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Research Article

The Connected Detour Numbers of Special Classes of Connected Graphs

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Simple finite connected graphs $G = (V, E)$ of $p \geq 2$ vertices are considered in this paper. A *connected detour set* of G is defined as a subset $S \subseteq V$ such that the induced subgraph $G[S]$ is connected and every vertex of G lies on a $u - v$ detour for some $u, v \in S$. The connected detour number $\mathrm{cdn}(G)$ of a graph G is the minimum order of the connected detour sets of G. In this paper, we determined $\mathrm{cdn}(G)$ for three special classes of graphs G, namely, unicyclic graphs, bicyclic graphs, and cog-graphs for C_p, K_p, and $K_{m,n}$.

1. Introduction

For basic definitions of the concepts of graphs we refer to [1–4], and for detour distance and related terminologies in graphs, we refer to [5–7]. Let $G = (V, E)$ be a connected simple graph of p vertices and q edges. We assume that p is finite and $p(G) \geq 2$. For $u, v \in V(G)$ the length of a maximum $u - v$ path is called *detour distance* $D(u, v)$. A $u - v$ path of length $D(u, v)$ is called $u - v$ *detour*. For vertex $v \in V$ the *detour eccentricity* $e_D(v)$ is defined by

$$e_D(v) = \max\{D(u, v) : u \in V\}. \tag{1}$$

The *detour radius* $\mathrm{rad}_D G$ and the *detour diameter* $\mathrm{diam}_D G$ (or $D(G)$) of G are defined as

$$\begin{aligned} \mathrm{rad}_D G &= \min\{e_D(v) : v \in V\}, \\ \mathrm{diam}_D G &= \max\{e_D(v) : v \in V\}. \end{aligned} \tag{2}$$

A vertex $w \in V(G)$ is said *to lie* on a $u - v$ detour Q if w is a vertex of $V(Q)$ including u and v. A *detour set* (denoted *d.s.*) is a subset S of $V(G)$ such that every vertex v of G lies on an $x - y$ detour of some $x, y \in S$. The *detour number* $\mathrm{dn}(G)$ of G is defined by

$$\mathrm{dn}(G) = \min\{|S| : S \text{ is a d.s. of } G\}. \tag{3}$$

A *detour basis* of G is a d.s. of G of order $\mathrm{dn}(G)$.

If S is a detour set of G and the induced subgraph $G[S]$ is connected, then S is called *connected detour set* (denoted c.d.s.) of G. The *connected detour number* of G denoted as $\mathrm{cdn}(G)$ is defined as

$$\mathrm{cdn}(G) = \min\{|S| : S \text{ is a c.d.s. of } G\}. \tag{4}$$

A *connected detour basis* of G is a c.d.s. of order $\mathrm{cdn}(G)$ (see [8, 9]).

A simple connected (p, q) graph G with $p \geq 3$ is called *unicyclic graph* iff $p = q$. The graph G is called *bicyclic* iff $q = p + 1$.

The concept of connected detour number was introduced and studied by Santhakumaran and Athisayanathan in [9]. They determined cdn for some special graphs such as K_p, C_p, $K_{m,n}$, trees, and Hamilton graph. There are many research papers on connected detour number and edge detour graphs (see [10–14]). Moreover, the concept of connected detour number and other related concepts have interesting applications in the channel assignment problem in radio technologies. This motivated us to determine connected detour number for other classes of graphs. Therefore, in this paper we determine the connected detour numbers for unicyclic graphs and bicyclic graphs. Moreover, the class of graphs called *cog-graphs* G^c will be explained and determined the $\mathrm{cnd}(G^c)$ if G is a complete graph, tree, cycle graph, and complete bipartite graph.

2. The Connected Detour Number of Unicyclic Graphs

Let G be a connected graph of order $p \ge 3$ and C the unique cycle in G, and let C be of length $l \ge 3$. It is clear that C has no chords, and every vertex of G, which is not on C, is either a cut-vertex or an end-vertex. We shall determine the connected detour number of such graphs in terms of l and p. Let n be the number of vertices of C that are not cut-vertices. Denote $T(G) = \{v \in V(G) : v \text{ is either a cut-vertex or an end-vertex}\}$ and $\overline{T}(G) = V(G) - T(G)$. Then, $n = |\overline{T}(G)|$ and $|T(G)| = p - n$.

If $l = p$, then $G = C_p$ so $\mathrm{cdn}(G) = 2$. If $p > l$, then G contains at least one cyclic cut-vertex. If $n = 0$, that is every vertex of C is a cut-vertex, then by Theorem 1.4 [Ref. 2] $\mathrm{cdn}(G) = p$. From now on, we assume $p > l$.

Proposition 1. *Let G be a connected unicyclic graph of order $p \ge 4$, and with l–cycle, $l = 3$. Then, $\mathrm{cdn}(G) = p - 1$ iff $n = 1$ or 2.*

Proof. If $n = 1$, then G contains exactly one vertex which is not a cut-vertex. It is clear that there is a detour joining the other two vertices of the triangle and v lies on it. Thus, $\mathrm{cdn}(G) = p - 1$. If $n = 2$, let u_1 and u_2 be cycle vertices which are not cut-vertices. Clearly, there is no path in G between two vertices of $T(G)$ that contains u_1 or u_2 (see Figure 1). Thus, every c.d.b. B of G must contain either u_1 or u_2. Therefore, $\mathrm{cdn}(G) \ge p - 1$. If u_3 is the third vertex of the triangle and $u_1 \in B$, then u_2 lies on the $u_1 - u_3$ detour. Therefore, $\mathrm{cdn}(G) \le p - 1$. Hence, $\mathrm{cdn}(G) = p - 1$.

