Research Article

A Leap-Frog Finite Difference Method for Strongly Coupled System from Sweat Transport in Porous Textile Media

Qian Zhang^{1,2} and Chao Huang^{1,3}

¹College of Mathematics and Statistics, Shenzhen University, China

²Shenzhen Key Laboratory of Advanced Machine Learning and Applications, China

³Guangdong Key Laboratory of Intelligent Information Processing, China

Correspondence should be addressed to Qian Zhang; mazhangq@szu.edu.cn

Received 25 March 2019; Accepted 15 May 2019; Published 12 June 2019

Academic Editor: Tepper L. Gill

Copyright © 2019 Qian Zhang and Chao Huang. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we present an uncoupled leap-frog finite difference method for the system of equations arising from sweat transport through porous textile media. Based on physical mechanisms, the sweat transport can be viewed as the multicomponent flow that coupled the heat and moisture transfer, such that the system is nonlinear and strongly coupled. The leap-frog method is proposed to solve this system, with the second order accuracy in both spatial and temporal directions. We prove the existence and uniqueness of the solution to the system with optimal error estimates in the discrete L^2 norm. Numerical simulations are presented and analyzed, respectively.

1. Introduction

Single/multicomponent flow in porous textile media attracted considerable attention in the last several decades. See [1–4] for the single-component models and [5–9] for the multicomponent models. In this paper, we study the multicomponent sweat transport coupled with vapor and heat in porous textile media. In [10], Ye et al. proposed a quasi-steady-state single-component model which consists of a steady-state air equation and dynamic state equations for other components. Under certain conditions, the multicomponent model reduces to a new single-component model, and the physical process can be viewed as sweat transport (vapor and heat flow) governed by the conservation of mass and energy:

$$\frac{\partial}{\partial t}(\epsilon C) + \frac{\partial}{\partial x}(u_g \epsilon C) = -\Gamma_{ce}, \quad (1)$$

$$\begin{aligned} \frac{\partial}{\partial t}(C_{vt}T) + \frac{\partial}{\partial x}(\epsilon u_g C_{mg} CT) - \frac{\partial}{\partial x}\left(\kappa \frac{\partial T}{\partial x}\right) \\ = \lambda M_w \Gamma_{ce}, \end{aligned} \quad (2)$$

where ϵ is the porosity of the media, C is the vapor concentration, T is the temperature, κ is the thermal conductivity, λ is the latent heat of evaporation/condensation, and M_w is the molecular weight of water. The effective volumetric heat capacity C_{vt} is defined by

$$C_{vt} = \epsilon C_{mg} C + (1 - \epsilon) C_{vf}, \quad (3)$$

where C_{mg} is the molar heat capacity and C_{vf} is the volumetric heat capacity of fiber.

By Darcy's law, the gas velocity u_g is defined as

$$u_g = -\frac{k}{\mu} \frac{\partial P}{\partial x}, \quad (4)$$

where k is the permeability and μ is the dynamic viscosity, which usually is density-dependent for the compressible flow. Here we choose a linear form of $\mu := \nu C$, where ν is a certain constant.

By the Hertz-Knudsen equation [11], the phase change rate Γ_{ce} is defined as

$$\Gamma_{ce} = \beta_\Gamma \left(\frac{P - P_{\text{sat}}(T)}{\sqrt{T}} \right), \quad (5)$$

where β_r is a positive constant, the saturation pressure P_{sat} is determined from experimental measurements [12], and the pressure P is given by $P = RCT$, where R is the universal gas constant.

With nondimensionalization, the sweat transport process (1)-(2) can be described by the following system:

$$c_t - ((c\theta)_x)_x = -\Gamma(c, \theta), \quad (6)$$

$$0 < x < L, \quad t > 0,$$

$$(c\theta)_t + (\sigma\theta)_t - ((c\theta)_x \theta)_x - \kappa\theta_{xx} = \lambda\Gamma(c, \theta), \quad (7)$$

$$0 < x < L, \quad t > 0,$$

where $(\cdot)_x = \partial/\partial x$, $(\cdot)_t = \partial/\partial t$, $\Gamma(c, \theta) = c\sqrt{\theta} - p_s(\theta)$, and $p_s(\theta) \sim P_{\text{sat}}(\theta)/\sqrt{\theta}$ is a smooth and increasing function satisfying $p_s(0) = 0$.

Since the right boundary is exposed to environment and the left boundary is connected to the body, we consider commonly used Robin type boundary conditions

$$(c\theta)_x = \alpha_1(c - \mu_1), \quad x = 0, \quad 0 < t \leq T, \quad (8)$$

$$(c\theta)_x = \alpha_2(\mu_2 - c), \quad x = L, \quad 0 < t \leq T, \quad (9)$$

$$\theta_x = \beta_1(\theta - \nu_1), \quad x = 0, \quad 0 < t \leq T, \quad (10)$$

$$\theta_x = \beta_2(\nu_2 - \theta), \quad x = L, \quad 0 < t \leq T, \quad (11)$$

and the initial conditions

$$c(x, 0) = c_0(x),$$

$$\theta(x, 0) = \theta_0(x), \quad (12)$$

$$0 \leq x \leq L.$$

Physically, parameters α_i , β_i , μ_i , ν_i , $i = 1, 2$, and σ are nonnegative constants [1, 2, 6]. We define initial condition parameters $c_0(x) \geq \underline{c}$, $\theta_0(x) \geq \underline{\theta}$ with $\underline{c}$ and $\underline{\theta}$ being positive constants.

Due to the strong nonlinearity and the coupling of the system, both theoretical and numerical analyses of the system are difficult. Numerical analysis for some related systems of parabolic/elliptic equations can be found in [13–20]. Existence and uniqueness of a classical solution for a steady-state model was given in [10]. Existence of a weak solution for the corresponding dynamic models was given in [21, 22]. Positivity of temperature and nonnegativity of vapor density were also proved here. Recently, a finite difference method second-order in space and first-order in time for the system (6)-(12) was presented in [23], where the backward semi-implicit Euler scheme is applied in the temporal direction and central finite difference approximations are used in the spatial direction. In [23], authors presented optimal error estimates under the assumption that the step size τ and h are smaller than a positive constant.

In this paper, we propose an uncoupled leap-frog finite difference method for the system (6)-(12) with second-order accuracy in both spatial and temporal directions. We prove the existence and uniqueness of a solution to the finite

difference system with optimal error estimates in the discrete L^2 norm, under the condition that the mesh size τ and h are smaller than a positive constant which depends solely upon the physical parameters involved in the equations. Due to the strong nonlinearity and the coupling of equations, the method presented in [23] does not apply to the leap-frog scheme directly. One of the difficulties is to show convergence of the numerical solution without restriction on the grid ratio. In this paper, we assume that the solution $(c(x, t), \theta(x, t))$ to the system (6)-(12) satisfies that

$$c_{\min} \leq c(x, t) \leq c_{\max},$$

$$\theta_{\min} \leq \theta(x, t) \leq \theta_{\max} \quad (13)$$

for some positive constants $c_{\min}$, $c_{\max}$, $\theta_{\max}$, and $\theta_{\min}$.

The manuscript is organized as follows: in Section 2, we present an uncoupled leap-frog finite difference method for the nonlinear sweat transport system. In Section 3, we prove the existence and uniqueness of the solution to the sweat transport system with the optimal error estimate in the discrete L^2 norm. Numerical results will be presented in Section 4 to support our theoretical results.

2. The Leap-Frog Finite Difference Scheme

For convenience of calculations, we add the equation (6) times θ into the equation (7); thus the governing system (6)-(7) can be modified as

$$c_t - ((c\theta)_x)_x = -\Gamma(c, \theta), \quad (14)$$

$$0 < x < L, \quad t > 0,$$

$$(c + \sigma)\theta_t - (c\theta)_x \theta_x - \kappa\theta_{xx} = (\lambda + \theta)\Gamma(c, \theta), \quad (15)$$

$$0 < x < L, \quad t > 0.$$

Due to the practical interest in a long time period, say 8–24 hours, we present an uncoupled leap-frog finite difference scheme in the temporal direction and the central finite difference (volume) scheme in the spatial direction for the above system with the initial/boundary conditions (8)-(12).

Let $\mathcal{T}$ be a positive number, let $\Omega_h = \{x_i \mid x_i = ih, 0 \leq i \leq M\}$ be a uniform partition in $[0, L]$, and let $\Omega_\tau = \{t_n \mid t_n = n\tau, 0 \leq n \leq N\}$ be a uniform partition in $[0, \mathcal{T}]$, where $h = L/M$ and $\tau = \mathcal{T}/N$ are the step size in the spatial and temporal directions, respectively. Denote $x_{i+1/2} = (x_i + x_{i+1})/2$ in the spatial cell and let $\{v_i^n \mid 0 \leq i \leq M, 0 \leq n \leq N\}$ be a mesh function defined on $\Omega_{h\tau}$, where $\Omega_{h\tau} = \Omega_h \times \Omega_\tau$. Some notations are introduced below:

$$\nabla_t v_i^n = \frac{1}{2\tau} (v_i^{n+1} - v_i^{n-1}),$$

$$v_i^{\hat{n}} = \frac{1}{2} (v_i^{n+1} + v_i^{n-1}),$$

$$v_{i+1/2}^{\hat{n}} = \frac{1}{2} (v_{i+1}^{n+1} + v_i^{n-1}),$$

$$v_{i+1/2}^n = \frac{1}{2} (v_i^n + v_{i+1}^n),$$

$$\begin{aligned}
\delta_x v_{i+1/2}^n &= \frac{1}{h} (v_{i+1}^n - v_i^n), \\
\delta_x^* v_i^{\hat{n}} &= \frac{1}{h} (v_{i+1/2}^{\hat{n}} - v_{i-1/2}^{\hat{n}}), \\
\delta_x v_{i+1/2}^{\hat{n}} &= \frac{1}{2} (\delta_x v_{i+1/2}^{n+1} + \delta_x v_{i+1/2}^{n-1}), \\
\delta_x^2 v_{i+1/2}^{\hat{n}} &= \frac{1}{h} (\delta_x v_{i+1/2}^{\hat{n}} - \delta_x v_{i-1/2}^{\hat{n}}),
\end{aligned} \tag{16}$$

from which

$$(\delta_x v_{i+1/2}^m) v_{i+1/2}^n + v_{i+1/2}^m \delta_x v_{i+1/2}^n = \delta_x (v_{i+1/2}^m v_{i+1/2}^n). \tag{17}$$

The discrete system is defined by

$$\begin{aligned}
\nabla_t c_i^n - \frac{1}{h} [(c_{i+1/2}^n \delta_x \theta_{i+1/2}^n + \theta_{i+1/2}^n \delta_x c_{i+1/2}^{\hat{n}}) \\
- (c_{i-1/2}^n \delta_x \theta_{i-1/2}^n + \theta_{i-1/2}^n \delta_x c_{i-1/2}^{\hat{n}})] = -\Gamma(c_i^n, \theta_i^n), \\
1 \leq i \leq M-1,
\end{aligned} \tag{18}$$

$$\begin{aligned}
\nabla_t c_0^n - \frac{2}{h} [(c_{1/2}^n \delta_x \theta_{1/2}^n + \theta_{1/2}^n \delta_x c_{1/2}^{\hat{n}}) - \alpha_1 (c_0^{\hat{n}} - \mu_1)] \\
= -\Gamma(c_0^n, \theta_0^n),
\end{aligned} \tag{19}$$

$$\begin{aligned}
\nabla_t c_M^n - \frac{2}{h} [\alpha_2 (\mu_2 - c_M^{\hat{n}}) \\
- (c_{M-1/2}^n \delta_x \theta_{M-1/2}^n + \theta_{M-1/2}^n \delta_x c_{M-1/2}^{\hat{n}})] \\
= -\Gamma(c_M^n, \theta_M^n),
\end{aligned} \tag{20}$$

$$\begin{aligned}
(c_i^{\hat{n}} + \sigma) \nabla_t \theta_i^n - \frac{1}{2} (u_{i-1/2}^{\hat{n}} \delta_x \theta_{i-1/2}^{\hat{n}} + u_{i+1/2}^{\hat{n}} \delta_x \theta_{i+1/2}^{\hat{n}}) \\
- \frac{\kappa}{h} (\delta_x \theta_{i+1/2}^{\hat{n}} - \delta_x \theta_{i-1/2}^{\hat{n}}) = (\lambda + \theta_i^n) \Gamma(c_i^n, \theta_i^n), \\
1 \leq i \leq M-1,
\end{aligned} \tag{21}$$

$$\begin{aligned}
(c_0^{\hat{n}} + \sigma) \nabla_t \theta_0^n - u_{1/2}^{\hat{n}} \delta_x \theta_{1/2}^{\hat{n}} - \frac{2\kappa}{h} [\delta_x \theta_{1/2}^{\hat{n}} \\
- \beta_1 (\theta_0^{\hat{n}} - \nu_1)] = (\lambda + \theta_0^n) \Gamma(c_0^n, \theta_0^n),
\end{aligned} \tag{22}$$

$$\begin{aligned}
(c_M^{\hat{n}} + \sigma) \nabla_t \theta_M^n - u_{M-1/2}^{\hat{n}} \delta_x \theta_{M-1/2}^{\hat{n}} \\
- \frac{2\kappa}{h} [\beta_2 (\gamma_2 - \theta_M^{\hat{n}}) - \delta_x \theta_{M-1/2}^{\hat{n}}] = (\lambda + \theta_M^n) \\
\cdot \Gamma(c_M^n, \theta_M^n), \quad 1 \leq n \leq N-1,
\end{aligned} \tag{23}$$

and the discrete initial conditions

$$c_i^0 = c_0(x_i), \tag{24}$$

$$\theta_i^0 = \theta_0(x_i),$$

$$c_i^1 = c_0(x_i) + \tau c_t(x_i, 0), \tag{25}$$

$$\theta_i^1 = \theta_0(x_i) + \tau \theta_t(x_i, 0), \quad 0 \leq i \leq M,$$

where

$$u_{i+1/2}^{\hat{n}} = c_{i+1/2}^n \delta_x \theta_{i+1/2}^n + \theta_{i+1/2}^n \delta_x c_{i+1/2}^{\hat{n}}. \tag{26}$$

The computational procedure of the uncoupled leap-frog scheme at each time step is listed below:

Step 1. The vapor concentration c_j^{n+1} can be calculated by solving the tridiagonal linear systems defined in (18)-(20).

