} - (1 + \sqrt{5})^{n-1}] C_3}{2^n} \right. \\
& \left. \left. - \frac{[(-1 + \sqrt{5})^n + (1 + \sqrt{5})^n] C_6}{2^{n+1}} \right] \right\} (y_n^2 - 1)
\end{aligned} \tag{54}$$

Therefore, we have six generators of Lie point symmetry

$$\begin{aligned}
\mathcal{X}_1 = & (x_n^2 - 1) \ln \frac{x_n + 1}{x_n - 1} \partial_{x_n} + (y_n^2 - 1) \ln \frac{y_n + 1}{y_n - 1} \partial_{y_n} \\
\mathcal{X}_2 = & (-1)^n (x_n^2 - 1) \ln \frac{x_n + 1}{x_n - 1} \partial_{x_n} - (-1)^n (y_n^2 - 1) \\
& \cdot \ln \frac{y_n + 1}{y_n - 1} \partial_{y_n} \\
\mathcal{X}_3 = & \frac{1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} + (1 + \sqrt{5})^{n-1}]}{2^n} \right] \\
& \cdot (x_n^2 - 1) \partial_{x_n} \\
& + \frac{-1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} - (1 + \sqrt{5})^{n-1}]}{2^n} \right] \\
& \cdot (y_n^2 - 1) \partial_{y_n} \\
\mathcal{X}_4 = & \frac{-1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^n + (1 + \sqrt{5})^n]}{2^{n+1}} \right] \\
& \cdot (x_n^2 - 1) \partial_{x_n}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^n - (1 + \sqrt{5})^n]}{2^{n+1}} \right] \\
& \cdot (y_n^2 - 1) \partial_{y_n} \\
\mathcal{X}_5 = & \frac{-1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} - (1 + \sqrt{5})^{n-1}]}{2^n} \right] \\
& \cdot (x_n^2 - 1) \partial_{x_n} \\
& + \frac{1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^{n-1} + (1 + \sqrt{5})^{n-1}]}{2^n} \right] \\
& \cdot (y_n^2 - 1) \partial_{y_n} \\
\mathcal{X}_6 = & \frac{1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^n - (1 + \sqrt{5})^n]}{2^{n+1}} \right] \\
& \cdot (x_n^2 - 1) \partial_{x_n} \\
& + \frac{-1 + (-1)^n}{\sqrt{5}} \left[\frac{[(-1 + \sqrt{5})^n + (1 + \sqrt{5})^n]}{2^{n+1}} \right] \\
& \cdot (y_n^2 - 1) \partial_{y_n}
\end{aligned} \tag{55}$$

Each generator in (55) can be used to reduce the order of (45).

Let us consider $\mathcal{X}_1$. By the characteristic method for Partial Differential Equations, the invariants are given by following equation:

$$\begin{aligned}
& \frac{dx_n}{(x_n^2 - 1) \ln((x_n + 1)/(x_n - 1))} \\
& = \frac{dy_n}{(y_n^2 - 1) \ln((y_n + 1)/(y_n - 1))} \\
& = \frac{dx_{n+1}}{(x_{n+1}^2 - 1) \ln((x_{n+1} + 1)/(x_{n+1} - 1))} \\
& = \frac{dy_{n+1}}{(y_{n+1}^2 - 1) \ln((y_{n+1} + 1)/(y_{n+1} - 1))} = \frac{V_n}{0}
\end{aligned} \tag{56}$$

We get

$$\begin{aligned}
C_1 = & \frac{\ln((x_n + 1)/(x_n - 1))}{\ln((y_{n+1} + 1)/(y_{n+1} - 1))}, \\
C_2 = & \frac{\ln((x_n + 1)/(x_n - 1))}{\ln((y_n + 1)/(y_n - 1))}, \\
C_3 = & \frac{\ln((x_n + 1)/(x_n - 1))}{\ln((x_{n+1} + 1)/(x_{n+1} - 1))}, \\
V_n = & f(C_1, C_2, C_3)
\end{aligned} \tag{57}$$

where C_1, C_2, C_3 are constants.