To prove the converse, let $\mathrm{cdn}(G) = p - 1$, and $B(G)$ is a c.d.b. of G, then $B(G)$ contains two vertices of the 3–cycle, one of them is a cut-vertex ($\because p \ge 4$). Thus, in view of Theorem 1.4 [9], the 3–cycle has one or two vertices in $\overline{T}(G)$, that is, $n = 1$ or 2. Hence, the proof is completed. □

Theorem 1. *Let G be a connected unicyclic graph of order $p \ge 5$ and with l–cycle, $l \ge 4$. Then, $\mathrm{cdn}(G) = p - 1$ iff the induced subgraph $G[T(G)]$ consists of exactly n components.*

Proof. Let m be the number of components of $G[T(G)]$. Let $m = n$, then since for every c.d.b., $B(G)$ and $G[B(G)]$ are connected, every connected component of $G[T(G)]$ contains at least one cycle cut-vertex, and $G[T(G)] \subset G[B(G)]$, then $G[B(G)]$ contains at least $n - 1$ vertices from $\overline{T}(G)$. Therefore, $|B(G)| = p - n + (n - 1) = p - 1$.

Conversely, let $\mathrm{cdn}(G) = p - 1$, and $B(G)$ is a c.d.b. of G. Since $G[B(G)]$ is connected and $G[T(G)]$ consists of m components and $G[T(G)] \subset G[B(G)]$, then $G[B(G)]$ contains at least $m - 1$ vertices from $\overline{T}(G)$. Because $B(G)$ is a connected detour set of minimum order, then $B(G)$ contains exactly $m - 1$ vertices from $\overline{T}(G)$. Thus,

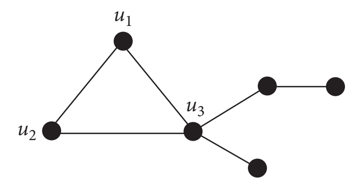

Figure 1

$$\mathrm{cdn}(G) = |B(G)| = |T(G)| + m - 1. \quad (5)$$

From the hypothesis

$$\mathrm{cdn}(G) = p - 1 = |\overline{T}(G)| + |T(G)| - 1 = |T(G)| + n - 1. \quad (6)$$

Therefore, $m = n$.

Hence, the proof of the theorem is completed. □

Example 1. For the unicyclic graph G in Figure 2, we have $m = n = 4$, so $\mathrm{cdn}(G) = p - 1$.

Now, we have the following result for the connected detour number of the unicyclic graph with exactly one cycle cut-vertex.

Proposition 2. *Let G be a connected unicyclic graph of order $p \ge 4$ and with exactly one cycle cut-vertex, say v, then $\mathrm{cdn}(G) = p - l + 2$, where l is the length of the unique cycle C of G.*

Proof. Let u be a vertex of C adjacent to v. Then, there is a $u - v$ detour consisting of all the vertices of C.

Thus, $T(G) \cup \{u\}$ is a c.d.s. of order $|T(G)| + 1 = [p - (l - 1)] + 1 = p - l + 2$.

It is clear that there are no $x - y$ detour containing vertices of $\overline{T}(G)$ for every pair $x, y \in T(G)$. Therefore, $T(G) \cup \{u\}$ is a connected detour basis of G, and hence $\mathrm{cdn}(G) = p - l + 2$.

For connected unicyclic graphs having more than one cycle cut-vertex, we need the following definition. □

Definition 1. Let G be a connected unicyclic graph of order $p \ge 4$ and with at least two cycle cut-vertices, and let C be the unique cycle of length $l \ge 4$. Moreover, let m be the number of components of the induced subgraph $G[T(G)]$. These components divide the cycle vertices which are not cut-vertices into m nonempty subsets $A_1, A_2, \ldots, A_m$, in successive order around C, as illustrated in Figure 3.

Example 2. Consider the unicyclic graph G shown in Figure 3. The set of cycle vertices which are not cut-vertices is $W(G) = \{w_1, w_2, \ldots, w_n\}$. It is clear that $n = 11$, $m = 4$, and $l = 18$. The set $W(G)$ is partitioned into $A_1 = \{w_1, w_2\}$, $A_2 = \{w_3, w_4, w_5, w_6, w_7\}$, $A_3 = \{w_8\}$, and $A_4 = \{w_9, w_{10}, w_{11}\}$.

The c.d.n. for unicyclic graphs having more than one cut-vertex is determined by the following theorem.

Theorem 2. *Let G be a connected unicyclic graph of order $p \ge 5$ and with at least two cycle cut-vertices, and the induced*

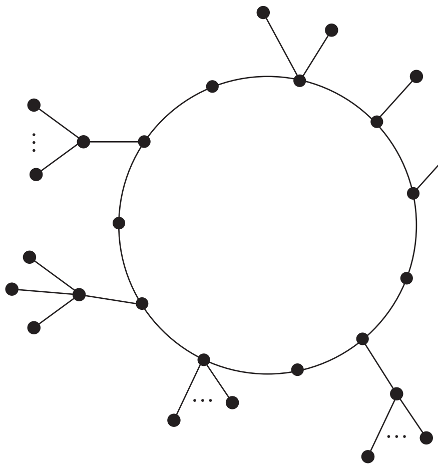

Figure 2

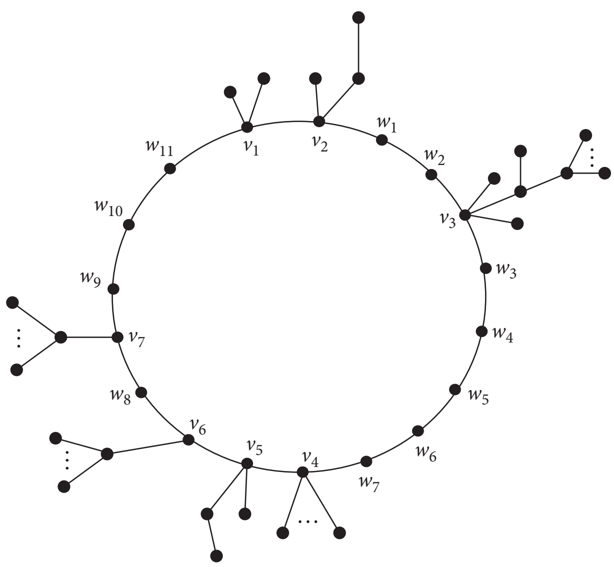

Figure 3: Unicyclic graph illustrating Def. 1.