Step 2. With the updated vapor concentration c_j^{n+1} , we can get $c_0^{\hat{n}}$ and $u_{i+1/2}^{\hat{n}}$ correspondingly.

Step 3. Finally, the temperature θ_j^{n+1} can be obtained by solving the tridiagonal linear system (21)-(23).

3. The Leap-Frog Scheme and the Optimal Error Estimate

In this section, we will show the existence and uniqueness of the solution to the system (18)-(26) with optimal error estimates in the discrete L^2 norm. Let $v = \{v_j^n\}_{j=0}^M$ and $z = \{z_j\}_{j=0}^M$ be two mesh functions on Ω_h . We define the inner product and norms by

$$\begin{aligned}
(v, z) &= h \left(\frac{1}{2} v_0 z_0 + \sum_{i=1}^{M-1} v_i z_i + \frac{1}{2} v_M z_M \right), \\
\|v\| &= \sqrt{h \left[\frac{1}{2} (v_0)^2 + \sum_{i=1}^{M-1} (v_i)^2 + \frac{1}{2} (v_M)^2 \right]}, \\
\|v\|_{\infty} &= \max_{0 \leq i \leq M} |v_i|,
\end{aligned} \tag{27}$$

$$\|\delta_x v\| = \sqrt{h \sum_{i=0}^{M-1} \left(\frac{v_{i+1} - v_i}{h} \right)^2},$$

$$\|\delta_x^2 v\| = \sqrt{h \sum_{i=1}^{M-1} (\delta_x^2 v_i)^2}.$$

Let (C, Θ) be the solution of the system (6)-(12) and $C_i^n = c(x_i, t_n)$, $\Theta_i^n = \theta(x_i, t_n)$. The error functions are defined by

$$\begin{aligned}
\tilde{c}_i^n &= C_i^n - c_i^n, \\
\tilde{\theta}_i^n &= \Theta_i^n - \theta_i^n,
\end{aligned} \tag{28}$$

$$0 \leq i \leq M, \quad 0 \leq n \leq N.$$

We state our main result in the theorem below:

Theorem 1. Suppose that the solution (c, v) of the system (6)-(12) is in $C^{4,3}([0, L] \times [0, \mathcal{T}])$, satisfying (13). Then there exist positive constants h_0 and E_0 , independent of h and τ , such that,

when $\tau \leq E_c h \leq h_0$, the finite difference scheme (18)-(26) is uniquely solvable and

$$\begin{aligned} & \|\tilde{c}^{n+1}\|^2 + \|\tilde{\theta}^{n+1}\|^2 + \tau \sum_{m=1}^n (\|\delta_x \tilde{c}^m\|^2 + \|\delta_x \tilde{\theta}^m\|^2) \\ & \leq E_0 (\tau^2 + h^2)^2, \quad 1 \leq n \leq N-1. \end{aligned} \quad (29)$$

To prove the theorem, we make a stronger assumption that there exists $E_0 > 0$, independent of n, h, τ , such that the inequality,

$$\|\tilde{c}^{n+1}\|^2 + \|\tilde{\theta}^{n+1}\|^2 + \|\delta_x \tilde{\theta}^{n+1}\|^2 \leq E_0 (\tau^2 + h^2)^2, \quad (30)$$

holds for $n \leq k-1$. We prove the assumption and the theorem by induction method. By the initial condition (26), this is true for $n = 0$. In the next subsection we will show that this is also true for $n = 1$. In this part, we let E_c be a generic positive constant, which is associated with the physical parameters $\sigma, \lambda, \kappa, c_{\min}, c_{\max}, \theta_{\min}, \theta_{\max}$, the parameters involved in initial and boundary conditions and the solution of the system (6)-(12). E_c is independent of time step n , mesh size h, τ , and constant E_0 .

3.1. The Leap-Frog Scheme and Preliminaries. For convenience of calculations, we further introduce some notations. Let $u = (c\theta)_x, w = \theta_x$; thus the sweat transport system (6)-(7) can be reduced to

$$\begin{aligned} c_t - u_x &= -\Gamma(c, \theta), \\ 0 \leq x \leq L, \quad 0 < t \leq \mathcal{T}, \end{aligned} \quad (31)$$

$$\begin{aligned} u &= (c\theta_x + \theta c_x), \\ 0 \leq x \leq L, \quad 0 < t \leq \mathcal{T}, \end{aligned} \quad (32)$$

$$\begin{aligned} (c + \sigma)\theta_t - u\theta_x - \kappa w_x &= (\lambda + \theta)\Gamma(c, \theta), \\ 0 \leq x \leq L, \quad 0 < t \leq \mathcal{T}, \end{aligned} \quad (33)$$

$$\begin{aligned} w &= \theta_x, \\ 0 \leq x \leq L, \quad 0 < t \leq \mathcal{T}, \end{aligned} \quad (34)$$

with the initial and boundary conditions

$$\begin{aligned} u &= \alpha_1 (c - \mu_1), \quad x = 0, \quad 0 < t \leq \mathcal{T}, \\ u &= \alpha_2 (\mu_2 - c), \quad x = L, \quad 0 < t \leq \mathcal{T}, \\ w &= \beta_1 (\theta - \nu_1), \quad x = 0, \quad 0 < t \leq \mathcal{T}, \\ w &= \beta_2 (\nu_2 - \theta), \quad x = L, \quad 0 < t \leq \mathcal{T}, \\ c(x, 0) &= c_0(x), \\ \theta(x, 0) &= \theta_0(x), \\ 0 \leq x \leq L. \end{aligned} \quad (35)$$

The discrete leap-frog system (18)-(23) is modified as

$$\begin{aligned} \nabla_t c_i^n - \delta_x^* u_i^{\hat{n}} &= -\Gamma(c_i^n, \theta_i^n), \quad 1 \leq i \leq M-1, \\ \nabla_t c_0^n - \frac{2}{h} [u_{1/2}^{\hat{n}} - \alpha_1 (c_0^{\hat{n}} - \mu_1)] &= -\Gamma(c_0^n, \theta_0^n), \\ \nabla_t c_M^n - \frac{2}{h} [\alpha_2 (\mu_2 - c_M^{\hat{n}}) - u_{M-1/2}^{\hat{n}}] &= -\Gamma(c_M^n, \theta_M^n), \\ u_{i+1/2}^{\hat{n}} &= c_{i+1/2}^n \delta_x \theta_{i+1/2}^n + \theta_{i+1/2}^n \delta_x c_{i+1/2}^{\hat{n}}, \\ 0 \leq i \leq M-1, \\ (c_i^{\hat{n}} + \sigma) \nabla_t \theta_i^n - \frac{1}{2} (u_{i-1/2}^{\hat{n}} \delta_x \theta_{i-1/2}^{\hat{n}} + u_{i+1/2}^{\hat{n}} \delta_x \theta_{i+1/2}^{\hat{n}}) \\ &\quad - \kappa \delta_x^* w_i^{\hat{n}} = (\lambda + \theta_i^n) \Gamma(c_i^n, \theta_i^n), \quad 1 \leq i \leq M-1, \\ (c_0^{\hat{n}} + \sigma) \nabla_t \theta_0^n - u_{1/2}^{\hat{n}} \delta_x \theta_{1/2}^{\hat{n}} \\ &\quad - \frac{2\kappa}{h} [w_{1/2}^{\hat{n}} - \beta_1 (\theta_0^{\hat{n}} - \nu_1)] = (\lambda + \theta_0^n) \Gamma(c_0^n, \theta_0^n), \\ (c_M^{\hat{n}} + \sigma) \nabla_t \theta_M^n - u_{M-1/2}^{\hat{n}} \delta_x \theta_{M-1/2}^{\hat{n}} \\ &\quad - \frac{2\kappa}{h} [\beta_2 (\nu_2 - \theta_M^{\hat{n}}) - w_{M-1/2}^{\hat{n}}] \\ &= (\lambda + \theta_M^n) \Gamma(c_M^n, \theta_M^n), \\ w_{i+1/2}^{\hat{n}} &= \delta_x \theta_{i+1/2}^{\hat{n}}, \quad 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1. \end{aligned} \quad (36)$$

Let $U_i^n = u(x_i, t_n) = (c\theta)_x(x_i, t_n)$ and $W_i^n = w(x_i, t_n) = \theta_x(x_i, t_n)$. We denote by u_i^n and w_i^n the corresponding finite difference solution and

$$\begin{aligned} \tilde{u}_{i+1/2}^n &= U_{i+1/2}^n - u_{i+1/2}^n, \\ \tilde{w}_{i+1/2}^n &= W_{i+1/2}^n - w_{i+1/2}^n, \\ 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1. \end{aligned} \quad (37)$$

We get

$$\begin{aligned} \nabla_t C_i^n - \delta_x^* U_i^{\hat{n}} &= -\Gamma(C_i^n, \Theta_i^n) + R_{c,i}^n, \quad 1 \leq i \leq M-1, \\ \nabla_t C_0^n - \frac{2}{h} [U_{1/2}^{\hat{n}} - \alpha_1 (C_0^{\hat{n}} - \mu_1)] &= -\Gamma(C_0^n, \Theta_0^n) + R_{c,0}^n, \\ \nabla_t C_M^n - \frac{2}{h} [\alpha_2 (\mu_2 - C_M^{\hat{n}}) - U_{M-1/2}^{\hat{n}}] &= -\Gamma(C_M^n, \Theta_M^n) \\ &\quad + R_{c,M}^n, \\ U_{i+1/2}^{\hat{n}} &= C_{i+1/2}^n \delta_x \Theta_{i+1/2}^n + \Theta_{i+1/2}^n \delta_x C_{i+1/2}^{\hat{n}} + R_{u,i+1/2}^n, \\ 0 \leq i \leq M-1, \\ (C_i^{\hat{n}} + \sigma) \nabla_t \Theta_i^n - \frac{1}{2} (U_{i-1/2}^{\hat{n}} \delta_x \Theta_{i-1/2}^{\hat{n}} + U_{i+1/2}^{\hat{n}} \delta_x \Theta_{i+1/2}^{\hat{n}}) \\ &\quad - \kappa \delta_x^* W_i^{\hat{n}} = (\lambda + \Theta_i^n) \Gamma(C_i^n, \Theta_i^n) + R_{\theta,i}^n, \\ 1 \leq i \leq M-1, \end{aligned}$$

$$\begin{aligned}
& (C_0^{\hat{n}} + \sigma) \nabla_t \Theta_0^n - U_{1/2}^n \delta_x \Theta_{1/2}^{\hat{n}} \\
& - \frac{2\kappa}{h} [W_{1/2}^{\hat{n}} - \beta_1 (\Theta_0^{\hat{n}} - \gamma_1)] \\
& = (\lambda + \Theta_0^n) \Gamma(C_0^n, \Theta_0^n) + R_{\theta,0}^n, \\
& (C_M^{\hat{n}} + \sigma) \nabla_t \Theta_M^n - U_{M-1/2}^n \delta_x \Theta_{M-1/2}^{\hat{n}} \\
& - \frac{2\kappa}{h} [\beta_2 (\gamma_2 - \Theta_M^{\hat{n}}) - W_{M-1/2}^{\hat{n}}] \\
& = (\lambda + \Theta_M^n) \Gamma(C_M^n, \Theta_M^n) + R_{\theta,M}^n, \\
& W_{i+1/2}^{\hat{n}} = \delta_x \Theta_{i+1/2}^{\hat{n}} + R_{w,i+1/2}^n, \\
& 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1,
\end{aligned} \tag{38}$$

and the initial conditions

$$\begin{aligned}
C_i^0 &= c_0(x_i), \\
\Theta_i^0 &= \theta_0(x_i), \\
C_i^1 &= c_0(x_i) + \tau c_t(x_i, 0) + \tau R_{c,i}^1, \\
\Theta_i^1 &= \theta_0(x_i) + \tau \theta_t(x_i, 0) + \tau R_{\theta,i}^1,
\end{aligned} \tag{39}$$

where

$$\begin{aligned}
& |R_{\theta,i}^1|, |R_{\theta,i}^1| \leq E_1(\tau + h^2), \quad 0 \leq i \leq M, \\
& |R_{c,i}^n|, |R_{\theta,i}^n| \leq E_2(\tau^2 + h^2), \quad 1 \leq i \leq M-1, \\
& |R_{u,i+1/2}^n|, |R_{w,i+1/2}^n| \leq E_2(\tau^2 + h^2), \quad 0 \leq i \leq M-1, \\
& |R_{c,0}^n|, |R_{c,M}^n| \leq E_2(\tau^2 + h^2), \\
& |R_{\theta,0}^n|, |R_{\theta,M}^n| \leq E_2(\tau^2 + h), \quad 1 \leq n \leq N-1.
\end{aligned} \tag{40}$$