If we choose $f(C_1, C_2, C_3) = C_1$, we have

$$u_n = \frac{\ln((x_n + 1)/(x_n - 1))}{\ln((y_{n+1} + 1)/(y_{n+1} - 1))} \quad (58)$$

and if we choose $f(C_1, C_2, C_3) = C_3/C_2$, we have

$$v_n = \frac{\ln((y_n + 1)/(y_n - 1))}{\ln((x_{n+1} + 1)/(x_{n+1} - 1))} \quad (59)$$

From (58) and (59) we deduce

$$\begin{aligned} u_{n+1} &= \frac{1}{1 + v_n}, \\ v_{n+1} &= \frac{1}{1 + u_n} \end{aligned} \quad (60)$$

Let us now consider the generator $\mathcal{X}_3$. The resulting invariants are

$$\begin{aligned} v_n &= \frac{[(x_n - 1)/(x_n + 1)]^{\alpha_n}}{(y_{n+1} - 1)/(y_{n+1} + 1)}, \\ u_n &= \frac{[(y_n - 1)/(y_n + 1)]^{\beta_n}}{(x_{n+1} - 1)/(x_{n+1} + 1)} \end{aligned} \quad (61)$$

where

$$\begin{aligned} \alpha_n &= \frac{(1 + \sqrt{5})^n - (-1 + \sqrt{5})^n}{2 \left[(1 + \sqrt{5})^{n-1} + (-1 + \sqrt{5})^{n-1} \right]}, \\ \beta_n &= \frac{(1 + \sqrt{5})^n + (-1 + \sqrt{5})^n}{2 \left[(1 + \sqrt{5})^{n-1} - (-1 + \sqrt{5})^{n-1} \right]} \end{aligned} \quad (62)$$

Note also the relationship between them

$$\begin{aligned} \alpha_{n+1} - 1 &= \frac{1}{\beta_n}, \\ \beta_{n+1} - 1 &= \frac{1}{\alpha_n} \end{aligned} \quad (63)$$

From (61), we deduce the following relation:

$$\begin{aligned} v_{n+1} &= \frac{1}{u_n^{1/\beta_n}}, \\ u_{n+1} &= \frac{1}{v_n^{1/\alpha_n}} \end{aligned} \quad (64)$$

One can readily check that the general solution to (64) is given by

$$\begin{aligned} u_n &= \frac{1 + (-1)^n}{2} \left[u_0^{\prod_{k=0}^{(n-2)/2} (1/\beta_{2k}\alpha_{2k+1})} \right] \\ &\quad + \frac{1 - (-1)^n}{2} \left[v_0^{\prod_{k=0}^{(n-1)/2} (1/\alpha_{2k}) \prod_{k=0}^{(n-3)/2} (1/\beta_{2k+1})} \right]^{-1} \\ v_n &= \frac{1 + (-1)^n}{2} \left[v_0^{\prod_{k=0}^{(n-2)/2} (1/\alpha_{2k}\beta_{2k+1})} \right] \\ &\quad + \frac{1 - (-1)^n}{2} \left[u_0^{\prod_{k=0}^{(n-1)/2} (1/\beta_{2k}) \prod_{k=0}^{(n-3)/2} (1/\alpha_{2k+1})} \right]^{-1} \end{aligned} \quad (65)$$

where α_n and β_n are defined in (62).

From (61) we obtain

$$v_n \frac{y_{n+1} - 1}{y_{n+1} + 1} = \left[\frac{x_n - 1}{x_n + 1} \right]^{\alpha_n}, \quad (66)$$

$$u_n \frac{x_{n+1} - 1}{x_{n+1} + 1} = \left[\frac{y_n - 1}{y_n + 1} \right]^{\beta_n}$$

which is a first-order system after substitution of u_n, v_n by the results given in (65). Its solutions can be obtained by using the following canonical coordinates:

$$\begin{aligned} s_n &= \ln \frac{x_n - 1}{x_n + 1}, \\ t_n &= \ln \frac{y_n - 1}{y_n + 1} \end{aligned} \quad (67)$$

This leads to the the following linear system with variable coefficients:

$$\begin{aligned} t_{n+1} &= \alpha_n s_n + \gamma_n \\ s_{n+1} &= \beta_n t_n + \varphi_n \end{aligned} \quad (68)$$

where $-\gamma_n = \ln u_n$ and $-\varphi_n = \ln v_n$.