subgraph $G[T(G)]$ *consists of* m *components. Then,* $\mathrm{cdn}(G) = p - \alpha$ *if and only if*

$$\alpha = \max\{|A_i| : i = 1, 2, \ldots, m\}. \tag{7}$$

Proof. Let α be as defined in (7) for the graph G, and let S be a c.d.b. for G. By Theorem 1.4 of Ref [9], S contains the set $T(G)$. Since the induced subgraph $G[S]$ is connected, then S must contain all the vertices of at least $(m-1)$ subsets from $\{A_1, A_2, \ldots, A_m\}$. Since S is of minimum order, then S does not contain the subset from $\{A_1, A_2, \ldots, A_m\}$ that has maximum order, say A_r. It is clear that there are two vertices $x, y \in T(G) \cup \{\cup_{i\neq r} A_i\}$ which are adjacent on C; hence, there is an $x - y$ detour containing all the vertices of C. Therefore, $S = T(G) \cup \{\cup_{i\neq r} A_i\}$.

Thus, $\mathrm{cdn}(G) = |S| = |T(G)| + (\sum_{i=1}^{m} |A_i| - |A_r|) = p - |A_r| = p - \alpha$.

To prove the converse, let $\mathrm{cdn}(G) = p - \beta$ and let S' be a c.d.b. of G. If β is not equal to $\max\{|A_i| : 1 \le i \le m\}$, then either S' is not of minimum order or the induced subgraph $G[S']$ is disconnected, contradicting the definition of connected detour basis.

Thus, $\beta = \max\{|A_i| : 1 \le i \le m\}$, and hence the proof of the theorem is completed. □

Remark 1. Clearly $\alpha = 1$, iff $m = n$. Thus, Theorem 1 follows from Theorem 2.

3. The Connected Detour Numbers of Connected Bicyclic Graphs

A (p, q) graph is bicyclic if and only if $q = p + 1$. Thus, if G is a connected bicyclic graph, then G contains either three cycles having some edges in common or contains exactly two cycles having no edges in common. The connected detour number for a block bicyclic graph is determined by the following result.

Proposition 3. *Let G be a 2-connected bicyclic graph of order $p \ge 5$ as shown in Figure 4. Then,*

(i) $\mathrm{cdn}(G) = 2$, *iff* $m = n \ge 1$ *and* $k \ge 1$.

(ii) $\mathrm{cdn}(G) = 3$, *if* $m, n, k \ge 1$ *and they are different.*

Proof.

(i) If $m = n$, then there are two $x - w_1$ detours, namely, $(x, v_1, v_2, \ldots, v_n, y, w_k, w_{k-1}, \ldots, w_2, w_1)$ and $(x, u_1, u_2, \ldots, u_m, y, w_k, w_{k-1}, \ldots, w_2, w_1)$. It is clear that each vertex of G lies on one of the two $x - w_1$ detours. Thus, $\{x, w_1\}$ is a c.d.b. of G, so $\mathrm{cnd}(G) = 2$.

Conversely, if $m \neq n$, say $m > n$ and $k \neq m, n$, then G does not contain adjacent vertices u, v such that $\{u, v\}$ is a detour set. Hence, the proof of Part (i) is completed.

(ii) If m, n, and k are different, say $m > n > k$, then it is clear that $\{w_1, x, v_1\}$ is a connected detour set of G. So, $\mathrm{cdn}(G) \le 3$. In view of Part (i), $\mathrm{cdn}(G) \ge 3$. Thus, $\mathrm{cdn}(G) = 3$. Hence, the proof of the proposition is completed. □

Remark 2. If G is a 2-connected bicyclic graph of order $p \ge 4$ with a cycle C and with exactly one chord, that is, an edge joining nonadjacent vertices of C, then $\mathrm{cdn}(G) = 2$.

This section is divided into two subsections according to the types of the bicyclic graphs.

3.1. The Connected Detour Numbers of Bicyclic Graphs of Three Cycles. Now assume that G is a connected bicyclic graph of order $p \ge 9$ with one or more cut-vertices and with three cycles, that is, three $x - y$ paths which are internally vertex disjoints denoted by Q_1, Q_2, and Q_3 as shown in Figure 5. We assume without loss of generality that

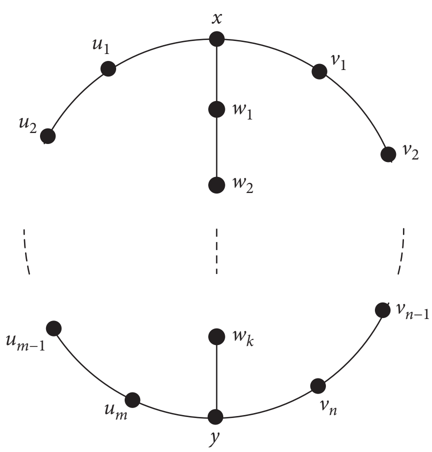

Figure 4: Bicyclic graph for the proof of Proposition 3.

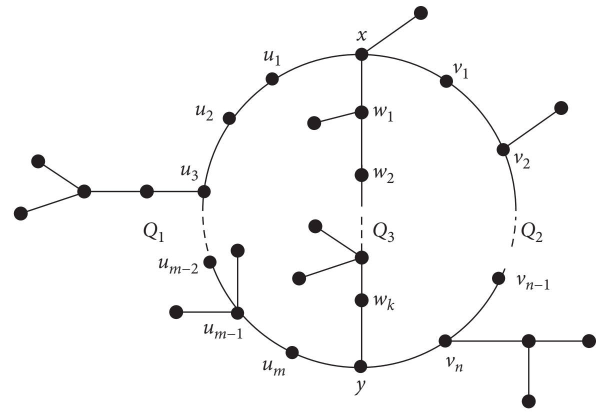

Figure 5: Bicyclic graph G.

$m \geq n \geq k \geq 1$. Let $\overline{T}$ be the set of all cycle vertices which are not cut-vertices in G, and let $T = V(G) - \overline{T}$.

We shall determine the connected detour number for three kinds of bicyclic graphs of three cycles.

