Research Article

Caristi-Type Fixed Point Theorem over Száz Principle in Quasi-Metric Space with a Graph

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The aim of this paper is to generalize Caristi’s fixed point theorem in a K-complete quasi-metric space endowed with a reflexive digraph by using Száz maximum principle. An example is given to support our main result.

1. Introduction

Let $X$ be a nonempty set. A binary relation “≤” on $X$ is said to be a preorder on $X$ if it is reflexive and transitive. In this case $(X, ≤)$ is called a preordered set. An element $x ∈ X$ is said to be maximal in $X$ if for all $y ∈ X$, $x ≤ y \Rightarrow x = y$. (1)

The set $S^+_x = \{z ∈ X; x ≤ z\}$ is called the final segment generated by $x$.

In 2007, Á. Száz (see [1]) generalized the Brézis-Browder principle in the setting of the preorder sets and gave a generalized version of Caristi’s theorem.

Theorem 1 (Száz [1]). Let $(X, ≤)$ be a preordered set and let $Φ : X × X → [0, +∞)$ be a function satisfying:

(S1) $x \mapsto \sup_{y ∈ S^+_x} Φ(x, y)$ is decreasing;

(S2) $−∞ < \sup_{y ∈ S^+_x} Φ(x, y)$ for all $x ∈ X$;

(S3) $\sup_{y ∈ S^+_x} Φ(a, y) < +∞$ for some $a ∈ X$;

(S4) For every nondecreasing sequence $\{x_n\}_{n ∈ \mathbb{N}} ⊆ X$ with $x_n = a$, there exists some $x ∈ X$ such that $x_n ≤ x$ for all $n ∈ \mathbb{N}$ and $\liminf_{n → +∞} Φ(x_n, x_{n+1}) = 0$;

(S5) $0 < Φ(x, y)$ for all $x, y ∈ X$ with $x < y$.

Then, there exists a maximal element $\hat{x} ∈ X$.

In 1976, Caristi (see [2]) gave a generalization of Banach contraction principle where the assumption “$T : X → X$ is continuous” is dropped and replaced by a weak assumption. Since then, various proofs, extensions, and generalizations are given by many authors (see [3–7]). It is worth mentioning that Caristi’s fixed point theorem is equivalent to the Ekeland’s variational principle [8].

In this work, we use the Száz principle to give a more generalized version of Caristi’s fixed point theorem in the setting of the quasimetric space with a graph. For that, we introduce a new class of functions called $K$-functions and $M$-functions which generalize the notion of dominated function in Caristi’s theorem. Also, we give an improved result in the framework of set-valued mappings and we derive some known results as corollaries.
2. Preliminaries

Definition 2. Let $X$ be a nonempty set; a function $\delta : X \times X \to [0, \infty)$ is quasidistance if we have

1. $\delta(x, y) = 0$ if and only if $x = y$
2. $\delta(x, z) \leq \delta(x, y) + \delta(y, z)$ for each $x, y, z \in X$.

The pair $(X, \delta)$ is called a quasimetric space.

Since $\delta(x, y) = \delta(y, x)$ is not necessarily satisfied in such spaces, there are many characterizations of completeness in this setting (e.g., [9]). Following the framework of [8], we have the following.

Definition 3. A sequence $\{x_n\}_n$ in $(X, \delta)$ is

1. left K-Cauchy if for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $\forall n, m \in \mathbb{N},$ with $n_0 \leq n \leq m, \delta(x_n, x_m) < \varepsilon$;
2. left K-converges to $x$, if $\lim_{n \to \infty} \delta(x_n, x) = 0$;
3. $(X, \delta)$ will be called left K-complete quasimetric if any left K-Cauchy sequence is left K-convergent.

Definition 4. Let $(X, \delta)$ be a quasimetric space. A mapping $\varphi : X \times X \times X \to [0, \infty)$ is said to be

1. lower semicontinuous if given any sequence $\{x_n\}_n$ in $X$, whenever $\lim_{n \to \infty} x_n = x$ and $\varphi(x_n) = r$, then $\varphi(x) \leq r$;
2. upper semicontinuous if $-\varphi$ is lower semicontinuous.

In the sequel, we recall some basic notions on graphs borrowed from [10].