The latter is a linear first-order system with variable coefficients. Its general solution is

$$\begin{aligned} s_{2n} &= \prod_{k=0}^{n-1} \alpha_{2k} \beta_{2k+1} s_0 + \sum_{r=0}^{n-1} \left(\gamma_{2r} \prod_{j=r+1}^{n-1} \alpha_{2j} \prod_{j=r}^{n-1} \beta_{2j+1} \right) \\ &\quad + \sum_{r=0}^{n-1} \left(\varphi_{2r+1} \prod_{j=r+1}^{n-1} \alpha_{2j} \beta_{2j+1} \right) \\ t_{2n} &= \prod_{k=0}^{n-1} \beta_{2k} \alpha_{2k+1} t_0 + \sum_{r=0}^{n-1} \left(\varphi_{2r} \prod_{j=r+1}^{n-1} \beta_{2j} \prod_{j=r}^{n-1} \alpha_{2j+1} \right) \\ &\quad + \sum_{r=0}^{n-1} \left(\gamma_{2r+1} \prod_{j=r+1}^{n-1} \beta_{2j} \alpha_{2j+1} \right) \\ t_{2n+1} &= \prod_{k=0}^n \alpha_{2k} \prod_{k=0}^{n-1} \beta_{2k+1} s_0 + \sum_{r=0}^n \left(\gamma_{2r} \prod_{j=r+1}^n \alpha_{2j} \prod_{j=r}^{n-1} \beta_{2j+1} \right) \\ &\quad + \sum_{r=0}^{n-1} \left(\varphi_{2r+1} \prod_{j=r+1}^n \alpha_{2j} \prod_{j=r+1}^{n-1} \beta_{2j+1} \right) \\ s_{2n+1} &= \prod_{k=0}^n \beta_{2k} \prod_{k=0}^{n-1} \alpha_{2k+1} t_0 + \sum_{r=0}^n \left(\varphi_{2r} \prod_{j=r+1}^n \beta_{2j} \prod_{j=r}^{n-1} \alpha_{2j+1} \right) \\ &\quad + \sum_{r=0}^{n-1} \left(\gamma_{2r+1} \prod_{j=r+1}^n \beta_{2j} \prod_{j=r+1}^{n-1} \alpha_{2j+1} \right) \end{aligned} \quad (69)$$

The general solution of (45) is obtained by substituting (69) into (67).

4. Conservation Laws

In Section 2, we have defined a first integral associated with a second-order SΔEs. It is given by (12)

$$\phi(n, x_n, y_n, x_{n+1}, y_{n+1}) = \phi(n+1, x_{n+1}, y_{n+1}, \omega_1, \omega_2) \quad (70)$$

Let

$$\begin{aligned} P_1 &= \frac{\partial \phi}{\partial x_n}, \\ P_2 &= \frac{\partial \phi}{\partial x_{n+1}}, \\ Q_1 &= \frac{\partial \phi}{\partial y_n}, \\ Q_2 &= \frac{\partial \phi}{\partial y_{n+1}} \end{aligned} \quad (71)$$