Subtracting the system (36) from the system (38), we get the error equations

$$\begin{aligned}
& \nabla_t \tilde{c}_i^n - \delta_x^* \tilde{w}_i^{\hat{n}} = -[\Gamma(C_i^n, \Theta_i^n) - \Gamma(c_i^n, \theta_i^n)] + R_{c,i}^n \\
& := \bar{R}_{c,i}^n, \quad 1 \leq i \leq M-1,
\end{aligned} \tag{41}$$

$$\begin{aligned}
& \nabla_t \tilde{c}_0^n - \frac{2}{h} [\tilde{u}_{1/2}^{\hat{n}} - \alpha_1 \tilde{c}_0^{\hat{n}}] = -[\Gamma(C_0^n, \Theta_0^n) - \Gamma(c_0^n, \theta_0^n)] \\
& + R_{c,0}^n := \bar{R}_{c,0}^n,
\end{aligned} \tag{42}$$

$$\begin{aligned}
& \nabla_t \tilde{c}_M^n - \frac{2}{h} [-\alpha_2 \tilde{c}_M^{\hat{n}} - \tilde{u}_{M-1/2}^{\hat{n}}] \\
& = -[\Gamma(C_M^n, \Theta_M^n) - \Gamma(c_M^n, \theta_M^n)] + R_{c,M}^n := \bar{R}_{c,M}^n,
\end{aligned} \tag{43}$$

$$\begin{aligned}
& \tilde{u}_{i+1/2}^{\hat{n}} = \tilde{c}_{i+1/2}^n \delta_x \Theta_{i+1/2}^n + \tilde{\theta}_{i+1/2}^n \delta_x C_{i+1/2}^{\hat{n}} \\
& + \tilde{c}_{i+1/2}^n \delta_x \tilde{\theta}_{i+1/2}^n + \tilde{\theta}_{i+1/2}^n \delta_x \tilde{c}_{i+1/2}^{\hat{n}} + R_{u,i+1/2}^n, \\
& 0 \leq i \leq M-1,
\end{aligned} \tag{44}$$

$$\begin{aligned}
& (c_i^{\hat{n}} + \sigma) \nabla_t \tilde{\theta}_i^n - \frac{1}{2} (u_{i-1/2}^{\hat{n}} \delta_x \tilde{\theta}_{i-1/2}^{\hat{n}} + u_{i+1/2}^{\hat{n}} \delta_x \tilde{\theta}_{i+1/2}^{\hat{n}}) \\
& - \kappa \delta_x^* \tilde{w}_i^{\hat{n}} = -\tilde{c}_i^{\hat{n}} \nabla_t \Theta_i^n \\
& + \frac{1}{2} (\tilde{u}_{i-1/2}^{\hat{n}} \delta_x \Theta_{i-1/2}^{\hat{n}} + \tilde{u}_{i+1/2}^{\hat{n}} \delta_x \Theta_{i+1/2}^{\hat{n}})
\end{aligned} \tag{45}$$

$$\begin{aligned}
& + [(\lambda + \Theta_i^n) \Gamma(C_i^n, \Theta_i^n) - (\lambda + \theta_i^n) \Gamma(c_i^n, \theta_i^n)] \\
& + R_{\theta,i}^n := \bar{R}_{\theta,i}^n, \quad 1 \leq i \leq M-1, \\
& (c_0^{\hat{n}} + \sigma) \nabla_t \tilde{\theta}_0^n - u_{1/2}^{\hat{n}} \delta_x \tilde{\theta}_{1/2}^{\hat{n}} - \frac{2\kappa}{h} [\tilde{w}_{1/2}^{\hat{n}} - \beta_1 \tilde{\theta}_0^{\hat{n}}] \\
& = -\tilde{c}_0^{\hat{n}} \nabla_t \Theta_0^n + \tilde{u}_{1/2}^{\hat{n}} \delta_x \Theta_{1/2}^{\hat{n}}
\end{aligned} \tag{46}$$

$$\begin{aligned}
& + [(\lambda + \Theta_0^n) \Gamma(C_0^n, \Theta_0^n) - (\lambda + \theta_0^n) \Gamma(c_0^n, \theta_0^n)] \\
& + R_{\theta,0}^n := \bar{R}_{\theta,0}^n,
\end{aligned}$$

$$\begin{aligned}
& (c_M^{\hat{n}} + \sigma) \nabla_t \tilde{\theta}_M^n - u_{M-1/2}^{\hat{n}} \delta_x \tilde{\theta}_{M-1/2}^{\hat{n}} \\
& - \frac{2\kappa}{h} [-\beta_2 \tilde{\theta}_M^{\hat{n}} - \tilde{w}_{M-1/2}^{\hat{n}}] = -\tilde{c}_M^{\hat{n}} \nabla_t \Theta_M^n \\
& + \tilde{u}_{M-1/2}^{\hat{n}} \delta_x \Theta_{M-1/2}^{\hat{n}}
\end{aligned} \tag{47}$$

$$\begin{aligned}
& + [(\lambda + \Theta_M^n) \Gamma(C_M^n, \Theta_M^n) - (\lambda + \theta_M^n) \Gamma(c_M^n, \theta_M^n)] \\
& + R_{\theta,M}^n := \bar{R}_{\theta,M}^n,
\end{aligned}$$

$$\begin{aligned}
& \tilde{w}_{i+1/2}^{\hat{n}} = \delta_x \tilde{\theta}_{i+1/2}^{\hat{n}} + R_{w,i+1/2}^n, \\
& 0 \leq i \leq M-1, \quad 1 \leq n \leq N-1,
\end{aligned} \tag{48}$$

and

$$\begin{aligned}
& \tilde{c}_i^0 = 0, \\
& \tilde{\theta}_i^0 = 0, \\
& \tilde{c}_i^1 = \tau R_{c,i}^1, \\
& \tilde{\theta}_i^1 = \tau R_{\theta,i}^1,
\end{aligned} \tag{49}$$

and by (40), we can directly derive the inequality

$$\|\tilde{c}^1\|^2 + \|\tilde{\theta}^1\|^2 + \|\delta_x \tilde{\theta}^1\|^2 \leq E_0(\tau^2 + h^2)^2. \tag{50}$$

To prove our main theorem, the following formula will be often used:

$$\begin{aligned}
& \left[v_{1/2} z_0 + h \sum_{i=1}^{M-1} \delta_x v_i z_i - v_{M-1/2} z_M \right] \\
& = \sum_{i=0}^{M-1} v_{i+1/2} (z_i^k - z_{i+1}^k).
\end{aligned} \tag{51}$$

In the following lemma, we present discrete Sobolev interpolation formulas and the proof can be found in [24].

Lemma 2. Let v and z be two mesh functions. Then for any positive constant ϵ ,

$$\|v\|_{\infty}^2 \leq \epsilon \|\delta_x v\|^2 + \left(\frac{1}{\epsilon} + \frac{1}{L}\right) \|v\|^2, \quad (52)$$

$$\|\delta_x v\|_{\infty} \leq \epsilon \|\delta_x^2 v\| + E_c \|\delta_x v\|^2. \quad (53)$$

Lemma 3.

$$\|\tilde{c}^n\|_{\infty}, \|\tilde{\theta}^n\|_{\infty} \leq 3E_0^{1/2} (\tau^{7/4} + h^{3/2}), \quad 1 \leq n \leq k, \quad (54)$$

$$\|\tilde{c}_x^{\hat{n}}\|, \|\tilde{\theta}_x^{\hat{n}}\| \leq E_0^{1/2} (\tau^{3/2} + h), \quad 1 \leq n \leq k-1. \quad (55)$$

Proof. From (30) for $0 \leq n \leq k-1$, we have

$$\begin{aligned} \|\tilde{c}^{n+1}\|^2 + \tau \|\delta_x \tilde{c}^{n+1}\|^2 &\leq E_0 (\tau^2 + h^2)^2, \\ \|\tilde{\theta}^{n+1}\|^2 + \tau \|\delta_x \tilde{\theta}^{n+1}\|^2 &\leq E_0 (\tau^2 + h^2)^2. \end{aligned} \quad (56)$$

When $\tau \leq h$, with the inverse inequality we have

$$\begin{aligned} |\tilde{c}_i^{n+1}|^2 &\leq \|\tilde{c}^{n+1}\|_{\infty}^2 \leq 2h^{-1} \|\tilde{c}^{n+1}\|^2 \leq 2h^{-1} E_0 (\tau^2 + h^2)^2 \\ &\leq 8E_0 h^3. \end{aligned} \quad (57)$$

When $h \leq \tau$, by taking $\epsilon = \tau^{1/2}$ in Lemma 2,

$$\begin{aligned} |\tilde{c}_i^{n+1}|^2 &\leq \|\tilde{c}^{n+1}\|_{\infty}^2 \\ &\leq \tau^{1/2} \|\delta_x \tilde{c}^{n+1}\|^2 + (\tau^{-1/2} + L^{-1}) \|\tilde{c}^{n+1}\|^2 \\ &\leq (\tau^{-1/2} + \tau^{1/2} L^{-1}) E_0 (\tau^2 + h^2)^2 \leq 8E_0 \tau^{7/2}, \\ &1 \leq i \leq M-1, \quad 0 \leq n \leq k-1. \end{aligned} \quad (58)$$

The first part of (54) is obtained and the second part and the inequality (55) can be proved similarly. $\square$

In addition, by Lemma 3, there exist constants $E_3 > 0$ and $s_0 > 0$ such that, when $h, \tau \leq s_0$,

$$\begin{aligned} \frac{c_{\min}}{2} &\leq c_i^n \leq 2c_{\max}, \\ \frac{\theta_{\min}}{2} &\leq \theta_i^n \leq 2\theta_{\max}, \\ &0 \leq i \leq M-1, \quad 1 \leq n \leq k, \end{aligned} \quad (59)$$

and

$$\begin{aligned} |\nabla_t \Theta_i^n|, |\nabla_t C_i^n| &\leq E_3, \quad 0 \leq i \leq M-1, \quad 1 \leq n \leq k, \\ |\delta_x C_{i+1/2}^n|, |\delta_x \Theta_{i+1/2}^n|, |\delta_x C_{i+1/2}^{\hat{n}}|, |\delta_x \Theta_{i+1/2}^{\hat{n}}| &\leq E_3, \\ &0 \leq i \leq M-1, \quad 1 \leq n \leq k, \\ |\tilde{c}_{i+1/2}^n|, |\tilde{\theta}_{i+1/2}^n| &\leq E_3, \quad 0 \leq i \leq M-1, \quad 1 \leq n \leq k, \end{aligned}$$

$$|\Gamma(\tilde{c}_{i+1/2}^n, \tilde{\theta}_{i+1/2}^n)| \leq E_3, \quad 0 \leq i \leq M-1, \quad 1 \leq n \leq k,$$

$$\begin{aligned} |(\lambda + \Theta_i^n) \Gamma(C_i^n, \Theta_i^n) - (\lambda + \theta_i^n) \Gamma(c_i^n, \theta_i^n)| \\ \leq E_3 (|\tilde{c}_i^n| + |\tilde{\theta}_i^n|), \quad 0 \leq i \leq M, \quad 1 \leq n \leq k, \end{aligned}$$

$$|\Gamma(C_i^n, \Theta_i^n) - \Gamma(c_i^n, \theta_i^n)| \leq E_3 (|\tilde{c}_i^n| + |\tilde{\theta}_i^n|),$$

$$0 \leq i \leq M, \quad 1 \leq n \leq k.$$

(60)

3.2. The Existence and Uniqueness. Since the coefficient matrix in the system (18)-(20) is strictly diagonally dominant, thus the system (18)-(20) has a unique solution c_i^{k+1} . Here we will discuss the boundedness of c_i^{k+1} .

Multiplying (41)-(43) by $h\tilde{c}_i^{\hat{n}}$, $h\tilde{c}_0^{\hat{n}}/2$, and $h\tilde{c}_M^{\hat{n}}/2$, respectively, we get

$$\begin{aligned} (\nabla_t \tilde{c}^{\hat{n}}, \tilde{c}^{\hat{n}}) + \alpha_1 |\tilde{c}_0^{\hat{n}}|^2 + \alpha_2 |\tilde{c}_M^{\hat{n}}|^2 &= \left[\tilde{u}_{1/2}^{\hat{n}} \tilde{c}_0^{\hat{n}} \right. \\ &+ h \sum_{i=1}^{M-1} \delta_x^* \tilde{u}_i^{\hat{n}} \tilde{c}_i^{\hat{n}} + \tilde{u}_{M-1/2}^{\hat{n}} \tilde{c}_M^{\hat{n}} \Big] + h \sum_{i=1}^{M-1} R_{c,i}^n \tilde{c}_i^{\hat{n}} + \frac{h}{2} \\ &\cdot R_{c,0}^n \tilde{c}_0^{\hat{n}} + \frac{h}{2} R_{c,M}^n \tilde{c}_M^{\hat{n}} \\ &- h \left\{ \frac{1}{2} [\Gamma(C_0^n, \Theta_0^n) - \Gamma(c_0^n, \theta_0^n)] \tilde{c}_0^{\hat{n}} \right. \\ &+ \sum_{i=1}^{M-1} [\Gamma(C_i^n, \Theta_i^n) - \Gamma(c_i^n, \theta_i^n)] \tilde{c}_i^{\hat{n}} \\ &+ \frac{1}{2} [\Gamma(C_M^n, \Theta_M^n) - \Gamma(c_M^n, \theta_M^n)] \tilde{c}_M^{\hat{n}} \Big\} := J_1 + J_2 \\ &+ J_3, \quad 1 \leq n \leq k. \end{aligned} \quad (61)$$

With (44), (51), (59), (60), and (40), we see that

$$\begin{aligned} -J_1 &= -\tilde{u}_{1/2}^{\hat{n}} \tilde{c}_0^{\hat{n}} - h \sum_{i=1}^{M-1} \delta_x \tilde{u}_i^{\hat{n}} \tilde{c}_i^{\hat{n}} + \tilde{u}_{M-1/2}^{\hat{n}} \tilde{c}_M^{\hat{n}} \\ &= h \sum_{i=0}^{M-1} \tilde{u}_{i+1/2}^{\hat{n}} \delta_x \tilde{c}_{i+1/2}^{\hat{n}} = h \sum_{i=0}^{M-1} [\tilde{c}_{i+1/2}^n \delta_x \Theta_{i+1/2}^n \\ &+ \tilde{\theta}_{i+1/2}^n \delta_x C_{i+1/2}^n + c_{i+1/2}^n \delta_x \tilde{\theta}_{i+1/2}^n + \theta_{i+1/2}^n \delta_x \tilde{c}_{i+1/2}^n \\ &+ R_{u,i+1/2}^n] \delta_x \tilde{c}_{i+1/2}^{\hat{n}} \geq \frac{\theta_{\min}}{2} \|\delta_x \tilde{c}^{\hat{n}}\|^2 - 2c_{\max} \|\delta_x \tilde{\theta}^n\| \\ &\cdot \|\delta_x \tilde{c}^{\hat{n}}\| - E_3 (\|\tilde{c}^n\| + \|\tilde{\theta}^n\|) \|\delta_x \tilde{c}^{\hat{n}}\| \\ &- h \sum_{i=0}^{M-1} |R_{u,i+1/2}^n| |\delta_x \tilde{c}_{i+1/2}^{\hat{n}}| \geq \frac{\theta_{\min}}{4} \|\delta_x \tilde{c}^{\hat{n}}\|^2 \\ &- E_c (\|\delta_x \tilde{\theta}^n\|^2 + \|\tilde{c}^n\|^2 + \|\tilde{\theta}^n\|^2) - E_c (\tau^2 + h^2)^2, \end{aligned} \quad (62)$$

and by using (60) again, we have

$$|J_3| \leq E_3 \left(\|\tilde{c}^n\| + \|\tilde{\theta}^n\| \right) \|\tilde{c}^{\hat{n}}\|, \quad (63)$$

and with (52),

$$\begin{aligned} |J_2| &\leq \frac{1}{2} h \sum_{i=1}^{M-1} \left[(R_{c,i}^n)^2 + (\tilde{c}_i^{\hat{n}})^2 \right] + \frac{h^2}{2} |R_{c,0}^n|^2 + \frac{1}{2} |\tilde{c}_0^{\hat{n}}|^2 \\ &\quad + \frac{h^2}{2} |R_{c,M}^n|^2 + \frac{1}{2} |\tilde{c}_M^{\hat{n}}|^2 \\ &\leq \|\tilde{c}^{\hat{n}}\|_{\infty}^2 + E_c \|\tilde{c}^{\hat{n}}\|^2 + \frac{E_2 L}{2} (\tau^2 + h^2)^2 \\ &\quad + E_2 h^2 (\tau^2 + h^2)^2 \\ &\leq \frac{\theta_{\min}}{8} \|\delta_x \tilde{c}^{\hat{n}}\|^2 + E_c \|\tilde{c}^{\hat{n}}\|^2 + E_c (\tau^2 + h^2)^2. \end{aligned} \quad (64)$$