Let $V$ be an arbitrary set. A directed graph, or digraph, is a pair $G = (V, E)$ where $E$ is a subset of the Cartesian product $V \times V$. The elements of $V$ are called vertices or nodes of $G$ and the elements of $E$ are the edges, also called oriented edges or arcs of $G$. An edge of the form $(v, v)$ is a loop on $v$. Another way to express that $E$ is a subset of $V \times V$ is to say that $E$ is a binary relation over $V$. Given a digraph $G$, the set of vertices (respectively, of edges) of $G$ is denoted by $V(G)$ (respectively, $E(G)$). The digraph $G = (V, E)$ is said to be

(i) transitive if whenever $(x, y) \in E$ and $(y, z) \in E$, then $(x, z) \in E$;
(ii) reflexive if $\Delta = \{(v, v) \mid v \in V\}$ is a subset of $E$.

A vertex $x$ is said to be isolated if for all vertex $y \neq x$, we have neither $(x, y) \in E$ nor $(y, x) \in E$. Given two vertices $x, y \in V$ a path in $G$, from (or joining) $x$ to $y$ is a sequence of vertices $p = \{a_i\}_{i=0}^{n}$ such that $a_0 = x, a_n = y$ and $(a_i, a_{i+1}) \in E$, for all $i \in \{0, 1, \ldots, n - 1\}$. The integer $n$ is the length of the path $p$. If $x = y$ and $n > 1$, the path $p$ is called a directed cycle. An acyclic digraph is a digraph which has no directed cycle.

We denote by $y \in [x]_G$ the fact that there is a directed path in $G$ joining $x$ to $y$.

A quasimetric space $(X, \delta)$ endowed with a digraph $G$ such that $V(G) = X$ is denoted by $(X, \delta, G)$.

As in [4], we use the following.

Definition 5. Let $(X, \delta, G)$ be a quasimetric space endowed with a digraph. We say that $X$ satisfies the property (OSC) if for any sequence $\{x_n\} \subseteq X$ that is convergent to $x \in X$ and for all $n \in \mathbb{N}$, $x_{n+1} \in [x_n]_G$, then $x \in [x_n]_G$ for all $n \in \mathbb{N}$.

Let $(X, \delta, G)$ be a reflexive digraph, and a function $\Phi : X \times X \to \mathbb{R}$. Having in mind $\Phi(x, y) = \varphi(x) - \varphi(y)$ for each $x, y \in X$, most dominated Caristi’s functions satisfies the following conditions:

(C1) N-superadditivity: $\Phi(x, x) = 0$ and $\Phi(x, y) + \Phi(y, z) \leq \Phi(x, z)$ for each $x, y, z \in X$ with $y \in [x]_G$ and $z \in [y]_G$.
(C2) $y \mapsto \Phi(x, y)$ is upper semicontinuous for each $x \in X$.
(C3) There exists $a \in X$ such that $\sup_{x \in [a]_G} \Phi(a, y) < \infty$.
(C4) There exists a function $\psi : X \to \mathbb{R}$ such that, for all $x, y \in X$,

$$y \in [x]_G \implies \Phi(x, y) \leq \psi(x).$$

Next, we introduce a new class of functions.

Definition 6. Let $(X, G)$ be a reflexive digraph; a real function $\Phi : X \times X \to \mathbb{R}$ is said to be

(i) K-function if (C1), (C2), and (C3) hold.
(ii) M-function if (C1), (C2), and (C4) hold.

Remark 7. Obviously, each $M$-function is a $K$-function. Indeed, let $a \in X$. Then for each $y \in [a]_G$, we have $\Phi(a, y) \leq \psi(a)$ which implies

$$\sup_{y \in [a]_G} \Phi(a, y) \leq \psi(a) < \infty.$$ (3)

3. Main Results

Let $(X, \delta, G)$ be a quasimetric space endowed with a digraph. Define a binary relation on $X$ by

$$x \equiv_{\Phi} y \iff \begin{cases} \delta(x, y) \leq \Phi(x, y) \\ y \in [x]_G. \end{cases}$$ (4)

We will use particularly the following fact.

Lemma 8. Let $(X, \delta, G)$ be a quasimetric space with a digraph and $\Phi : X \times X \to \mathbb{R}$ a function satisfying condition (CI); then $(X, \delta, \equiv_{\Phi})$ is a preordered quasimetric space.

The following result is a generalization and an extension of Caristi’s theorem in the setting of the quasimetric spaces with a graph.

Theorem 9. Let $(X, \delta, G)$ be a left K-complete quasimetric space endowed with a reflexive digraph satisfying the (OSC) property and $T : X \to 2^X$ a set-valued map. If there exists a $K$-function $\Phi : X \times X \to \mathbb{R}$ such that for each $x \in X$, there exists $y \in Tx \cap [x]_G$ with $\delta(x, y) \leq \Phi(x, y)$, then $T$ has a fixed point in $X$. 