By differentiating (70) with respect to x_n, y_n, x_{n+1} and y_{n+1} we obtain

$$\begin{aligned} P_1 &= \mathcal{S}(P_2) \omega_{1,x_n} + \mathcal{S}(Q_2) \omega_{2,x_n} \\ Q_1 &= \mathcal{S}(P_2) \omega_{1,y_n} + \mathcal{S}(Q_2) \omega_{2,y_n} \end{aligned} \quad (72)$$

and

$$\begin{aligned} P_2 &= \mathcal{S}(P_1) + \mathcal{S}(P_2) \omega_{1,x_{n+1}} + \mathcal{S}(Q_2) \omega_{2,x_{n+1}} \\ Q_2 &= \mathcal{S}(Q_1) + \mathcal{S}(P_2) \omega_{1,y_{n+1}} + \mathcal{S}(Q_2) \omega_{2,y_{n+1}} \end{aligned} \quad (73)$$

The substitution of (72) in (73) leads to the following second-order system of functional equations:

$$\begin{aligned} &\mathcal{S}^2(P_2) \mathcal{S}(\omega_{1,x_n}) + \mathcal{S}^2(Q_2) \mathcal{S}(\omega_{2,x_n}) \\ &+ \mathcal{S}(P_2) \omega_{1,x_{n+1}} + \mathcal{S}(Q_2) \omega_{2,x_{n+1}} - P_2 = 0 \\ &\mathcal{S}^2(P_2) \mathcal{S}(\omega_{1,y_n}) + \mathcal{S}^2(Q_2) \mathcal{S}(\omega_{2,y_n}) \\ &+ \mathcal{S}(P_2) \omega_{1,y_{n+1}} + \mathcal{S}(Q_2) \omega_{2,y_{n+1}} - Q_2 = 0 \end{aligned} \quad (74)$$

As for SLSC, we differentiate repeatedly to obtain a system of DEs for P_2 and Q_2 . Given the solutions P_2, Q_2 of (74) we easily construct P_1, Q_1 . For consistency of our solutions, we must check the integrability conditions

$$\frac{\partial P_1}{\partial x_{n+1}} = \frac{\partial P_2}{\partial x_n} \quad (75)$$

and

$$\frac{\partial Q_1}{\partial y_{n+1}} = \frac{\partial Q_2}{\partial y_n} \quad (76)$$

The first integral is then given by

$$\phi = \int (P_1 dx_n + P_2 dx_{n+1} + Q_1 dy_n + Q_2 dy_{n+1}) + F(n) \quad (77)$$

The constant of integration $F(n)$ which is a function depending on n is determined by substituting (77) in (70).

4.1. Applications. Let us consider the second-order SΔEs

$$\begin{aligned} x_{n+2} &= a(n) y_n, \\ y_{n+2} &= b(n) x_n \end{aligned} \quad (78)$$

By choosing the ansatz $P_2(n, x_n, y_n)$ and $Q_2(n, x_n, y_n)$ one can readily check that the determining system (74) is simplified to

$$\begin{aligned} Q_2(n+2, \omega_1, \omega_2) b(n+1) - P_2(n, x_n, y_n) &= 0, \\ P_2(n+2, \omega_1, \omega_2) a(n+1) - Q_2(n, x_n, y_n) &= 0. \end{aligned} \quad (79)$$

where ω_1 and ω_2 denote the right-hand side of (78).

Differentiating (79) with respect to x_n and y_n leads to

$$\begin{aligned} P_2 &= x_n \psi_1(n) + y_n \psi_2(n) + \psi_3(n), \\ Q_2 &= x_n \psi_4(n) + y_n \psi_5(n) + \psi_6(n) \end{aligned} \quad (80)$$

Thus, we have from (72)

$$\begin{aligned} P_1 &= b(n) [x_{n+1} \psi_4(n+1) + y_{n+1} \psi_5(n+1) + \psi_6(n+1)] \\ Q_1 &= a(n) [x_{n+1} \psi_1(n+1) + y_{n+1} \psi_2(n+1) + \psi_3(n+1)] \end{aligned} \quad (81)$$