Substituting the last three equations into (61) results in

$$\begin{aligned} \frac{1}{2} \nabla_t \|\tilde{c}^n\|^2 + \frac{\theta_{\min}}{8} \|\delta_x \tilde{c}^{\hat{n}}\|^2 + \alpha_1 |\tilde{c}_0^{\hat{n}}|^2 + \alpha_2 |\tilde{c}_M^{\hat{n}}|^2 \\ \leq E_c \left(\|\delta_x \tilde{\theta}^n\|^2 + \|\tilde{c}^n\|^2 + \|\tilde{\theta}^n\|^2 + \|\tilde{c}^{\hat{n}}\|^2 \right) \\ + E_c (\tau^2 + h^2)^2, \quad 1 \leq n \leq k, \end{aligned} \quad (65)$$

where we have noted $(\nabla_t \tilde{c}^n, \tilde{c}^{\hat{n}}) = (1/4\tau)(\|\tilde{c}^{n+1}\|^2 - \|\tilde{c}^{n-1}\|^2) = (1/2)\nabla_t \|\tilde{c}^n\|^2$. Moreover, by the assumption of the induction,

$$\begin{aligned} \|\tilde{c}^{k+1}\|^2 + 4\tau \left(\frac{\theta_{\min}}{8} \|\delta_x \tilde{c}^{\hat{k}}\|^2 + \alpha_1 |\tilde{c}_0^{\hat{k}}|^2 + \alpha_2 |\tilde{c}_M^{\hat{k}}|^2 \right) \\ \leq \|\tilde{c}^{k-1}\|^2 \\ + 4E_c \tau \left(\|\delta_x \tilde{\theta}^k\|^2 + \|\tilde{c}^k\|^2 + \|\tilde{\theta}^k\|^2 + \|\tilde{c}^{\hat{k}}\|^2 \right) \\ + 4\tau E_c (\tau^2 + h^2)^2. \end{aligned} \quad (66)$$

Since we have the fact that $\|\tilde{c}^{\hat{k}}\|^2 \leq (1/2)(\|\tilde{c}^{k+1}\|^2 + \|\tilde{c}^{k-1}\|^2)$, thus

$$\begin{aligned} (1 - 2\tau E_c) \|\tilde{c}^{k+1}\|^2 + \frac{\tau \theta_{\min}}{2} \|\delta_x \tilde{c}^{\hat{k}}\|^2 \\ \leq (E_0 + 4\tau E_c + 4E_c E_0 + 6\tau E_c E_0) (\tau^2 + h^2)^2. \end{aligned} \quad (67)$$

When $\tau E_c < 1/4$, we can get the inequality as

$$\|\tilde{c}^{k+1}\|^2 + \tau \theta_{\min} \|\delta_x \tilde{c}^{\hat{k}}\|^2 \leq E_4 (\tau^2 + h^2)^2. \quad (68)$$

Since E_4 are independent of k , by (13) when h and τ are small enough,

$$c_i^{k+1} \geq 0, \quad 0 \leq i \leq M. \quad (69)$$

Now we try to prove our main theorem. By noting (44), (60), (40), and Lemma 3,

$$\begin{aligned} \|\tilde{u}^{\hat{n}}\|^2 &= \sum_{i=0}^{M-1} h \left[c_{i+1/2}^n \delta_x \tilde{\theta}_{i+1/2}^n + \theta_{i+1/2}^n \delta_x \tilde{c}_{i+1/2}^{\hat{n}} \right. \\ &\quad \left. + \tilde{c}_{i+1/2}^n \delta_x \Theta_{i+1/2}^n + \tilde{\theta}_{i+1/2}^n \delta_x C_{i+1/2}^n + R_{u,i+1/2}^n \right]^2 \\ &\leq 5 \sum_{i=0}^{M-1} h \left(4c_{\max}^2 |\delta_x \tilde{\theta}_{i+1/2}^n|^2 + 4\theta_{\max}^2 |\delta_x \tilde{c}_{i+1/2}^{\hat{n}}|^2 \right. \\ &\quad \left. + E_3 |\tilde{c}_{i+1/2}^n|^2 + E_3 |\tilde{\theta}_{i+1/2}^n|^2 + |R_{u,i+1/2}^n|^2 \right) \\ &\leq 20c_{\max}^2 \|\delta_x \tilde{\theta}^n\|^2 + 20\theta_{\max}^2 \|\delta_x \tilde{c}^{\hat{n}}\|^2 + 5E_3 (\|\tilde{c}^n\|^2 \\ &\quad + \|\tilde{\theta}^n\|^2) + 5E_2 L (\tau^2 + h^2)^2, \quad 1 \leq n \leq k \end{aligned} \quad (70)$$

We can see that, when $\tau \leq h$, the assumption of induction and (68) show that

$$\begin{aligned} \|\tilde{u}^{\hat{k}}\|^2 &\leq 40\theta_{\max}^2 \frac{\|\tilde{c}^{k+1}\|^2}{h^2} + E_c h^4 \\ &\leq 160\theta_{\max}^2 E_4 h^2 + E_c h^4, \end{aligned} \quad (71)$$

and when $h \leq \tau$, by (68),

$$\begin{aligned} \|\tilde{u}^{\hat{k}}\|^2 &\leq 20\theta_{\max}^2 \frac{E_4 (\tau^2 + h^2)^2}{\tau \theta_{\min}} + E_c \tau^4 \\ &\leq 80 \frac{\theta_{\max}^2}{\theta_{\min}} E_4 \tau^3 + E_c \tau^4, \end{aligned} \quad (72)$$

which means there exists an E_5 independent of k , such that

$$\|\tilde{u}^{\hat{k}}\| \leq E_5 (\tau^{3/2} + h). \quad (73)$$

Multiplying the error equation (41) by $\delta_x^* u_i^{\hat{k}}$ leads to

$$\begin{aligned} \|\delta_x^* \tilde{u}^{\hat{k}}\|^2 &\leq \left| \sum_{i=1}^{M-1} \nabla_t \tilde{c}_i^k \delta_x^* u_i^{\hat{k}} \right| + \left| \sum_{i=1}^{M-1} \tilde{R}_{c,i} \delta_x^* u_i^{\hat{k}} \right| \\ &\leq 4 \|\nabla_t \tilde{c}^k\|^2 + 8E_3^2 (\|\tilde{\theta}^k\|^2 + \|\tilde{c}^k\|^2) \\ &\quad + \frac{3}{4} \|\delta_x^* \tilde{u}^{\hat{k}}\|^2 + 4E_2^2 L (\tau^2 + h^2)^2, \end{aligned} \quad (74)$$

that is,

$$\begin{aligned} \|\delta_x^* \tilde{u}^{\hat{k}}\|^2 &\leq 16 \|\nabla_t \tilde{c}^k\|^2 + 32E_3^2 (\|\tilde{\theta}^k\|^2 + \|\tilde{c}^k\|^2) \\ &\quad + 16E_2^2 L (\tau^2 + h^2)^2. \end{aligned} \quad (75)$$

We can see that, when $\tau \leq h$,

$$\|\tilde{u}^{\hat{k}}\|_{\infty} \leq h^{-1/2} \|\tilde{u}^{\hat{k}}\| \leq E_5 (h + h^{1/2}), \quad (76)$$

and when $h \leq \tau$, with Lemma 2,

$$\|\tilde{u}^k\|_\infty^2 \leq \|\delta_x^* \tilde{u}^k\|^2 + (1 + L^{-1}) \|\tilde{u}^k\|^2 \leq E_6 \tau^2, \quad (77)$$

where E_6 is independent of k . Then there exists $s_0 > 0$, when $h, \tau \leq s_0$,

$$\|\tilde{u}^k\|_\infty = \max_{1 \leq i \leq M} |u_{i-1/2}^k| \leq 2E_3. \quad (78)$$

With a time step condition $\tau \leq E_c h$, we can see that the coefficient matrix of the system (21)-(23) is strictly diagonally dominant. Thus this system has a unique solution θ_i^{k+1} .

3.3. The Optimal Error Estimate. We have proved the existence and uniqueness of the solution to the system and have derived the estimate (65) for $\tilde{c}^{n+1}$. In this part, we try to derive an estimate for $\tilde{\theta}^{n+1}$.

Multiplying (45)-(47) by $h\tilde{\theta}_i^n$, $h\tilde{\theta}_0^n/2$, and $h\tilde{\theta}_M^n/2$, respectively, we try to estimate each term below:

$$\begin{aligned} & h \left[\frac{1}{2} (c_0^n + \sigma) (\nabla_t \tilde{\theta}_0^n) \tilde{\theta}_0^n + \sum_{i=1}^{M-1} (c_i^n + \sigma) (\nabla_t \tilde{\theta}_i^n) \tilde{\theta}_i^n \right. \\ & \quad + \frac{1}{2} (c_M^n + \sigma) (\nabla_t \tilde{\theta}_M^n) \tilde{\theta}_M^n \Big] - \frac{h}{2} \left[u_{1/2}^n (\delta_x \tilde{\theta}_{1/2}^n) \tilde{\theta}_0^n \right. \\ & \quad + \sum_{i=1}^{M-1} (u_{i-1/2}^n \delta_x \tilde{\theta}_{i-1/2}^n + u_{i+1/2}^n \delta_x \tilde{\theta}_{i+1/2}^n) \tilde{\theta}_i^n \\ & \quad + u_{M-1/2}^n (\delta_x \tilde{\theta}_{M-1/2}^n) \tilde{\theta}_M^n \Big] - \kappa \left[\tilde{w}_{1/2}^n \tilde{\theta}_0^n \right. \\ & \quad + h \sum_{i=1}^{M-1} \delta_x^* \tilde{w}_{i+1/2}^n \tilde{\theta}_i^n - \tilde{w}_{M-1/2}^n \tilde{\theta}_M^n \Big] + \kappa \beta_1 |\tilde{\theta}_0^n|^2 \\ & \quad + \kappa \beta_2 |\tilde{\theta}_M^n|^2 = -h \left[\frac{1}{2} \tilde{c}_0^n (\nabla_t \Theta_0^n) \tilde{\theta}_0^n \right. \\ & \quad + \sum_{i=1}^{M-1} \tilde{c}_i^n (\nabla_t \Theta_i^n) \tilde{\theta}_i^n + \frac{1}{2} \tilde{c}_M^n (\nabla_t \Theta_M^n) \tilde{\theta}_M^n \Big] \\ & \quad + \frac{h}{2} \left[\tilde{u}_{1/2}^n (\delta_x \Theta_{1/2}^n) \tilde{\theta}_0^n \right. \\ & \quad + \sum_{i=1}^{M-1} (\tilde{u}_{i-1/2}^n \delta_x \Theta_{i-1/2}^n + \tilde{u}_{i+1/2}^n \delta_x \Theta_{i+1/2}^n) \tilde{\theta}_i^n \\ & \quad + \tilde{u}_{M-1/2}^n (\delta_x \Theta_{M-1/2}^n) \tilde{\theta}_M^n \Big] + \frac{h}{2} [(\lambda + \Theta_0^n) \\ & \quad \cdot \Gamma(C_0^n, \Theta_0^n) - (\lambda + \Theta_0^n) \Gamma(c_0^n, \theta_0^n)] \tilde{\theta}_0^n \\ & \quad + h \sum_{i=1}^{M-1} [(\lambda + \Theta_i^n) \Gamma(C_i^n, \Theta_i^n) - (\lambda + \Theta_i^n) \Gamma(c_i^n, \theta_i^n)] \tilde{\theta}_i^n \end{aligned}$$

$$\begin{aligned} & \cdot \tilde{\theta}_i^n + \frac{h}{2} [(\lambda + \Theta_M^n) \Gamma(C_M^n, \Theta_M^n) - (\lambda + \Theta_M^n) \\ & \cdot \Gamma(c_M^n, \theta_M^n)] \tilde{\theta}_M^n + \left[h \sum_{i=1}^{M-1} R_{\theta,i}^n \tilde{\theta}_i^n + \frac{h}{2} R_{\theta,0}^n \tilde{\theta}_0^n + \frac{h}{2} \right. \\ & \cdot R_{\theta,M}^n \tilde{\theta}_M^n \Big] := J_3 + J_4 + J_5 + J_6, \quad 1 \leq n \leq k. \end{aligned} \quad (79)$$