Case 1. Assume that each Q_i, $1 \leq i \leq 3$, contains at least one cut-vertex other than x and y. Moreover, let T' be the set of all cycle cut-vertices in G. Then, we have the following proposition which determines the c.d.b. of such kind of bicyclic graph G.

Proposition 4. *Let G be a connected bicyclic graph of three cycles and with one or more cut-vertices on each Q_i, $i = 1, 2, 3$, other than x and y as explained above and shown in Figure 5. Then,*

$$\mathrm{cdn}(G) = |T| + |S|, \tag{8}$$

where S is a subset of $\overline{T}$ of minimum order such that the induced subgraph $G[T' \cup S]$ is connected.

Proof. Since $T' \subset T$ and G is connected, then the induced subgraph $G[T \cup S]$ is connected. Because each Q_i, $1 \leq i \leq 3$, contains a vertex of T, then $[T \cup S]$ contains x or y and each Q_i contains two adjacent vertices from $[T \cup S]$. Therefore, every vertex of the $x - y$ paths lies on an $u - v$ detour for some $u, v \in [T \cup S]$. Thus, $[T \cup S]$ is a c.d.s. of G. Moreover, from the minimalist of S we deduce that $[T \cup S]$ is a c.d.b. of G. Therefore, $\mathrm{cdn}(G) = |T| + |S|$. Hence, the proof is completed.

The following example illustrates Proposition 3.2. □

Example 3. Consider the bicyclic graph G shown in Figure 6.

It is clear that $p(G) = 32$ and $S = \{u_1, u_2, v_1, v_3, w_1, w_2\}$:

$$T = V(G) - \{u_1, u_2, u_5, u_6, u_7, v_1, v_3, v_5, v_6, w_1, w_2, w_4, w_5, y\}. \tag{9}$$

$$\text{Thus, } |T| = p(G) - 14 = 32 - 14 = 18$$
$$\therefore \mathrm{cdn}(G) = 18 + 6 = 24. \tag{10}$$

Case 2. Assume that G contains exactly one $x - y$ path that does not contain cut-vertices, other than x and y. So we have three possibilities for such bicyclic graph G:

(i) Let Q_1 and Q_3 each contains at least one internal cut-vertex, and Q_2 does not contain an internal cut-vertex. Then, $G - \{v_1, v_2, \ldots, v_n\} (= H_{1,3})$ is a unicyclic graph. By Theorem 2, $\mathrm{cdn}(H_{1,3}) = p(H_{1,3}) - \alpha_{1,3}$, in which $\alpha_{1,3}$ is defined in Definition 1 for the graph $H_{1,3}$. We can easily verify that if $B_{1,3}$ is a c.d.b. of $H_{1,3}$, then it is a c.d.b. of G because $m \geq n \geq k$. Therefore,

$$\mathrm{cdn}(G) = p(G) - n - \alpha_{1,3}. \tag{11}$$

(ii) Let Q_2 and Q_3 each contains at least one internal cut-vertex and Q_1 does not contain an internal cut-vertex. Then, $G - \{u_1, u_2, \ldots, u_m\} (= H_{2,3})$ is a unicyclic graph. By Theorem 2, $\mathrm{cdn}(H_{2,3}) = p(H_{2,3}) - \alpha_{2,3}$, where $\alpha_{2,3}$ is the number defined in Definition 1 for the unicyclic graph $H_{2,3}$. Thus, as in (i),

$$\mathrm{cdn}(G) = p(G) - m - \alpha_{2,3}. \tag{12}$$

(iii) Let Q_1 and Q_2 each contains at least one internal cut-vertex, and Q_3 does not contain an internal cut-vertices. Then, $G - \{w_1, w_2, \ldots, w_k\} (= H_{1,2})$ is a unicyclic graph. By Theorem 2, $\mathrm{cdn}(H_{1,2}) = p(H_{1,2}) - \alpha_{1,2}$, where the number $\alpha_{1,2}$ is explained in Definition 1. If $k = n$, then every c.d.b. of $H_{1,2}$ is a c.d.b. of G. Therefore,

$$\mathrm{cdn}(G) = p(G) - k - \alpha_{1,2}. \tag{13a}$$

If $k < n$, then any c.d.b. $B_{1,2}$ of $H_{1,2}$ is not c.d.s. of G because each vertex w_i $(i = 1, 2, \ldots, k)$ of Q_3 does not lie on a u-v detour for every pair of vertices $u, v \in B_{1,2}$. But it is clear that either $x \in B_{1,2}$ or $y \in B_{1,2}$. Thus, if $x \in B_{1,2}$ then $B_{1,2} \cup \{w_1\}$ is a c.d.b. of G; and if $y \in B_{1,2}$ then $B_{1,2} \cup \{w_k\}$ is a c.d.b. of G. Therefore,

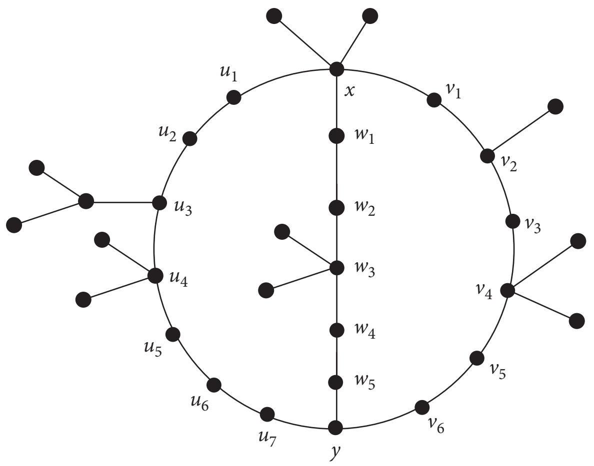

Figure 6: Graph G for Example 3.

$$\operatorname{cdn}(G) = p\left(H_{1,2}\right) - \alpha_{1,2} + 1 = p(G) - \left(k + \alpha_{1,2} - 1\right). \tag{13b}$$

The following example illustrates formulas (11)–(13b).

Example 4. Consider the graphs G_i, $1 \le i \le 4$, as shown in Figure 7.