Substituting (80) in (79) and separating with respect to x_n and y_n we obtain the system

$$\begin{aligned} a(n) b(n+1) \psi_4(n+2) - \psi_2(n) &= 0, \\ b(n) a(n+1) \psi_2(n+2) - \psi_4(n) &= 0, \\ b(n) b(n+1) \psi_5(n+2) - \psi_1(n) &= 0, \\ a(n) a(n+1) \psi_1(n+2) - \psi_5(n) &= 0, \\ b(n+1) \psi_6(n+2) - \psi_3(n) &= 0, \\ a(n+1) \psi_3(n+2) - \psi_6(n) &= 0. \end{aligned} \quad (82)$$

The solutions to (82) will provide us with the explicit form of ψ_i , $i = 1, \dots, 6$

The first integral is then given by

$$\begin{aligned}
& \phi \\
& = \int b(n) [x_{n+1} \psi_4(n) \\
& \quad + 1) + y_{n+1} \psi_5(n+1) + \psi_6(n+1)] dx_n \\
& \quad + (x_n \psi_1(n) + y_n \psi_2(n) + \psi_3(n)) dx_{n+1} a(n) [x_{n+1} \psi_1(n+1) + y_{n+1} \psi_2(n+1) + \psi_3(n+1)] dy_n \\
& \quad + (x_n \psi_4(n) + y_n \psi_5(n) + \psi_6(n)) dy_{n+1} \\
& \quad + K_i
\end{aligned} \tag{83}$$

for some constants K_i .

For clarification, let us consider $a(n) = b(n) = 1$, that is,

$$\begin{aligned}
x_{n+2} &= y_n, \\
y_{n+2} &= x_n
\end{aligned} \tag{84}$$

The solutions to (82) will be

$$\begin{aligned}
\psi_1(n) &= \frac{C_1 [1 + (-1)^n + i^n + (-i)^n]}{4} \\
& \quad + \frac{C_2 [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\
& \quad + \frac{C_3 [1 + (-1)^n - i^n - (-i)^n]}{4} \\
& \quad + \frac{C_4 [1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \\
\psi_5(n) &= \frac{C_3 [1 + (-1)^n + i^n + (-i)^n]}{4} \\
& \quad + \frac{C_4 [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\
& \quad + \frac{C_1 [1 + (-1)^n - i^n - (-i)^n]}{4} \\
& \quad + \frac{C_2 [1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \\
\psi_2(n) &= \frac{C_5 [1 + (-1)^n + i^n + (-i)^n]}{4} \\
& \quad + \frac{C_6 [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\
& \quad + \frac{C_7 [1 + (-1)^n - i^n - (-i)^n]}{4} \\
& \quad + \frac{C_8 [1 - (-1)^n + i(i)^n - i(-i)^n]}{4}
\end{aligned}$$

$$\begin{aligned}
\psi_4(n) &= \frac{C_7 [1 + (-1)^n + i^n + (-i)^n]}{4} \\
& \quad + \frac{C_8 [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\
& \quad + \frac{C_5 [1 + (-1)^n - i^n - (-i)^n]}{4} \\
& \quad + \frac{C_6 [1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \\
\psi_3(n) &= \frac{C_9 [1 + (-1)^n + i^n + (-i)^n]}{4} \\
& \quad + \frac{C_{10} [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\
& \quad + \frac{C_{11} [1 + (-1)^n - i^n - (-i)^n]}{4} \\
& \quad + \frac{C_{12} [1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \\
\psi_6(n) &= \frac{C_{11} [1 + (-1)^n + i^n + (-i)^n]}{4} \\
& \quad + \frac{C_{12} [1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \\
& \quad + \frac{C_9 [1 + (-1)^n - i^n - (-i)^n]}{4} \\
& \quad + \frac{C_{10} [1 - (-1)^n + i(i)^n - i(-i)^n]}{4}
\end{aligned} \tag{85}$$

where $C_i, i = 1, \dots, 12$ are constants. We have twelve solutions for P_2 and Q_2 . That is,

(1) If

$$\begin{aligned}
P_2 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) x_n, \\
Q_2 &= \left(\frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \right) y_n,
\end{aligned} \tag{86}$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) x_{n+1}, \\ P_1 &= \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) y_{n+1} \end{aligned} \quad (87)$$