According to Lemma 2, (40), (51), and (60), three terms on the left can be bounded by

$$\begin{aligned} & h \left[\frac{1}{2} (c_0^n + \sigma) (\nabla_t \tilde{\theta}_0^n) \tilde{\theta}_0^n + \sum_{i=1}^{M-1} (c_i^n + \sigma) (\nabla_t \tilde{\theta}_i^n) \tilde{\theta}_i^n \right. \\ & \quad + \frac{1}{2} (c_M^n + \sigma) (\nabla_t \tilde{\theta}_M^n) \tilde{\theta}_M^n \Big] = \frac{h}{2} \left\{ \frac{1}{2} (c_0^n + \sigma) \right. \\ & \quad \cdot \nabla_t [(\tilde{\theta}_0^n)^2] + \sum_{i=1}^{M-1} (c_i^n + \sigma) \nabla_t [(\tilde{\theta}_i^n)^2] + \frac{1}{2} (c_M^n \\ & \quad + \sigma) \nabla_t [(\tilde{\theta}_M^n)^2] \Big\}, \end{aligned} \quad (80)$$

$$\begin{aligned} & \left| -\frac{h}{2} \left[u_{1/2}^n (\delta_x \tilde{\theta}_{1/2}^n) \tilde{\theta}_0^n \right. \right. \\ & \quad + \sum_{i=1}^{M-1} (u_{i-1/2}^n \delta_x \tilde{\theta}_{i-1/2}^n + u_{i+1/2}^n \delta_x \tilde{\theta}_{i+1/2}^n) \tilde{\theta}_i^n \\ & \quad + u_{M-1/2}^n (\delta_x \tilde{\theta}_{M-1/2}^n) \tilde{\theta}_M^n \Big] \Big| \\ & = \left| -h \sum_{i=0}^{M-1} u_{i+1/2}^n \delta_x \tilde{\theta}_{i+1/2}^n \tilde{\theta}_{i+1/2}^n \right| \\ & \leq 2E_3 \left| h \sum_{i=0}^{M-1} \delta_x \tilde{\theta}_{i+1/2}^n \tilde{\theta}_{i+1/2}^n \right| \leq \frac{\kappa}{4} \|\delta_x \tilde{\theta}^n\|^2 + E_c \|\tilde{\theta}^n\|^2, \end{aligned}$$

and

$$\begin{aligned} & -\kappa \left[\tilde{w}_{1/2}^n \tilde{\theta}_0^n + h \sum_{i=1}^{M-1} \delta_x^* \tilde{w}_i^n \tilde{\theta}_i^n - \tilde{w}_{M-1/2}^n \tilde{\theta}_M^n \right] \\ & = \kappa h \sum_{i=0}^{M-1} (\delta_x \tilde{\theta}_{i+1/2}^n)^2 + \kappa h \sum_{i=0}^{M-1} R_{w,i+1/2} \delta_x \tilde{\theta}_{i+1/2}^n \\ & \geq \kappa \|\delta_x \tilde{\theta}^n\|^2 - E_c h \sum_{i=0}^{M-1} |R_{w,i+1/2}^n|^2 - \frac{\kappa}{4} \|\delta_x \tilde{\theta}^n\|^2 \\ & \geq \frac{3\kappa}{4} \|\delta_x \tilde{\theta}^n\|^2 - E_c (\tau^2 + h^2)^2, \quad 1 \leq n \leq k. \end{aligned} \quad (81)$$

By (70), for those terms in the right hand side, we obtain

$$\begin{aligned} |J_3| & \leq E_3 (\|\tilde{c}^n\|^2 + \|\tilde{\theta}^n\|^2), \\ |J_4| & \leq hE_3 \left(|\tilde{u}_{1/2}^n| \cdot |\tilde{\theta}_0^n| + \sum_{i=1}^{M-1} (|\tilde{u}_{i-1/2}^n| + |\tilde{u}_{i+1/2}^n|) |\tilde{\theta}_i^n| \right) \end{aligned}$$

$$\begin{aligned}
& + \left| \tilde{u}_{M-1/2}^{\hat{n}} \cdot \left| \tilde{\theta}_M^{\hat{n}} \right| \right) \leq 2E_3 \left(\left\| \tilde{\theta}^{\hat{n}} \right\|^2 \right. \\
& + \left. \sum_{i=0}^{M-1} h \left| \tilde{u}_{i+1/2}^{\hat{n}} \right|^2 \right) \leq 40E_3 \theta_{\max}^2 \left\| \delta_x \tilde{c}^{\hat{n}} \right\|^2 \\
& + E_c \left(\left\| \delta_x \tilde{\theta}^{\hat{n}} \right\|^2 + \left\| \tilde{\theta}^{\hat{n}} \right\| + \left\| c^n \right\|^2 \right) + E_2 \left(\tau^2 + h^2 \right)^2, \\
|J_5| & \leq E_3 \left(\left\| \tilde{c}^{\hat{n}} \right\| + \left\| \tilde{\theta}^{\hat{n}} \right\| \right) \left\| \tilde{\theta}^{\hat{n}} \right\|. \\
|J_6| & \leq \frac{1}{2} h \sum_{i=1}^{M-1} \left[\left(R_{\theta,i}^n \right)^2 + \left(\tilde{\theta}_i^n \right)^2 \right] + \frac{h^2}{2} \left| R_{\theta,0}^n \right|^2 + \frac{1}{2} \left| \tilde{\theta}_0^n \right|^2 \\
& + \frac{h^2}{2} \left| R_{\theta,M}^n \right|^2 + \frac{1}{2} \left| \tilde{\theta}_M^n \right|^2 \leq \left\| \tilde{\theta}^{\hat{n}} \right\|_{\infty}^2 + E_c \left\| \tilde{\theta}^{\hat{n}} \right\|^2 \\
& + \frac{E_2 L}{2} \left(\tau^2 + h^2 \right)^2 + E_2 h^2 \left(\tau^2 + h \right)^2 \leq \frac{\kappa}{4} \left\| \delta_x \tilde{\theta}^{\hat{n}} \right\|^2 \\
& + \left\| \tilde{\theta}^{\hat{n}} \right\|^2 + E_c \left(\tau^2 + h^2 \right)^2.
\end{aligned} \tag{82}$$

Taking the last six equations into (79), we obtain

$$\begin{aligned}
& \frac{h}{2} \left\{ \frac{1}{2} \left(c_0^{\hat{n}} + \sigma \right) \nabla_t \left[\left(\tilde{\theta}_0^n \right)^2 \right] + \sum_{i=1}^{M-1} \left(c_i^{\hat{n}} + \sigma \right) \nabla_t \left[\left(\tilde{\theta}_i^n \right)^2 \right] \right. \\
& + \left. \frac{1}{2} \left(c_M^{\hat{n}} + \sigma \right) \nabla_t \left[\left(\tilde{\theta}_M^n \right)^2 \right] \right\} + \frac{\kappa}{4} \left\| \delta_x \tilde{\theta}^{\hat{n}} \right\|^2 \\
& + \kappa \beta_1 \left| \tilde{\theta}_0^{\hat{n}} \right|^2 + \kappa \beta_2 \left| \tilde{\theta}_M^{\hat{n}} \right|^2 \leq 40E_3 \theta_{\max}^2 \left\| \delta_x \tilde{c}^{\hat{n}} \right\|^2 \\
& + E_c \left(\left\| \delta_x \tilde{\theta}^{\hat{n}} \right\|^2 + \left\| \tilde{c}^{\hat{n}} \right\|^2 + \left\| \tilde{\theta}^{\hat{n}} \right\|^2 + \left\| \tilde{c}^{\hat{n}} \right\|^2 + \left\| \tilde{\theta}^{\hat{n}} \right\|^2 \right) \\
& + E_c \left(\tau^2 + h^2 \right)^2, \quad 1 \leq n \leq k.
\end{aligned} \tag{83}$$

Besides, we introduce a notation as $\tilde{\theta}_i^* = (1/2)[(\tilde{\theta}_i^{n+1})^2 + (\tilde{\theta}_i^{n-1})^2]$, and adding the first three equations into (36) by $h\tilde{\theta}_i^*/2$, $h\tilde{\theta}_0^*/4$, and $h\tilde{\theta}_M^*/4$, respectively, we have

$$\begin{aligned}
& \frac{h}{2} \left[\frac{1}{2} \left(\nabla_t c_0^n \right) \tilde{\theta}_0^* + \sum_{i=1}^{M-1} \left(\nabla_t c_i^n \right) \tilde{\theta}_i^* + \frac{1}{2} \left(\nabla_t c_M^n \right) \tilde{\theta}_M^* \right] \\
& - \frac{1}{2} \left[u_{1/2}^{\hat{n}} \tilde{\theta}_0^* + h \sum_{i=1}^{M-1} \delta_x^* u_{i+1/2}^{\hat{n}} \tilde{\theta}_i^* - u_{M-1/2}^{\hat{n}} \tilde{\theta}_M^* \right] \\
& + \frac{\alpha_1}{2} \left(c_0^{\hat{n}} - \mu_1 \right) \tilde{\theta}_0^* + \frac{\alpha_2}{2} \left(c_M^{\hat{n}} - \mu_2 \right) \tilde{\theta}_M^* \\
& = -\frac{h}{2} \left[\frac{1}{2} \tilde{\theta}_0^* \Gamma \left(c_0^n, \theta_0^n \right) + \sum_{i=1}^{M-1} \tilde{\theta}_i^* \Gamma \left(c_i^n, \theta_i^n \right) \right. \\
& + \left. \frac{1}{2} \tilde{\theta}_M^* \Gamma \left(c_M^n, \theta_M^n \right) \right], \quad 1 \leq n \leq k.
\end{aligned} \tag{84}$$

We now estimate the terms in (84). By (51), we denote

$$\begin{aligned}
|J_7| & := \left| -\frac{1}{2} \left[u_{1/2}^{\hat{n}} \tilde{\theta}_0^* + h \sum_{i=1}^{M-1} \delta_x^* u_{i+1/2}^{\hat{n}} \tilde{\theta}_i^* - u_{M-1/2}^{\hat{n}} \tilde{\theta}_M^* \right] \right| \\
& = \left| \frac{1}{2} \sum_{i=0}^{M-1} u_{i+1/2}^{\hat{n}} \left(\tilde{\theta}_{i+1}^* - \tilde{\theta}_i^* \right) \right| \\
& \leq \left| E_3 h \sum_{i=0}^{M-1} \left(\tilde{\theta}_{i+1/2}^{n+1} \delta_x \tilde{\theta}_{i+1/2}^{n+1} + \tilde{\theta}_{i+1/2}^{n-1} \delta_x \tilde{\theta}_{i+1/2}^{n-1} \right) \right| \\
& \leq \frac{E_3}{2} \left(\left\| \delta_x \tilde{\theta}^{n+1} \right\|^2 + \left\| \delta_x \tilde{\theta}^{n-1} \right\|^2 + \left\| \tilde{\theta}^{n+1} \right\|^2 + \left\| \tilde{\theta}^{n-1} \right\|^2 \right).
\end{aligned} \tag{85}$$

Using (60) again, we get

$$\begin{aligned}
& \left| -\frac{h}{2} \left[\frac{1}{2} \tilde{\theta}_0^* \Gamma \left(c_0^n, \theta_0^n \right) + \sum_{i=1}^{M-1} \tilde{\theta}_i^* \Gamma \left(c_i^n, \theta_i^n \right) \right. \right. \\
& + \left. \left. \frac{1}{2} \tilde{\theta}_M^* \Gamma \left(c_M^n, \theta_M^n \right) \right] \right| \leq \frac{E_3}{4} \left(\left\| \tilde{\theta}^{n+1} \right\|^2 + \left\| \tilde{\theta}^{n-1} \right\|^2 \right),
\end{aligned} \tag{86}$$

and with Lemma 2, we have

$$\begin{aligned}
& \frac{h}{2} \left[\frac{1}{2} \left(\nabla_t c_0^n \right) \tilde{\theta}_0^* + \sum_{i=1}^{M-1} \left(\nabla_t c_i^n \right) \tilde{\theta}_i^* + \frac{1}{2} \left(\nabla_t c_M^n \right) \tilde{\theta}_M^* \right] \\
& \leq \frac{\alpha_1 \mu_1}{2} \tilde{\theta}_0^* + \frac{\alpha_2 \mu_2}{2} \tilde{\theta}_M^* \\
& + E_c \left(\left\| \tilde{\theta}^{n+1} \right\|^2 + \left\| \tilde{\theta}^{n-1} \right\|^2 + \left\| \delta_x \tilde{\theta}^{n+1} \right\|^2 + \left\| \delta_x \tilde{\theta}^{n-1} \right\|^2 \right) \\
& \leq \frac{\alpha_1 \mu_1 + \alpha_2 \mu_2}{2} \left(\left\| \tilde{\theta}^{n+1} \right\|_{\infty}^2 + \left\| \tilde{\theta}^{n-1} \right\|_{\infty}^2 \right) \\
& + E_c \left(\left\| \tilde{\theta}^{n+1} \right\|^2 + \left\| \tilde{\theta}^{n-1} \right\|^2 + \left\| \delta_x \tilde{\theta}^{n+1} \right\|^2 + \left\| \delta_x \tilde{\theta}^{n-1} \right\|^2 \right) \\
& \leq E_c \left(\left\| \tilde{\theta}^{n+1} \right\|^2 + \left\| \tilde{\theta}^{n-1} \right\|^2 + \left\| \delta_x \tilde{\theta}^{n+1} \right\|^2 + \left\| \delta_x \tilde{\theta}^{n-1} \right\|^2 \right).
\end{aligned} \tag{87}$$

Moreover, by noting the fact that

$$\left(c_i^{\hat{n}} + \sigma \right) \nabla_t \left[\left(\tilde{\theta}_i^n \right)^2 \right] + \tilde{\theta}_i^* \nabla_t c_i^n = \nabla_t \left[\left(c_i^n + \sigma \right) \left(\tilde{\theta}_i^n \right)^2 \right], \tag{88}$$

adding (87) in (84) and using Lemma 2 again, we further get

$$\begin{aligned}
& \frac{h}{2} \left\{ \frac{1}{2} \nabla_t \left[\left(c_0^n + \sigma \right) \left(\tilde{\theta}_0^n \right)^2 \right] + \sum_{i=1}^{M-1} \nabla_t \left[\left(c_i^n + \sigma \right) \left(\tilde{\theta}_i^n \right)^2 \right] \right. \\
& + \left. \frac{1}{2} \nabla_t \left[\left(c_M^n + \sigma \right) \left(\tilde{\theta}_M^n \right)^2 \right] \right\} + \frac{\kappa}{4} \left\| \delta_x \tilde{\theta}^{\hat{n}} \right\|^2 + \kappa \beta_1 \left| \tilde{\theta}_0^{\hat{n}} \right|^2 \\
& + \kappa \beta_2 \left| \tilde{\theta}_M^{\hat{n}} \right|^2 \leq 40E_3 \theta_{\max}^2 \left\| \delta_x \tilde{c}^{\hat{n}} \right\|^2 + E_c \left(\left\| \delta_x \tilde{\theta}^{\hat{n}} \right\|^2 \right. \\
& + \left\| \tilde{\theta}^{n+1} \right\|^2 + \left\| \tilde{\theta}^{n-1} \right\|^2 + \left\| \tilde{\theta}^{\hat{n}} \right\|^2 + \left\| \tilde{c}^{\hat{n}} \right\|^2 + \left\| \tilde{\theta}^{n-1} \right\|^2 \\
& + \left\| \tilde{c}^{n-1} \right\|^2 \left. \right) + E_c \left(\tau^2 + h^2 \right)^2, \quad 1 \leq n \leq k.
\end{aligned} \tag{89}$$