It is easy to verify that:

$$\begin{aligned} &p\left(G_1\right) = 20,\ \alpha_{1,3} = 6,\ n = 5 \Longrightarrow \operatorname{cdn}\left(G_1\right) = 20 - 6 - 5 = 9, \\ &p\left(G_2\right) = 24,\ \alpha_{2,3} = 3,\ m = 7 \Longrightarrow \operatorname{cdn}\left(G_2\right) = 24 - 3 - 7 = 14, \\ &p\left(G_3\right) = 25,\ \alpha_{1,2} = 6,\ k = 5 = n \Longrightarrow \operatorname{cdn}\left(G_3\right) = 25 - 6 \\ &\qquad - 5 = 14, \\ &p\left(G_4\right) = 23,\ \alpha_{1,2} = 6,\ k = 3 < 5 = n \Longrightarrow \operatorname{cdn}\left(G_4\right) = 23 - 3 \\ &\qquad - 6 + 1 = 15. \end{aligned} \tag{14}$$

Case 3. Assume that the connected bicyclic graph G consists of two $x - y$ paths, and each path does not contain cut-vertices but only one $x - y$ path contains internal cut-vertices.

If Q_1 contains at least two internal cut-vertices, and Q_2 and Q_3 have no cut-vertices, $n = k$, then $Q_1 \cup Q_2$ is a unicyclic graph, denoted $H'_{1,2}$. It is clear that

$$\operatorname{cdn}(G) = \operatorname{cdn}\left(H'_{1,2}\right) = p\left(H'_{1,2}\right) - \alpha'_{1,2}, \tag{15}$$

where $\alpha'_{1,2}$ is given for $H'_{1,2}$ as defined in Definition 1. Thus,

$$\operatorname{cdn}(G) = p(G) - n - \alpha'_{1,2}. \tag{16}$$

Similar results we have if Q_i $(i = 2, 3)$ has at least two internal cut-vertices and the other $x - y$ paths have no cut-vertices. Therefore,

$$\begin{aligned} \operatorname{cdn}(G) &= p(G) - m - \alpha'_{2,3}, \quad \text{for } i = 2 \text{ and } m = k, \\ \operatorname{cdn}(G) &= p(G) - n - \alpha'_{1,3}, \quad \text{for } i = 3 \text{ and } m = n, \end{aligned} \tag{17}$$

where $\alpha'_{2,3}$ is for the unicyclic graph $H'_{2,3}$ and $\alpha'_{1,3}$ is for $H'_{1,3}$.

Remark 3. If the bicyclic graph G depicted in Figure 5 has exactly one cycle cut-vertex which is a vertex of the $x - y$ path Q_i $(1 \le i \le 3)$ including x and y, and the other two $x - y$ paths have equal lengths, then

$$\operatorname{cdn}(G) = 1 + |T|. \tag{18}$$

From now on, we assume that $m > n > k \ge 1$ (see Figure 5). If Q_1 contains internal cut-vertices and Q_2 and Q_3 contain no cut-vertices, then we may assume that the distance from x to the first cut-vertex along the $x - y$ path Q_1 is not more than the distance from y to the last cut-vertex along Q_1. Let $H''_{1,3}$ be the unicyclic graph constructed from $Q_1 \cup Q_3 \cup \{w_1, z\}$, where $w_1 z$ is an end-edge incident to vertex w_1 of Q_3. It is clear $H''_{1,3}$ contains vertices x and w_1 in addition to vertices from Q_1, and so

$$\begin{aligned} \operatorname{cdn}(G) &= \operatorname{cdn}\left(H''_{1,3}\right) - 1 = p\left(H''_{1,3}\right) - \alpha''_{1,3} - 1 = p(G) \\ &\quad + 1 - \alpha''_{1,3} - 1 - n. \end{aligned} \tag{19}$$

Therefore,

$$\operatorname{cdn}(G) = p(G) - n - \alpha''_{1,3}. \tag{20a}$$

If the distance from y to the last cut-vertex along Q_1 is less than the distance from x to the first cut-vertex along Q_1, then we have the unicyclic graph $H'''_{1,3} = Q_1 \cup Q_3 \cup \{w_k z\}$. Hence,

$$\operatorname{cdn}(G) = p(G) - n - \alpha'''_{1,3}, \tag{20b}$$

where $\alpha'''_{1,3}$ is the number defined for $H'''_{1,3}$ (Definition 1).

We have results similar to (20a) and (20b) for the cases where Q_i $(i = 2, 3)$ has internal cut-vertices, the other two $x - y$ paths have no internal cut-vertices and $m > n > k \ge 1$. Namely, $\operatorname{cdn}(G) = p(G) - m - \alpha''_{2,3}$ or $\operatorname{cdn}(G) = p(G) - m - \alpha'''_{2,3}$ for $i = 2$ or 3 and the unicyclic graphs $H''_{2,3}$ or $H'''_{2,3}$.

Remark 4. If the vertex x or the vertex y is the only cycle cut-vertex of the bicyclic graph G shown in Figure 5 and $m > n > k \ge 1$, then

$$\operatorname{cdn}(G) = 2 + |T|. \tag{21}$$

3.2. The Connected Detour Numbers of Bicyclic Graphs of Two Cycles. Let G be a bicyclic graph containing exactly two cycles C_1 and C_2, either having one vertex in common or there is a path joining a vertex of C_1 to a vertex of C_2. Thus, G is considered to consist of two unicyclic graphs G_1 and G_2 having exactly one vertex v in common.

Let G'_i $(i = 1, 2)$ be a uncyclic graph obtained from G_i by adding to it an end-edge vw_i. The connected detour number of G is determined by the following theorem.