(2) If

$$\begin{aligned} P_2 &= \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) x_n, \\ Q_2 &= \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) y_n, \end{aligned} \quad (88)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) x_{n+1}, \\ P_1 &= \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) y_{n+1} \end{aligned} \quad (89)$$

(3) If

$$\begin{aligned} P_2 &= \frac{[1 + (-1)^n - i^n - (-i)^n]}{4} x_n, \\ Q_2 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) y_n, \end{aligned} \quad (90)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) x_{n+1}, \\ P_1 &= \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) y_{n+1} \end{aligned} \quad (91)$$

(4) If

$$\begin{aligned} P_2 &= \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) x_n, \\ Q_2 &= \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) y_n, \end{aligned} \quad (92)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) x_{n+1}, \\ P_1 &= \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) y_{n+1} \end{aligned} \quad (93)$$

(5) If

$$\begin{aligned} P_2 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) y_n, \\ Q_2 &= \left(\frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \right) x_n, \end{aligned} \quad (94)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) y_{n+1}, \\ P_1 &= \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) x_{n+1} \end{aligned} \quad (95)$$

(6) If

$$\begin{aligned} P_2 &= \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) y_n, \\ Q_2 &= \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) x_n, \end{aligned} \quad (96)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) y_{n+1}, \\ P_1 &= \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) x_{n+1} \end{aligned} \quad (97)$$

(7) If

$$\begin{aligned} P_2 &= \frac{[1 + (-1)^n - i^n - (-i)^n]}{4} y_n, \\ Q_2 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) x_n, \end{aligned} \quad (98)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) y_{n+1}, \\ P_1 &= \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) x_{n+1} \end{aligned} \quad (99)$$

(8) If

$$\begin{aligned} P_2 &= \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) y_n, \\ Q_2 &= \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) x_n, \end{aligned} \quad (100)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) y_{n+1}, \\ P_1 &= \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) x_{n+1} \end{aligned} \quad (101)$$

(9) If

$$\begin{aligned} P_2 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right), \\ Q_2 &= \left(\frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \right), \end{aligned} \quad (102)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 - (-1)^n + ii^n - i(-i)^n]}{4} \right), \\ P_1 &= \left(\frac{[1 - (-1)^n - ii^n + i(-i)^n]}{4} \right) \end{aligned} \quad (103)$$

(10) If

$$\begin{aligned} P_2 &= \left(\frac{[1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \right), \\ Q_2 &= \left(\frac{[1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \right), \end{aligned} \quad (104)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right), \\ P_1 &= \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) \end{aligned} \quad (105)$$

(11) If

$$\begin{aligned} P_2 &= \frac{[1 + (-1)^n - i^n - (-i)^n]}{4}, \\ Q_2 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right), \end{aligned} \quad (106)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 - (-1)^n - ii^n + i(-i)^n]}{4} \right), \\ P_1 &= \left(\frac{[1 - (-1)^n + ii^n - i(-i)^n]}{4} \right) \end{aligned} \quad (107)$$

(12) If

$$\begin{aligned} P_2 &= \left(\frac{[1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \right), \\ Q_2 &= \left(\frac{[1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \right), \end{aligned} \quad (108)$$

then

$$\begin{aligned} Q_1 &= \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right), \\ P_1 &= \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) \end{aligned} \quad (109)$$

Therefore, we obtain twelve conservation laws for the system (84). They are given by