Multiplying the last equation with $s_1 = \theta_{\min}/320E_3\theta_{\max}^2$ and adding it into equation (65), we get

$$\begin{aligned} & \nabla_t \|\tilde{c}^n\|^2 + \frac{\theta_{\min}}{8} \|\delta_x \tilde{c}^n\|^2 + [2\alpha_1 |\tilde{c}_0^n|^2 + 2\alpha_2 |\tilde{c}_M^n|^2 \\ & + s_1 \kappa \beta_1 |\tilde{\theta}_0^n|^2 + s_1 \kappa \beta_2 |\tilde{\theta}_M^n|^2] \\ & + \frac{s_1 h}{2} \left[\frac{1}{2} \nabla_t [(c_0^n + \sigma) |\tilde{\theta}_0^n|^2] \right. \\ & \left. + \sum_{i=1}^{M-1} \nabla_t [(c_i^n + \sigma) |\tilde{\theta}_i^n|^2] + \frac{1}{2} \nabla_t [(c_M^n + \sigma) |\tilde{\theta}_M^n|^2] \right] \quad (90) \\ & + \frac{\kappa s_1}{4} \|\delta_x \tilde{\theta}^n\|^2 \leq E_c (\|\delta_x \tilde{\theta}^n\|^2 + \|\tilde{\theta}^{n+1}\|^2 + \|\tilde{c}^{n+1}\|^2 \\ & + \|\tilde{\theta}^n\|^2 + \|\tilde{c}^n\|^2 + \|\tilde{\theta}^{n-1}\|^2 + \|\tilde{c}^{n-1}\|^2) + E_c (\tau^2 \\ & + h^2)^2, \quad 1 \leq n \leq k. \end{aligned}$$

Finally we estimate $\|\delta_x \tilde{\theta}^n\|$. Multiplying the error equation (45) by $-h\delta_x^2 \tilde{\theta}^n / (c_i^n + \sigma)$ and summing up the resulting equations for $i = 1, 2, \dots, M-1$, we have

$$\begin{aligned} & -h \sum_{i=1}^{M-1} (\nabla_t \tilde{\theta}_i^n) \delta_x^2 \tilde{\theta}_i^n + \frac{\kappa}{c_i^n + \sigma} h \sum_{i=1}^{M-1} \delta_x^* \tilde{w}_i^n \delta_x^2 \tilde{\theta}_i^n = \frac{h}{c_i^n + \sigma} \\ & \cdot \sum_{i=1}^{M-1} \tilde{c}_i^n (\nabla_t \Theta_i^n) \delta_x^2 \tilde{\theta}_i^n - \frac{h}{2(c_i^n + \sigma)} \\ & \cdot \sum_{i=1}^{M-1} (\tilde{u}_{i-1/2}^n \delta_x \Theta_{i-1/2}^n + \tilde{u}_{i+1/2}^n \delta_x \Theta_{i+1/2}^n) \delta_x^2 \tilde{\theta}_i^n \\ & - \frac{h}{c_i^n + \sigma} \sum_{i=1}^{M-1} [(\lambda + \Theta_i^n) \Gamma(C_i^n, \Theta_i^n) \\ & - (\lambda + \theta_i^n) \Gamma(c_i^n, \theta_i^n) + R_{\theta,i}^n] \delta_x^2 \tilde{\theta}_i^n - \frac{h}{2(c_i^n + \sigma)} \\ & \cdot \sum_{i=1}^{M-1} (\tilde{u}_{i-1/2}^n \delta_x \tilde{\theta}_{i-1/2}^n + \tilde{u}_{i+1/2}^n \delta_x \tilde{\theta}_{i+1/2}^n) \delta_x^2 \tilde{\theta}_i^n, \quad (91) \end{aligned}$$

$$1 \leq n \leq k.$$

For the first term, we have

$$\begin{aligned} & -h \sum_{i=1}^{M-1} (\nabla_t \tilde{\theta}_i^n) \delta_x^2 \tilde{\theta}_i^n \\ & = -\frac{h}{2\tau} \sum_{i=1}^{M-1} \tilde{\theta}_i^{n+1} \delta_x^2 \tilde{\theta}_i^n + \frac{h}{2\tau} \sum_{i=1}^{M-1} \tilde{\theta}_i^{n-1} \delta_x^2 \tilde{\theta}_i^n \\ & = -\frac{1}{2\tau} \sum_{i=1}^{M-1} \tilde{\theta}_i^{n+1} (\delta_x \tilde{\theta}_{i+1/2}^n - \delta_x \tilde{\theta}_{i-1/2}^n) \\ & + \frac{1}{2\tau} \sum_{i=1}^{M-1} \tilde{\theta}_i^{n-1} (\delta_x \tilde{\theta}_{i+1/2}^n - \delta_x \tilde{\theta}_{i-1/2}^n). \quad (92) \end{aligned}$$

By (51), we have

$$\begin{aligned} & -h \sum_{i=1}^{M-1} (\nabla_t \tilde{\theta}_i^n) \delta_x^2 \tilde{\theta}_i^n = -\frac{1}{2\tau} \left(\tilde{\theta}_M^{n+1} \delta_x \tilde{\theta}_{M-1/2}^n \right. \\ & \left. - \tilde{\theta}_1^{n+1} \delta_x \tilde{\theta}_{1/2}^n - \sum_{i=1}^{M-1} \delta_x \tilde{\theta}_{i+1/2}^n (\tilde{\theta}_{i+1}^{n+1} - \tilde{\theta}_i^{n+1}) \right) \\ & + \frac{1}{2\tau} \left(\tilde{\theta}_M^{n-1} \delta_x \tilde{\theta}_{M-1/2}^n - \tilde{\theta}_1^{n-1} \delta_x \tilde{\theta}_{1/2}^n \right. \\ & \left. - \sum_{i=1}^{M-1} \delta_x \tilde{\theta}_{i+1/2}^n (\tilde{\theta}_{i+1}^{n-1} - \tilde{\theta}_i^{n-1}) \right) = -\nabla_t \tilde{\theta}_M^n \delta_x \tilde{\theta}_{M-1/2}^n \quad (93) \\ & + \nabla_t \tilde{\theta}_1^n \delta_x \tilde{\theta}_{1/2}^n + \frac{1}{4\tau} \left(h \sum_{i=1}^{M-1} (\delta_x \tilde{\theta}_{i+1/2}^{n+1})^2 \right. \\ & \left. - h \sum_{i=1}^{M-1} (\delta_x \tilde{\theta}_{i+1/2}^{n-1})^2 \right) = \frac{1}{2} \nabla_t \|\delta_x \tilde{\theta}^n\|^2 \\ & - \nabla_t \tilde{\theta}_M^n \delta_x \tilde{\theta}_{M-1/2}^n + \nabla_t \tilde{\theta}_0^n \delta_x \tilde{\theta}_{1/2}^n. \end{aligned}$$

For the second term, we have

$$\begin{aligned} & \frac{\kappa}{c_i^n + \sigma} h \sum_{i=1}^{M-1} \delta_x^* \tilde{w}_i^n \delta_x^2 \tilde{\theta}_i^n \\ & \geq \frac{\kappa}{c_{\max} + \sigma} h \sum_{i=1}^{M-1} |\delta_x^2 \tilde{\theta}_i^n|^2 \\ & + \frac{\kappa}{c_{\max} + \sigma} h \sum_{i=1}^{M-1} \delta_x^* R_{w,i}^n \delta_x^2 \tilde{\theta}_i^n \\ & \geq \frac{\kappa}{2(c_{\max} + \sigma)} h \sum_{i=1}^{M-1} |\delta_x^2 \tilde{\theta}_i^n|^2 - E_c (\tau^2 + h^2)^2, \quad (94) \end{aligned}$$

where we noted the fact that $|\delta_x^* R_{w,i}^n| \leq E_c (\tau^2 + h^2)$. From (94) we can get

$$\begin{aligned} & \frac{1}{2} \nabla_t \|\delta_x \tilde{\theta}^n\|^2 + \frac{\kappa}{2(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^n\|^2 + \nabla_t \tilde{\theta}_0^n \delta_x \tilde{\theta}_{1/2}^n \\ & - \nabla_t \tilde{\theta}_M^n \delta_x \tilde{\theta}_{M-1/2}^n \leq \frac{h}{c_i^n + \sigma} \sum_{i=1}^{M-1} \tilde{c}_i^n (\nabla_t \Theta_i^n) \delta_x^2 \tilde{\theta}_i^n \\ & - \frac{h}{2(c_i^n + \sigma)} \sum_{i=1}^{M-1} (\tilde{u}_{i-1/2}^n \delta_x \Theta_{i-1/2}^n + \tilde{u}_{i+1/2}^n \delta_x \Theta_{i+1/2}^n) \\ & \cdot \delta_x^2 \tilde{\theta}_i^n - \frac{h}{c_i^n + \sigma} \sum_{i=1}^{M-1} [(\lambda + \Theta_i^n) \Gamma(C_i^n, \Theta_i^n) \\ & - (\lambda + \theta_i^n) \Gamma(c_i^n, \theta_i^n) + R_{\theta,i}^n] \delta_x^2 \tilde{\theta}_i^n - \frac{h}{2(c_i^n + \sigma)} \end{aligned}$$

$$\begin{aligned} & \cdot \sum_{i=1}^{M-1} \left(u_{i-1/2}^{\hat{n}} \delta_x \tilde{\theta}_{i-1/2}^{\hat{n}} + u_{i+1/2}^{\hat{n}} \delta_x \tilde{\theta}_{i+1/2}^{\hat{n}} \right) \delta_x^2 \tilde{\theta}_i^{\hat{n}} \\ & + E_c \left(\tau^2 + h^2 \right)^2 := J_8 + J_9 + J_{10} + J_{11}, \quad 1 \leq n \leq k. \end{aligned} \quad (95)$$

Then we estimate the term $\nabla_t \tilde{\theta}_0^n \delta_x \tilde{\theta}_{1/2}^{\hat{n}}$, $-\nabla_t \tilde{\theta}_M^n \delta_x \tilde{\theta}_{M-1/2}^{\hat{n}}$ and J_8 to J_{11} , respectively. From (46), we have

$$\begin{aligned} \delta_x \tilde{\theta}_{1/2}^{\hat{n}} &= \beta_1 \tilde{\theta}_0^{\hat{n}} \\ &+ \frac{h}{2\kappa} \left[\left(\hat{c}_0^n + \sigma \right) \nabla_t \tilde{\theta}_0^n - u_{1/2}^{\hat{n}} \delta_x \tilde{\theta}_{1/2}^{\hat{n}} - \bar{R}_{\theta,0}^n \right] \\ &- R_{w,1/2}^n. \end{aligned} \quad (96)$$

A straightforward calculation with Lemma 2 leads to

$$\begin{aligned} \nabla_t \tilde{\theta}_0^n \delta_x \tilde{\theta}_{1/2}^{\hat{n}} &= \beta_1 \nabla_t \tilde{\theta}_0^n \tilde{\theta}_0^{\hat{n}} + \frac{h}{2\kappa} \left[\left(\hat{c}_0^n + \sigma \right) \nabla_t \tilde{\theta}_0^n \right. \\ &- u_{1/2}^{\hat{n}} \delta_x \tilde{\theta}_{1/2}^{\hat{n}} - \bar{R}_{\theta,0}^n \left. \right] \nabla_t \tilde{\theta}_0^n - R_{w,1/2}^n \nabla_t \tilde{\theta}_0^n \geq \frac{\beta_1}{2} \\ &\cdot \nabla_t \left(|\tilde{\theta}_0^n|^2 \right) + \frac{h\sigma}{2\kappa} |\nabla_t \tilde{\theta}_0^n|^2 - \frac{h}{2\kappa} u_{1/2}^{\hat{n}} \delta_x \tilde{\theta}_{1/2}^{\hat{n}} \nabla_t \tilde{\theta}_0^n \\ &- \frac{h}{2\kappa} \bar{R}_{\theta,0}^n \nabla_t \tilde{\theta}_0^n - \nabla_t \left(R_{w,1/2}^n \tilde{\theta}_0^n \right) + \frac{R_{w,1/2}^{n+1} - R_{w,1/2}^n}{2\tau} \\ &\cdot \tilde{\theta}_0^{n+1} + \frac{R_{w,1/2}^n - R_{w,1/2}^{n-1}}{2\tau} \tilde{\theta}_0^{n-1} \geq \frac{\beta_1}{2} \nabla_t \left(|\tilde{\theta}_0^n|^2 \right) \\ &+ \frac{h\sigma}{4\kappa} |\nabla_t \tilde{\theta}_0^n|^2 - \frac{h}{2\kappa\sigma} \left[|u_{1/2}^{\hat{n}} \delta_x \tilde{\theta}_{1/2}^{\hat{n}}|^2 + |\bar{R}_{\theta,0}^n|^2 \right] \\ &- \nabla_t \left(R_{w,1/2}^n \tilde{\theta}_0^n \right) - E_c \left(\tau^2 + h^2 \right) \left(\|\tilde{\theta}^{n+1}\|_{\infty} \right. \\ &+ \|\tilde{\theta}^{n-1}\|_{\infty} \left. \right) \geq \frac{\beta_1}{2} \nabla_t \left(|\tilde{\theta}_0^n|^2 \right) + \frac{h\sigma}{4\kappa} |\nabla_t \tilde{\theta}_0^n|^2 \\ &- \nabla_t \left(R_{w,1/2}^n \tilde{\theta}_0^n \right) - \frac{2E_3^2}{\kappa\sigma} \|\delta_x \tilde{\theta}^{\hat{n}}\|^2 \\ &- \frac{50E_3^2\theta_{\max}^2}{\kappa\sigma} \|\delta_x \tilde{c}^{\hat{n}}\|^2 - E_c \left(\|\tilde{\theta}^{n+1}\|^2 + \|\tilde{\theta}^{n-1}\|^2 \right. \\ &+ \|\delta_x \tilde{\theta}^{n+1}\|^2 + \|\delta_x \tilde{\theta}^n\|^2 + \|\delta_x \tilde{\theta}^{n-1}\|^2 + \|\tilde{c}^n\|^2 \\ &+ \|\tilde{\theta}^n\|^2 + \|\tilde{c}^{\hat{n}}\|^2 \left. \right) - E_c \left(\tau^2 + h^2 \right)^2, \end{aligned} \quad (97)$$