Theorem 3. *Let G be a connected bicyclic graph of order $p \ge 5$ and consist of two edge-disjoint unicyclic graphs G_1 and G_2 having one vertex v in common. Then,* $\operatorname{cdn}(G) = p - \alpha_1 - \alpha_2$, *in which α_i $(i = 1, 2)$ is the number defined in Definition 1 for the unicyclic graph G'_i.*

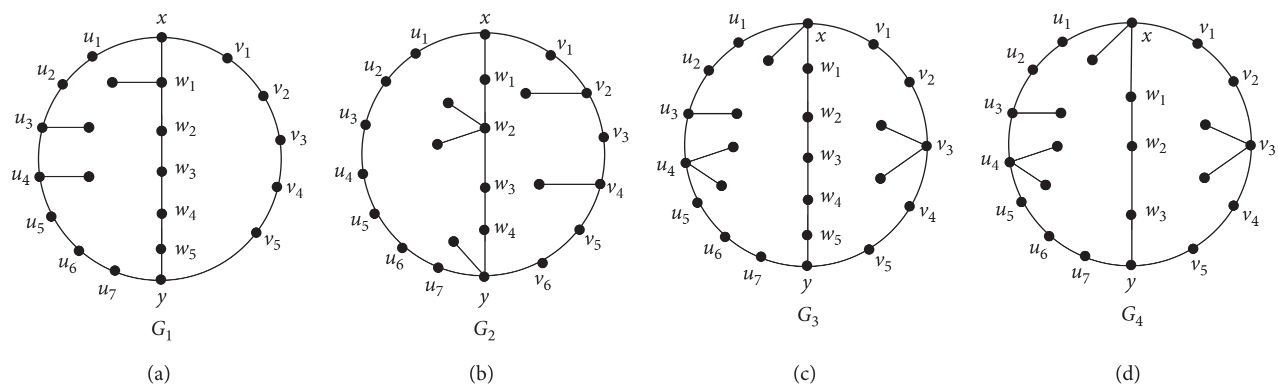

Figure 7: Graphs of Example 4.

Proof. Let B be a c.d.b. of G, then B contains v. Moreover, let B_i be the subset of B consisting of the vertices of G_i $(i = 1, 2)$. It is clear that each vertex of G_i lies on $u - v$ detour for some pair $u, v \in B_i$. Therefore, B_i is a c.d.s. of G_i $(i = 1, 2)$, that is because the connectedness of the induced subgraph $G[B]$ implies that $G_i[B_i]$ is connected. Since B is of minimum order, then $B_i \cup \{w_i\}$ is a c.d.b. of G'_i $(i = 1, 2)$. Conversely, it is clear that if B'_i $(i = 1, 2)$ is a c.d.b. of G'_i, then $(B'_1 \cup B'_2) - \{v, w_1, w_2\}$ is a c.d.b. of G. By Theorem 2, $\mathrm{cdn}(G'_i) = |B'_i| = p'_i - \alpha'_i$ $(i = 1, 2)$, in which p_i is the order of G'_i. Therefore, $\mathrm{cdn}(G) = p'_1 + p'_2 - (\alpha_1 + \alpha_2 + 3)$. Since $p'_1 + p'_2 = p + 3$, then $\mathrm{cdn}(G) = p - (\alpha_1 + \alpha_2)$. □

4. The Connected Detour Numbers of Cog-Graphs

Let G be a connected (p, q)-graph, then $G^{(c)}$ is the graph constructed from the graph G with q additional vertices $u_1, u_2, \ldots, u_q$ corresponding to the edges $e_1, e_2, \ldots, e_q$ of G and $2q$ additional edges obtained from joining u_i to the two vertices of e_i for all $i = 1, 2, \ldots, q$. Such class of graphs are called cog-graphs of G. For example, let G be a star of order five, then $G^{(c)}$ is cog-star of order nine shown in Figure 8.

Clearly if G is (p, q)–graph then $G^{(c)}$ is $(p + q, 3q)$-graph. The proofs of the following elementary results are obvious.

Proposition 5

(1) The cog-graph $G^{(c)}$ does not contain end-vertices.

(2) If the graph G has n end-vertices, then $G^{(c)}$ contains exactly $(q + n)$ vertices of degree 2.

(3) For every vertex $v \in V(G)$, $\deg_{G^{(c)}} v = 2\deg_G v$.

(4) Let $v \in V(G)$, then v is a cut-vertex in $G^{(c)}$ iff it is a cut-vertex in G.

Let $V(G) = \{v_1, v_2, \ldots, v_p\}$ and $V(G^{(c)}) = V(G) \cup \{u_1, u_2, \ldots, u_q\}$. If $(x_1, x_2, \ldots, x_{k-1}, x_k)$ is an $x_1 - x_k$ detour in G, then $(x_1, y_1, x_2, y_2, \ldots, x_{k-1}, y_{k-1}, x_k)$ is an $x_1 - x_k$ detour in $G^{(c)}$, in which $y_i \in U = \{u_1, u_2, \ldots, u_q\}$ for $1 \le i \le k - 1$ and y_i is the vertex that corresponds to edge $x_i x_{i+1}$ of G. Therefore, $D_{G^{(c)}}(x, y) = 2D_G(x, y)$, $\forall x, y \in V(G)$.

Moreover, if Q is an $y - y'$ detour in $G^{(c)}$ $y, y' \in U$, (as shown in Figure 9), then

$$D_{G^{(c)}}(y, y') = D_{G^{(c)}}(y', x) = D_{G^{(c)}}(y, x') = 2D_G(x, x'). \tag{22}$$

Any way, if S is a detour set of G, then S may not be a detour set of $G^{(c)}$. Also for some graphs G, $\mathrm{cdn}(G) \neq \mathrm{cdn}(G^{(c)})$. For example, if G is an odd cycle graph C_p with exactly one chord, then $\mathrm{cdn}(G) = 2$ and $\mathrm{cdn}(G^{(c)}) = 3$. But there are special graphs G such that $\mathrm{cdn}(G) = \mathrm{cdn}(G^{(c)})$, as given in the following proposition.

Proposition 6. *Let G be a connected graph. If G is a tree or a cycle graph, then*

$$\mathrm{cdn}(G) = \mathrm{cdn}\left(G^{(c)}\right) = \begin{cases} 2, & \text{if G is a cycle graph,} \\ p(G), & \text{G is a tree.} \end{cases} \tag{23}$$

Proof. It is obvious.