$$\begin{aligned} \phi_1 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) x_n x_{n+1} \\ &+ \left(\frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \right) y_n y_{n+1} \\ &+ \left(\frac{[1 - (-1)^n + ii^n - i(-i)^n]}{4} \right) x_{n+1} y_n \\ &+ \left(\frac{[1 - (-1)^n - ii^n + i(-i)^n]}{4} \right) y_{n+1} x_n \\ &+ K_1 \\ \phi_2 &= \left(\frac{[1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \right) x_n x_{n+1} \\ &+ \left(\frac{[1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \right) y_n y_{n+1} \\ &+ \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) x_{n+1} y_n \\ &+ \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) y_{n+1} x_n \\ &+ K_2 \\ \phi_3 &= \frac{[1 + (-1)^n - i^n - (-i)^n]}{4} x_n x_{n+1} \\ &+ \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) y_n y_{n+1} \\ &+ \left(\frac{[1 - (-1)^n - ii^n + i(-i)^n]}{4} \right) x_{n+1} y_n \\ &+ \left(\frac{[1 - (-1)^n + ii^n - i(-i)^n]}{4} \right) y_{n+1} x_n \\ &+ K_3 \\ \phi_4 &= \left(\frac{[1 - (-1)^n + i(i)^n - i(-i)^n]}{4} \right) x_n x_{n+1} \\ &+ \left(\frac{[1 - (-1)^n - i(i)^n + i(-i)^n]}{4} \right) y_n y_{n+1} \\ &+ \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) x_{n+1} y_n \\ &+ \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) y_{n+1} x_n \\ &+ K_4 \end{aligned}$$

$$\begin{aligned}
\phi_5 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) y_n x_{n+1} \\
&+ \left(\frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \right) x_n y_{n+1} \\
&+ \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) y_{n+1} y_n \\
&+ \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) x_{n+1} x_n \\
&+ K_5 \\
\phi_6 &= \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) y_n x_{n+1} \\
&+ \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) x_n y_{n+1} \\
&+ \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) y_{n+1} y_n \\
&+ \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) x_{n+1} x_n \\
&+ K_6 \\
\phi_7 &= \frac{[1 + (-1)^n - i^n - (-i)^n]}{4} y_n x_{n+1} \\
&+ \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) x_n y_{n+1} \\
&+ \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) y_{n+1} y_n \\
&+ \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) x_{n+1} x_n \\
&+ K_7 \\
\phi_8 &= \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) y_n x_{n+1} \\
&+ \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) x_n y_{n+1} \\
&+ \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) y_{n+1} y_n \\
&+ \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) x_{n+1} x_n \\
&+ K_8
\end{aligned}$$

$$\begin{aligned}
\phi_9 &= \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) x_{n+1} \\
&+ \left(\frac{[1 + (-1)^n - i^n - (-i)^n]}{4} \right) y_{n+1} \\
&+ \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) y_n \\
&+ \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) x_n + K_9 \\
\phi_{10} &= \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) x_{n+1} \\
&+ \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) y_{n+1} \\
&+ \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) y_n \\
&+ \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) x_n + K_{10} \\
\phi_{11} &= \frac{[1 + (-1)^n - i^n - (-i)^n]}{4} x_{n+1} \\
&+ \left(\frac{[1 + (-1)^n + i^n + (-i)^n]}{4} \right) y_{n+1} \\
&+ \left(\frac{[1 - (-1)^n - i i^n + i (-i)^n]}{4} \right) y_n \\
&+ \left(\frac{[1 - (-1)^n + i i^n - i (-i)^n]}{4} \right) x_n + K_{11} \\
\phi_{12} &= \left(\frac{[1 - (-1)^n + i (i)^n - i (-i)^n]}{4} \right) x_{n+1} \\
&+ \left(\frac{[1 - (-1)^n - i (i)^n + i (-i)^n]}{4} \right) y_{n+1} \\
&+ \left(\frac{[1 + (-1)^n - (i)^n - (-i)^n]}{4} \right) y_n \\
&+ \left(\frac{[1 + (-1)^n + (i)^n + (-i)^n]}{4} \right) x_n + K_{12}
\end{aligned}$$

(110)

5. Conclusion and Discussions

We have presented a method for obtaining nontrivial symmetries and how to use them for solving a second-order SΔEs. Each symmetry can be used to reduce the order. However, different symmetries lead to different reductions (see (60) and (64)), but the same solution. We also proposed a technique to construct first integral associated to second-order systems of difference equations.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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