where we noted the fact that $(R_{w,1/2}^{n+1} - R_{w,1/2}^n)/2\tau \leq E_c(\tau^2 + h^2)$ and $(R_{w,1/2}^n - R_{w,1/2}^{n-1})/2\tau \leq E_c(\tau^2 + h^2)$. Similarly we estimate $-\nabla_t \tilde{\theta}_M^n \delta_x \tilde{\theta}_{M-1/2}^{\hat{n}}$. From (47), we have

$$\begin{aligned} & -\delta_x \tilde{\theta}_{M-1/2}^{\hat{n}} \\ &= \beta_2 \tilde{\theta}_M^{\hat{n}} \\ &+ \frac{h}{2\kappa} \left[\left(\hat{c}_M^n + \sigma \right) \nabla_t \tilde{\theta}_M^n - u_{M-1/2}^{\hat{n}} \delta_x \tilde{\theta}_{M-1/2}^{\hat{n}} - \bar{R}_{\theta,M}^n \right] \\ &+ R_{w,M-1/2}^n, \end{aligned} \quad (98)$$

such that

$$\begin{aligned} & -\nabla_t \tilde{\theta}_M^n \delta_x \tilde{\theta}_{M-1/2}^{\hat{n}} \geq \frac{\beta_2}{2} \nabla_t \left(\left(\tilde{\theta}_M^n \right)^2 \right) + \frac{h\sigma}{4\kappa} |\nabla_t \tilde{\theta}_M^n|^2 \\ &+ \nabla_t \left(R_{w,M-1/2}^n \tilde{\theta}_M^n \right) - \frac{2E_3^2}{\kappa\sigma} \|\delta_x \tilde{\theta}^{\hat{n}}\|^2 \\ &- \frac{50E_3^2\theta_{\max}^2}{\kappa\sigma} \|\delta_x \tilde{c}^{\hat{n}}\|^2 - E_c \left(\tau^2 + h^2 \right)^2 - E_c \left(\|\tilde{\theta}^{n+1}\|^2 \right. \\ &+ \|\tilde{\theta}^{n-1}\|^2 + \|\delta_x \tilde{\theta}^{n+1}\|^2 + \|\delta_x \tilde{\theta}^n\|^2 + \|\delta_x \tilde{\theta}^{n-1}\|^2 \\ &+ \|\tilde{c}^n\|^2 + \|\tilde{\theta}^n\|^2 + \|\tilde{c}^{\hat{n}}\|^2 \left. \right). \end{aligned} \quad (99)$$

For those terms in the right hand side, with (70), we have

$$|J_8| \leq E_c \|\tilde{c}^{\hat{n}}\|^2 + \frac{\kappa}{16(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^{\hat{n}}\|^2, \quad (100)$$

$$\begin{aligned} |J_9| &\leq \frac{E_3}{2(c_{\min} + \sigma)} \left| h \sum_{i=1}^{M-1} \left(\tilde{u}_{i-1/2}^{\hat{n}} \delta_x^2 \tilde{\theta}_i^{\hat{n}} + \tilde{u}_{i+1/2}^{\hat{n}} \delta_x^2 \tilde{\theta}_i^{\hat{n}} \right) \right| \\ &\leq \frac{\kappa}{16(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^{\hat{n}}\|^2 + \frac{16E_3^2(c_{\max} + \sigma)}{(c_{\min} + \sigma)^2 \kappa} \\ &\cdot h \sum_{i=1}^{M-1} |\tilde{u}_{i+1/2}^{\hat{n}}|^2 \leq \frac{\kappa}{16(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^{\hat{n}}\|^2 \\ &+ \frac{320E_3^2(c_{\max} + \sigma)\theta_{\max}^2}{(c_{\min} + \sigma)^2 \kappa} \|\delta_x \tilde{c}^{\hat{n}}\|^2 + E_c \left(\|\delta_x \tilde{\theta}^n\|^2 \right. \\ &+ \|\tilde{c}^n\|^2 + \|\tilde{\theta}^n\|^2 \left. \right) + E_c \left(\tau^2 + h^2 \right)^2, \end{aligned} \quad (101)$$

$$\begin{aligned} |J_{10}| &\leq \frac{\kappa}{16(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^{\hat{n}}\|^2 + E_c \left(\|\tilde{c}^n\|^2 + \|\tilde{\theta}^n\|^2 \right) \\ &+ E_c \left(\tau^2 + h^2 \right)^2, \end{aligned} \quad (102)$$

$$\begin{aligned} |J_{11}| &\leq E_3 \left| h \sum_{i=1}^{M-1} \left(\delta_x \tilde{\theta}_{i-1/2}^{\hat{n}} \delta_x^2 \tilde{\theta}_{i-1/2}^{\hat{n}} \right. \right. \\ &\left. \left. + \delta_x \tilde{\theta}_{i+1/2}^{\hat{n}} \delta_x^2 \tilde{\theta}_{i+1/2}^{\hat{n}} \right) \right| \\ &\leq 2E_3 \left| h \sum_{i=0}^{M-1} \left[\left(\sqrt{\frac{32E_3(c_{\max} + \sigma)}{\kappa}} \delta_x \tilde{\theta}_{i+1/2}^{\hat{n}} \right) \right. \right. \\ &\left. \left. \cdot \left(\sqrt{\frac{\kappa}{32E_3(c_{\max} + \sigma)}} \delta_x^2 \tilde{\theta}_{i+1/2}^{\hat{n}} \right) \right] \right| \\ &\leq \frac{\kappa}{16(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^{\hat{n}}\|^2 \\ &+ \frac{64E_3^2(c_{\max} + \sigma)}{\kappa} \|\delta_x \tilde{\theta}^{\hat{n}}\|^2. \end{aligned} \quad (103)$$

Plugging the last six equations into (95), we get

$$\begin{aligned}
& \frac{1}{2} \nabla_t \|\delta_x \tilde{\theta}^n\|^2 + \frac{\kappa}{4(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^n\|^2 + \frac{\beta_1}{2} \nabla_t (|\tilde{\theta}_0^n|^2) \\
& + \frac{\beta_2}{2} \nabla_t (|\tilde{\theta}_M^n|^2) + \nabla_t (R_{w,M-1/2}^n \tilde{\theta}_M^n - R_{w,1/2}^n \tilde{\theta}_0^n) \\
& \leq E_7 \|\delta_x \tilde{c}^n\|^2 + E_8 \|\delta_x \tilde{\theta}^n\|^2 + E_c (\tau^2 + h^2)^2 \\
& + E_c (\|\delta_x \tilde{\theta}^n\|^2 + \|\tilde{c}^n\|^2 + \|\tilde{c}^n\|^2 + \|\tilde{\theta}^n\|^2 + \|\tilde{\theta}^{n+1}\|^2 \\
& + \|\tilde{\theta}^{n-1}\|^2 + \|\delta_x \tilde{\theta}^{n+1}\|^2 + \|\delta_x \tilde{\theta}^{n-1}\|^2).
\end{aligned} \tag{104}$$

Multiplying the last equation with $0 \leq s_2 \leq E_9$ and adding it into (90), we have

$$\begin{aligned}
& \frac{s_2}{2} \nabla_t \|\delta_x \tilde{\theta}^n\|^2 + \frac{s_2 \kappa}{4(c_{\max} + \sigma)} \|\delta_x^2 \tilde{\theta}^n\|^2 + \frac{\theta_{\min}}{16} \|\delta_x \tilde{c}^n\|^2 \\
& + \frac{\kappa s_1}{8} \|\delta_x \tilde{\theta}^n\|^2 + \nabla_t \|\tilde{c}^n\|^2 + \frac{\beta_1 s_2}{2} \nabla_t (|\tilde{\theta}_0^n|^2) \\
& + \frac{\beta_2 s_2}{2} \nabla_t (|\tilde{\theta}_M^n|^2) + s_2 \nabla_t (-R_{w,1/2}^n \tilde{\theta}_0^n \\
& + R_{w,M-1/2}^n \tilde{\theta}_M^n) + \frac{s_1 h}{2} \left[\frac{1}{2} \nabla_t [(c_0^n + \sigma) |\tilde{\theta}_0^n|^2] \right. \\
& \left. + \sum_{i=1}^{M-1} \nabla_t [(c_i^n + \sigma) |\tilde{\theta}_i^n|^2] + \frac{1}{2} \nabla_t [(c_M^n + \sigma) |\tilde{\theta}_M^n|^2] \right] \\
& \leq E_c (\|\delta_x \tilde{\theta}^{n+1}\|^2 + \|\tilde{c}^{n+1}\|^2 + \|\tilde{\theta}^{n+1}\|^2) \\
& + E_c (\|\delta_x \tilde{\theta}^n\|^2 + \|\delta_x \tilde{\theta}^{n-1}\|^2 + \|\tilde{c}^n\|^2 + \|\tilde{c}^{n-1}\|^2 \\
& + \|\tilde{\theta}^n\|^2 + \|\tilde{\theta}^{n-1}\|^2) + E_c (\tau^2 + h^2)^2.
\end{aligned} \tag{105}$$

Letting $s = \min(s_2/2, s_2 \kappa/2(c_{\max} + \sigma), \theta_{\min}/8, \kappa s_1/4, 1, s_1, s_2 \beta_1/2, s_2 \beta_2/2)$, we get

$$\begin{aligned}
& s \nabla_t \|\delta_x \tilde{\theta}^n\|^2 + \frac{s}{2} \|\delta_x^2 \tilde{\theta}^n\|^2 + \frac{s}{2} \|\delta_x \tilde{c}^n\|^2 + \frac{s}{2} \|\delta_x \tilde{\theta}^n\|^2 \\
& + s \nabla_t \|\tilde{c}^n\|^2 + s \nabla_t (|\tilde{\theta}_0^n|^2) + s \nabla_t (|\tilde{\theta}_M^n|^2) \\
& + s \nabla_t (-R_{w,1/2}^n \tilde{\theta}_0^n + R_{w,M-1/2}^n \tilde{\theta}_M^n) \\
& + \frac{s h}{2} \left[\frac{1}{2} \nabla_t [(c_0^n + \sigma) |\tilde{\theta}_0^n|^2] \right. \\
& \left. + \sum_{i=1}^{M-1} \nabla_t [(c_i^n + \sigma) |\tilde{\theta}_i^n|^2] \right. \\
& \left. + \frac{1}{2} \nabla_t [(c_M^n + \sigma) |\tilde{\theta}_M^n|^2] \right] \leq E_c (\|\delta_x \tilde{\theta}^{n+1}\|^2 \\
& + \|\delta_x \tilde{\theta}^n\|^2 + \|\delta_x \tilde{\theta}^{n-1}\|^2 + \|\tilde{c}^{n+1}\|^2 + \|\tilde{c}^n\|^2
\end{aligned}$$

$$\begin{aligned}
& + \|\tilde{c}^{n-1}\|^2 + \|\tilde{\theta}^{n+1}\|^2 + \|\tilde{\theta}^n\|^2 + \|\tilde{\theta}^{n-1}\|^2) \\
& + E_c (|\tilde{\theta}_0^{n+1}|^2 + |\tilde{\theta}_0^n|^2 + |\tilde{\theta}_0^{n-1}|^2 + |\tilde{\theta}_M^{n+1}|^2 + |\tilde{\theta}_M^n|^2 \\
& + |\tilde{\theta}_M^{n-1}|^2) + E_c (\tau^2 + h^2)^2.
\end{aligned} \tag{106}$$

Letting

$$\begin{aligned}
F_{n+1} &= s \|\delta_x \tilde{\theta}^{n+1}\|^2 + s \|\tilde{c}^{n+1}\|^2 + |\tilde{\theta}_0^n|^2 + |\tilde{\theta}_M^n|^2 \\
& + \frac{s h}{2} \left(\frac{1}{2} (c_0^{n+1} + \sigma) |\tilde{\theta}_0^{n+1}|^2 \right. \\
& \left. + \sum_{i=1}^{M-1} ((c_i^{n+1} + \sigma) |\tilde{\theta}_i^{n+1}|^2) \right. \\
& \left. + \frac{1}{2} (c_M^{n+1} + \sigma) |\tilde{\theta}_M^{n+1}|^2 \right),
\end{aligned} \tag{107}$$

we have

$$\begin{aligned}
F_{n+1} - F_{n-1} + s \tau (\|\delta_x \tilde{c}^n\|^2 + \|\delta_x \tilde{\theta}^n\|^2 + \|\delta_x^2 \tilde{\theta}^n\|^2) \\
\leq 2\tau E_c (F_{n+1} + F_n + F_{n-1}) + 2\tau E_c (\tau^2 + h^2)^2,
\end{aligned} \tag{108}$$

from which we can find $\alpha = (-\tau E_c + \sqrt{1 - 3(\tau E_c)^2})/(1 - 2\tau E_c)$ and $\lambda = (\tau E_c + \sqrt{1 - 3(\tau E_c)^2})/(1 - 2\tau E_c)$, such that

$$\begin{aligned}
F_{n+1} + \alpha F_n \\
+ \frac{s \tau}{1 - 2\tau E_c} (\|\delta_x \tilde{c}^n\|^2 + \|\delta_x \tilde{\theta}^n\|^2 + \|\delta_x^2 \tilde{\theta}^n\|^2) \\
\leq \lambda (F_n + \alpha F_{n-1}) + \frac{2\tau E_c}{1 - 2\tau E_c} (\tau^2 + h^2)^2.
\end{aligned} \tag{109}$$