The following concepts were introduced by Santhakumaran and Athisayanathan in [12]. □

Definition 2. [12, 15] "Any edge e of G is said to lie on an $x - y$ detour Q, if e is an edge of Q. A set $S \subseteq V(G)$ is called an edge detour set of G if every edge of G lies on a detour joining a pair of vertices of S. The edge detour number $\mathrm{dn}_1(G)$ of G is the minimum order of its edge detour set. Any edge detour set of order $\mathrm{dn}_1(G)$ is called an edge detour basis of G. A graph which has an edge detour set is called a edge detour graph (denoted E.D. graph)."

There are graphs which are not E.D. graphs because they do not have edge detour sets [12]. For E.D. graphs we give the following definition.

Definition 3. Let S be an edge detour set (will be denoted e.d.s.) of an E.D. graph G. If $G[S]$ is connected then S is called a connected edge detour set (denoted c.e.d.s.). The connected detour number $\mathrm{cdn}_1(G)$ of G is defined by

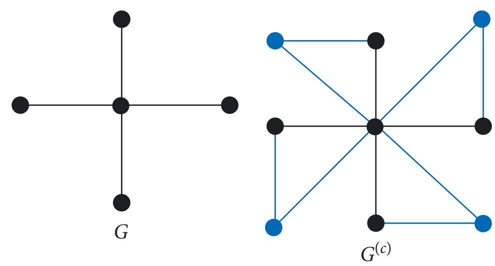

Figure 8

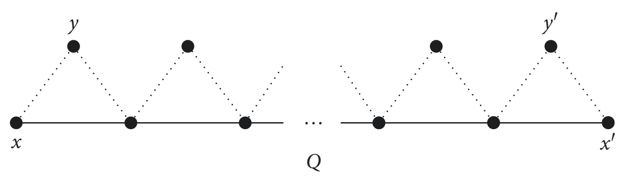

Figure 9: $x, x' \in V(G)$.

$$\mathrm{cdn}_1(G) = \min\{|S| : S \text{ is c.e.d.s. of } G\}. \tag{24}$$

Any c.e.d.s. of order $\mathrm{cdn}_1(G)$ is called connected edge detour basis (denoted c.e.d.b.) of G.

It can easily be proved that if G is an E.D. graph, then every c.e.d.s. of G contains all the end-vertices and all the cut-vertices of G. Thus, for every tree T, $\mathrm{cdn}_1(T) = p(T)$.

Now, we shall determine c.e.d.n. for some special classes of connected graphs.

Proposition 7. *For every cycle graph C_p with $p \geq 3$, $\mathrm{cdn}_1(C_p) = 3$.*

Proof. Let $C_p = (v_1, v_2, \ldots, v_p, v_1)$, then it is clear that every edge of C_p other than v_1v_2 lies on the $v_1 - v_2$ detour. Moreover, the edge v_1v_2 lies on the $v_2 - v_3$ detour. Thus, $\{v_1, v_2, v_3\}$ is a c.e.d.b. of C_p, and hence $\mathrm{cdn}_1(C_p) = 3$. □

Proposition 8. *Let K_p be a complete graph of order $p \geq 3$, then for every pair u, v of vertices, every edge other than uv lies on a $u - v$ detour of K_p.*

Proof. One can easily check that the statement is true for $p = 3, 4$, and 5. Now assume that the statement is true for $p = r \geq 5$, and consider K_{r+2}. Let x, y be any pair of vertices of K_{r+2}, and let $K_r = K_{r+2} - \{x, y\}$ and $V(K_r) = \{v_1, v_2, \ldots, v_{r-1}, v_r\}$ as shown in Figure 10.

By induction hypothesis for every pair v_i, v_j of vertices of K_r, every edge other than v_iv_j of K_r lies on a $v_i - v_j$ detour Q in K_r.

It is clear that the two edges xv_i, yv_j or (xv_j, yv_i) with Q produce an $x - y$ detour in K_{r+2}. This is true for all $i, j = 1, 2, \ldots, r$, $i \neq j$. Thus, every edge of K_{r+2} other than xy lies on some $x - y$ detour in K_{r+2}. Therefore, the proposition is true for K_{r+2}. Hence, by induction on p the proposition is true for K_p, $p \geq 3$. □

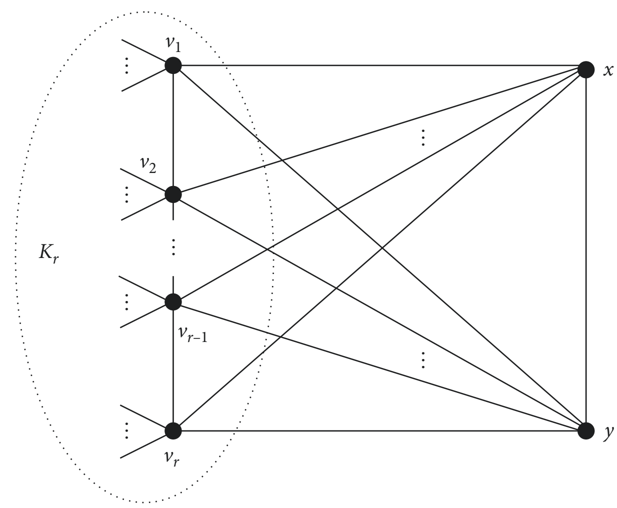

Figure 10: The graph K_{r+2}, $r \geq 5$.