When $\tau E_c < 1/4$,

$$\begin{aligned}
F_{n+1} + \alpha F_n + \sum_{m=1}^n \tau s (\|\delta_x \tilde{c}^m\|^2 + \|\delta_x \tilde{\theta}^m\|^2 + \|\delta_x^2 \tilde{\theta}^m\|^2) \\
\leq e^{6E_c \mathcal{T}} (F_1 + F_0 + (\tau^2 + h^2)^2),
\end{aligned} \tag{110}$$

and

$$\begin{aligned}
s \|\delta_x \tilde{\theta}^{n+1}\|^2 + s \|\tilde{c}^{n+1}\|^2 + \frac{s h}{2} \left(\frac{1}{2} (c_0^{n+1} + \sigma) |\tilde{\theta}_0^{n+1}|^2 \right. \\
+ \sum_{i=1}^{M-1} (c_i^{n+1} + \sigma) |\tilde{\theta}_i^{n+1}|^2 + \frac{1}{2} (c_M^{n+1} + \sigma) |\tilde{\theta}_M^{n+1}|^2 \Big) \\
+ \sum_{m=1}^n \tau s (\|\delta_x \tilde{c}^m\|^2 + \|\delta_x \tilde{\theta}^m\|^2 + \|\delta_x^2 \tilde{\theta}^m\|^2) \\
\leq e^{6E_c \mathcal{T}} (\tau^2 + h^2)^2.
\end{aligned} \tag{111}$$

TABLE 1: Numerical results of Example 1 with $\tau = h$.

t	$h = L/200$	$h = L/400$	$h = L/800$	Order r
$t = 0.25$	2.914E-03	1.004E-03	0.254E-03	1.99
$t = 0.50$	8.163E-03	2.178E-03	0.527E-03	2.03
$t = 0.75$	1.397E-02	0.350E-02	0.084E-02	2.04
$t = 1.00$	2.032E-02	0.496E-02	0.118E-02	2.05

t	$h = L/200$	$h = L/400$	$h = L/800$	Order r
$t = 0.25$	9.509E-03	3.795E-03	0.947E-03	2.00
$t = 0.50$	3.860E-03	1.042E-03	0.253E-03	2.03
$t = 0.75$	7.115E-02	1.799E-02	0.439E-02	2.02
$t = 1.00$	1.069E-01	0.265E-01	0.065E-01	2.02

Since E_c is independent of E_0 , with

$$E_0 = \frac{e^{6E_c \mathcal{T}}}{s}, \quad (112)$$

we find that (30) holds for $k = n$. The induction and the proof of the theorem are completed.

4. Numerical Examples

We now numerically evaluate the performance of the proposed leap-frog scheme.

Example 1. First, we test the accuracy of our algorithm in an artificial example which is taken from [23]. The system is

$$\begin{aligned} C_t + (u_g C)_x &= -\Gamma + f_C(x, t), \\ (C + \sigma) T_t - (\kappa T_x)_x + u_g C_x T_x &= [\lambda + \sigma T] \Gamma + f_T(x, t), \end{aligned} \quad (113)$$

with the boundary conditions (8)-(12), where f_C , f_T , μ_i , and v_i are coefficients decided by the exact solution

$$\begin{aligned} C(x, t) &= e^{-0.72t} x^2 (1-x)^2 + a(1-x) + bx, \\ T(x, t) &= 50e^{-0.72t} x^2 (1-x)^2 + c(1-x) + dx, \end{aligned} \quad (114)$$

with a, b, c, d being constants.

We apply the uncoupled leap-frog finite difference method to solve the artificial example. We choose $\mathcal{T} = 1$ and $L = 1$. Since the proposed scheme is of the second order in both spatial and temporal directions, we take $\tau = h$ such that the error bound is proportional to h^2 . We present the L^2 -norm errors and the order of convergence h' in Table 1 with $h = L/200, L/400, L/800$ at different time level. We can see clearly from Table 1 that the L^2 -norm errors for both components are proportional to h^2 , which confirms our theoretical analysis.

Example 2. In the second example, we discuss a typical clothing assembly in the textile industry [2, 4, 25]. The

TABLE 2: Physical parameters for batting materials.

parameter	polyester	unit
ρ_f	1.39×10^3	kgm^{-3}
ρ_w	1×10^3	kgm^{-3}
ϵ'	0.993	
κ_f	1×10^{-1}	$\text{W m}^{-1} \text{K}^{-1}$
κ_w	5.7×10^{-1}	$\text{W m}^{-1} \text{K}^{-1}$
C_{vf}	1.17×10^6	$\text{J m}^{-1} \text{K}^{-1}$
L	4.92×10^{-2}	m

clothing assembly consists of three layers, in the middle is porous fibrous media, and the outside cover is exposed to a cold environment with fixed temperature and relative humidity while the inside cover is exposed to a mixture of air and vapor at higher temperature and relative humidity. In this paper, polyester porous media with laminated or nylon cover materials are tested. To compare with the experimental data in [12], a water equation is added to equations (1)-(2):

$$\frac{\partial}{\partial t} (\rho_w (1 - \epsilon') W) = M_w \Gamma_{ce} \quad (115)$$

where W is water content, ρ_w is the density of water, ϵ is the porosity with liquid water content, and ϵ' is the porosity without liquid water content. We have

$$\epsilon = \epsilon' - \frac{\rho_f}{\rho_w} W (1 - \epsilon') \quad (116)$$

and the effective heat conductivity is defined by

$$\kappa = \epsilon \kappa_g + (1 - \epsilon) \kappa_s \quad (117)$$

where κ_g is the thermal conductivity of gas and κ_s is the thermal conductivity of the fiber-water mixture [2, 6, 7], given by

$$\kappa_s = \left(\frac{1}{\rho_f} + \frac{W}{\rho_w} \right) \left(\frac{1}{\rho_f \kappa_f} + \frac{W}{\rho_w \kappa_w} \right)^{-1}. \quad (118)$$

The values of these physical parameters for polyester media are presented in Table 2. Other parameters values can be found in [2, 6, 7].

FIGURE 1: Numerical results for 10-pile polyester batting sandwiched by two layers of laminated fabric.

The initial conditions for the vapor, temperature, and water content are given by

$$C = 65\% \frac{P_{sat}(T)}{RT}, \quad T = 25^\circ\text{C}, \quad W = 0, \quad \text{at } t = 0. \quad (119)$$

We apply the uncoupled leap-frog finite difference method for solving the sweat transport system defined in (1)-(2) coupled with the water equation (115). Since only the right side of the water equation includes c and θ , therefore, the water equation is calculated separately. Numerically, at each time step, we first find solution $c_j^{n+1}, \theta_j^{n+1}$ by procedure (18)-(26), and then W_j^{n+1} can be solved by following nonnormalized discrete format:

$$\frac{\rho_f (1 - \epsilon_j^n)}{\tau} (W_j^{n+1} - W_j^n) = M_w \Gamma_{ce,j}^n. \quad (120)$$

Then we evaluate the parameters explicitly in (18)-(26) based on W_j^{n+1} . Here all numerical results are obtained by taking the time step size $\tau = 20\text{s}$ and spatial mesh size $h = L/100$. We present numerical results of vapor, temperature, and water content at 8 hours and 24 hours, respectively, for

the porous polyester media assembly with laminated cover in Figure 1 and with nylon cover in Figure 2. The comparisons between numerical results of water content and experimental measurements [12] are given in last two subfigures, where the blue lines represent the numerical solution and the red line is given by experimental measurement.

5. Conclusion

As a subsequent work of [23], we have presented an uncoupled leap-frog finite difference method for the sweat transport system in porous textile media, which is governed by a strongly coupled, nonlinear parabolic system. Optimal L^2 error estimates were presented, which imply that the numerical scheme is unconditionally stable. Both theoretical analysis and numerical example indicate that the current scheme is second order accurate in both the temporal and spatial directions. Since the scheme is decoupled for the system, the method can be applied efficiently for problems in higher-dimensional space. Under certain time-step restrictions, the analysis can also be extended to the multidimensional problems.

FIGURE 2: Numerical results for 10-pile polyester batting sandwiched by two layers of nylon fabric.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors wish to thank Professors Z. Sun and W. Sun for helpful discussions. This research was partially supported by National Natural Science Foundation of China (Nos. 11601346, 11501377), Shenzhen Key Laboratory of Advanced Machine Learning and Applications, Guangdong Key Laboratory of Intelligent Information Processing, and Interdisciplinary Innovation Team of Shenzhen University.

References

- [1] M. K. Choudhary, K. C. Karki, and S. V. Patankar, "Mathematical modeling of heat transfer, condensation, and capillary flow in porous insulation on a cold pipe," *International Journal of Heat and Mass Transfer*, vol. 47, pp. 5629–5638, 2004.
- [2] J. Fan, X. Cheng, X. Wen, and W. Sun, "An improved model of heat and moisture transfer with phase change and mobile condensates in fibrous insulation and comparison with experimental results," *International Journal of Heat and Mass Transfer*, vol. 47, no. 10-11, pp. 2343–2352, 2004.
- [3] C. V. le, N. G. ly, and R. Postle, "Heat and Moisture Transfer in Textile Assemblies: Part I: Steaming of Wool, Cotton, Nylon, and Polyester Fabric Beds," *Textile Research Journal*, vol. 65, no. 4, pp. 203–212, 1995.
- [4] Y. li and Q. Zhu, "Simultaneous Heat and Moisture Transfer with Moisture Sorption, Condensation, and Capillary Liquid Diffusion in Porous Textiles," *Textile Research Journal*, vol. 73, no. 6, pp. 515–524, 2003.
- [5] P. W. Gibson and M. Charmchi, "Modeling convection/diffusion processes in porous textiles with inclusion of humidity-dependent air permeability," *International Communications in Heat and Mass Transfer*, vol. 24, no. 5, pp. 709–724, 1997.
- [6] H. Huang, C. Ye, and W. Sun, "Moisture transport in fibrous clothing assemblies," *Journal of Engineering Mathematics*, vol. 61, no. 1, pp. 35–54, 2008.
- [7] C. Ye, H. Huang, J. Fan, and W. Sun, "Numerical study of heat and moisture transfer in textile materials by a finite volume method," *Communications in Computational Physics*, vol. 4, no. 4, pp. 929–948, 2008.

- [8] Q. Zhang, B. Li, and W. Sun, "Heat and sweat transport through clothing assemblies with phase changes, condensation/evaporation and absorption," *Proceedings of the Royal Society A Mathematical, Physical and Engineering Sciences*, vol. 467, no. 2136, pp. 3469–3489, 2011.
- [9] Q. Zhang, "Mathematical modeling and numerical study of carbonation in porous concrete materials," *Applied Mathematics and Computation*, vol. 281, pp. 16–27, 2016.
- [10] C. Ye, B. Li, and W. Sun, "Quasi-steady-state and steady-state models for heat and moisture transport in textile assemblies," *Proceedings of the Royal Society A Mathematical, Physical and Engineering Sciences*, vol. 466, no. 2122, pp. 2875–2896, 2010.
- [11] F. E. Jones, *Evaporation of Water*, Lewis Publishers Inc, Michigan, Mich, USA, 1992.
- [12] J. Fan, X. Cheng, and Y. S. Chen, "An experimental investigation of moisture absorption and condensation in fibrous insulations under low temperature," *Experimental Thermal and Fluid Science*, vol. 27, pp. 723–729, 2002.
- [13] W. Dai and G. Li, "A finite difference scheme for solving parabolic two-step micro-heat transport equations in a double-layered micro-sphere heated by ultrashort-pulsed lasers," *Numerical Methods for Partial Differential Equations*, vol. 22, no. 6, pp. 1396–1417, 2006.
- [14] R. Eymard, J. Fuhrmann, and K. Gartner, "A finite volume scheme for nonlinear parabolic equations derived from one-dimensional local Dirichlet problems," *Numerische Mathematik*, vol. 102, pp. 463–495, 2006.
- [15] R. Eymard, T. Gallouët, R. Herbin, and A. Michel, "Convergence of a finite volume scheme for nonlinear degenerate parabolic equations," *Numerische Mathematik*, vol. 92, pp. 41–82, 2002.
- [16] C. V. Pao, "Numerical analysis of coupled systems of nonlinear parabolic equations," *SIAM Journal on Numerical Analysis*, vol. 36, no. 2, pp. 393–416, 1999.
- [17] H. J. Schroll, "Convergence of implicit finite difference methods applied to nonlinear mixed systems," *SIAM Journal on Numerical Analysis*, vol. 33, no. 3, pp. 997–1013, 1996.
- [18] W. Sun and G. Yuan, "Stability condition for difference schemes for parabolic systems," *SIAM Journal on Numerical Analysis*, vol. 38, no. 2, pp. 548–555, 2000.
- [19] H. Wang, "An optimal-order error estimate for a family of ELLAM-MFEM approximations to porous medium flow," *SIAM Journal on Numerical Analysis*, vol. 46, no. 4, pp. 2133–2152, 2008.
- [20] Y. Yuan, "The upwind finite difference fractional steps methods for two-phase compressible flow in porous media," *Numerical Methods for Partial Differential Equations*, vol. 19, no. 1, pp. 67–88, 2003.
- [21] B. Li, W. Sun, and Y. Wang, "Global existence of weak solution to the heat and moisture transport system in fibrous porous media," *Journal of Differential Equations*, vol. 249, no. 10, pp. 2618–2642, 2010.
- [22] B. Li and W. Sun, "Global existence of weak solution for nonisothermal multicomponent flow in porous textile media," *SIAM Journal on Mathematical Analysis*, vol. 42, no. 6, pp. 3076–3102, 2010.
- [23] W. Sun and Z. Sun, "Finite difference methods for a nonlinear and strongly coupled heat and moisture transport system in textile materials," *Numerische Mathematik*, 2011.
- [24] A. A. Samarskii and B. B. Andreev, *Finite Difference Methods for Elliptic Equation*, Nauka, Moscow, Russia, 1976.
- [25] J. Fan, Z. Luo, and Y. Li, "Heat and moisture transfer with sorption and condensation in porous clothing assemblies and numerical simulation," *International Journal of Heat and Mass Transfer*, vol. 43, pp. 2989–3000, 2000.

Submit your manuscripts at
www.hindawi.com