Corollary 1. *For each complete graph K_p with $p \geq 3$, $cdn_1(K_p) = 3$.*

Proof. Let u, v, and w be any three vertices in K_p. By Proposition 8 every edge of K_p other than uv (resp., uw) lies on an $u - v$ detour (resp., $u - w$ detour). Thus, $\{u, v, w\}$ is a c.e.d.s. of K_p. Clearly, $\mathrm{cdn}_1(K_p) > 2$, and hence $\mathrm{cdn}_1(K_p) = 3$. □

Corollary 2. *For every complete graph K_p with $p \geq 2$, $cdn(K_p^{(c)}) = 2$.*

Proof. Let x, y be a pair of vertices of K_p, then by Proposition 8 every edge other than xy of K_p lies on an $x - y$ detour in K_p. Thus, every vertex of $K_p^{(c)}$ other than u lies on an $x - y$ detour in $K_p^{(c)}$, in which vertex u corresponds to the edge xy in $K_p^{(c)}$. Since vertex y is adjacent to u, then every vertex of $K_p^{(c)}$ lies on an $x - u$ detour. Therefore, $\{x, u\}$ is a c.d.b. of $K_p^{(c)}$, and hence $\mathrm{cdn}(K_p^{(c)}) = 2$. □

Proposition 9. *Let $K_{m,n}$, $m, n \geq 2$ be a complete bipartite graph, then for any pair of adjacent vertices u, v every edge other than uv lies on a $u - v$ detour in $K_{m,n}$.*

Proof. One can easily check that the proposition holds for $K_{2,2}$, $K_{2,3}$, and $K_{3,3}$. Now assume that it holds for

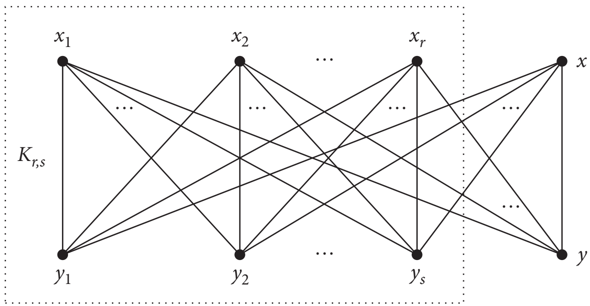

Figure 11: $K_{r+1,s+1}$, $r, s \geq 3$.

$K_{r,s}$, $r, s \geq 3$, and consider $K_{r+1,s+1}$. Let xy be any edge of $K_{r+1,s+1}$, and let $K_{r,s} = K_{r+1,s+1} - \{x, y\}$ as shown in Figure 11 in which its vertex set is $X \cup Y$, $X = \{x_1, x_2, \ldots, x_r\}$, and $Y = \{y_1, y_2, \ldots, y_s\}$. By induction hypothesis, every edge of $K_{r,s}$ other than $x_i y_j$ $(1 \leq i \leq r, 1 \leq j \leq s)$ lies on $x_i - y_j$ detour Q in $K_{r,s}$. Clearly, each $x_i - y_j$ detour Q in $K_{r,s}$ implies $x - y$ detour Q' (namely, x, $y_j - x_i$ detour, and y) in $K_{r+1,s+1}$. Moreover, each edge of $K_{r,s}$ with edges xy_j and yx_i lie on Q'. Since this holds for $i = 1, 2, \ldots, r$ and $j = 1, 2, \ldots, s$, then every edge of $K_{r+1,s+1}$ (other than xy) lies on an $x - y$ detour in $K_{r+1,s+1}$. Thus, by induction the proposition holds for every $K_{m,n}$, $m, n \geq 2$. □

Corollary 3. *For every complete bipartite graph $K_{m,n}$, $m, n \geq 2$, then* $\mathrm{cdn}_1(K_{m,n}) = 3$.

Proof. Consider the vertices x_1, x_2, and y_1 of $K_{m,n}$ where $x_1 x_2 \notin E(K_{m,n})$ and $x_1 y_1, x_2 y_1 \in E(K_{m,n})$. Then, by Proposition 9 every edge of $K_{m,n}$ (other than $x_1 y_1$) lies on an $x_1 - y_1$ detour, and $x_1 y_1$ lies on an $x_2 - y_1$ detour in $K_{m,n}$. Therefore, $\{x_1, x_2, y_1\}$ is a c.e.d.s. of $K_{m,n}$, and hence $\mathrm{cdn}_1(K_{m,n}) = 3$. □

Corollary 4. *For every complete bipartite graph $K_{m,n}$, $m.n \geq 2$, then* $\mathrm{cdn}(K_{m,n}^{(c)}) = 2$.

Proof. Let xy be an edge of $K_{m,n}$, then by Proposition 9 every edge other than xy lies on an $x - y$ detour in $K_{m,n}$. From the definition of cog-graphs, every vertex other than z lies on an $x - y$ detour in $K_{m,n}^{(c)}$, where z is the vertex that corresponds to the edge xy in $K_{m,n}^{(c)}$. Adding the edge yz to every such $x - y$ detour in $K_{m,n}^{(c)}$ we obtain $x - z$ detours, and hence every $K_{m,n}^{(c)}$ lies on an $x - z$ detour in $K_{m,n}^{(c)}$. Hence, $\mathrm{cdn}(K_{m,n}^{(c)}) = 2$. □

Proposition 10. *Let G be an E.D. graph of order $p \geq 2$, then* $\mathrm{cdn}(G^{(c)}) \leq \mathrm{cdn}_1(G)$.

Proof. Let B be a c.e.d.b. of G. If $u, v \in B$ and uv is an edge of G, and w is the vertex in $G^{(c)}$ that corresponds to the edge uv, then we interchange vertex v to vertex w in B. We repeat such interchange for every edge $G[B]$ to get the set B' of vertices in $G^{(c)}$. By Definitions 2 and 3, B' is a c.d.s. of $G^{(c)}$, and $|B| = |B'|$. Thus, $\mathrm{cdn}(G^{(c)}) \leq |B'| = \mathrm{cdn}_1(G)$. □

5. Conclusions

The connected detour numbers for three classes of connected simple graphs are determined in this research paper. The three classes are unicyclic graphs, bicyclic graphs, and cog-graphs for C_p^c, K_p^c, and $K_{m,n}^c$. We think that the methods used in proving the results in Section 3 can be used to determine the connected detour numbers for bridge graphs and chain graphs (defined in [16]) that are constructed from finite pairwise disjoint unicyclic graphs.

It is shown that $\mathrm{cdn}(G^c)$ is related to $\mathrm{cdn}_1(G)$, and in view of Proposition 10 we suggest the following problem: characterize edge detour graphs G such that $\mathrm{cdn}(G^{(c)}) = \mathrm{cdn}_1(G)$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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