Proof. Firstly, we show that for each increasing sequence \( \{x_n\}_n \) with respect to \( \preceq \Phi \) where \( x_0 = a \), there exists \( \overline{x} \) such that for each \( n \in \mathbb{N} \) we get \( x_n \preceq \Phi \overline{x} \).

According to the definition of \( \preceq \Phi \), we have \( x_n \preceq \Phi x_{n+1} \iff \delta(x_n, x_{n+1}) \leq \Phi(x_n, x_{n+1}) \); hence,

\[
\sum_{i=0}^{n} \delta(x_i, x_{i+1}) \leq \sum_{i=0}^{n} \Phi(x_i, x_{i+1}) \leq \Phi(a, x_{n+1}) \leq \sup_{y \in S \preceq \Phi (a)} \Phi(a, y) \quad (5)
\]

Then \( \sum_{i=0}^{n} \Phi(x_i, x_{i+1}) \) is a real convergent sequence, so \( \lim_{n \to \infty} \Phi(x_n, x_{n+1}) = 0 \). And \( \{x_n\}_n \) is a left K-Cauchy sequence in \( X \) since \( \sum_{i=0}^{\infty} \delta(x_i, x_{i+1}) \) is convergent. Then, \( \{x_n\}_n \) left K-converges to some \( \overline{x} \in [x_n]_G \) for all \( n \in \mathbb{N} \).

Note that for each \( n, m \in \mathbb{N} \) with \( n \leq m \), we get

\[
x_n \preceq \Phi x_m \iff \delta(x_n, x_m) \leq \Phi(x_n, x_m) \quad (6)
\]

which by left K-convergence of \( \{x_n\}_n \) and upper semicontinuity of \( y \mapsto \Phi(x, y) \), we have \( \lim_{m \to \infty} \sup_{m \to \infty} \Phi(x_n, x_m) \leq \Phi(x_n, \overline{x}) \). Then, \( \delta(x_n, \overline{x}) \leq \Phi(x_n, \overline{x}) \) and \( \overline{x} \in [x_n]_G \) for all \( n \in \mathbb{N} \), which leads to

\[
x_n \preceq \Phi \overline{x}, \quad \text{for every } n \in \mathbb{N}. \quad (7)
\]

Thus, (S4) holds.

Define a function \( \varphi : X \to [0, \infty] \) for each \( x \in X \) by

\[
\varphi(x) = \sup_{y \in [x]_G} \Phi(x, y) \quad (8)
\]

\( \varphi \) is nonnegative function, since for each \( x \in X \), there is \( y \in Tx \) such that \( \delta(x, y) \leq \Phi(x, y) \); then \( 0 \leq \varphi(x) \) that is (S2) holds.

If \( x \preceq \Phi y \); that is, \( (x \preceq \Phi y \text{ and } x \neq y) \), then \( 0 < \delta(x, y) \leq \Phi(x, y) \), which implies (S5).

Let \( x \preceq \Phi y \); then for each \( z \in [y]_G \), we get

\[
\Phi(x, y) + \Phi(y, z) \leq \Phi(x, z) \quad (9)
\]

and since \( \Phi(x, y) \geq 0 \), we have

\[
\Phi(y, z) \leq \Phi(x, z) \quad (10)
\]

Then

\[
\sup_{z \in [y]_G} \Phi(y, z) \leq \sup_{z \in [x]_G} \Phi(x, z) \quad (11)
\]

i.e., \( \varphi(y) \leq \varphi(x) \), which implies that \( \varphi \) is nonincreasing function; that is, (S1) holds.

All assumptions of Szávas principle hold; then \( X \) has a maximal element \( \overline{x} \). By hypothesis, there exists \( \overline{y} \in T \overline{x} \) such that \( \overline{x} \preceq \Phi \overline{y}; \) then we get \( \overline{x} = \overline{y} \), which implies that \( \overline{x} \in T \overline{x} \). \( \Box \)

We support our result by the following example.

Example 10. Consider the digraph \( G = (X, E) \) represented in Figure 1, where

\[
X = \{0, 1\} \cup \left\{1 - \frac{1}{2^n} : n \in \mathbb{N}\right\},
\]

\[
E = \Delta \cup \{(0, 1)\} \cup \{(1 - \frac{1}{2^n}, 1) : n \in \mathbb{N}^*\} \quad (12)
\]

Define on \( X \) the quasimetric \( \delta \) as follows:

\[
\delta(x, y) = \begin{cases} 
1 & \text{if } y \in [x]_G; \\
y - x & \text{if not} \end{cases} \quad (13)
\]

Consider the \( K \)-function \( \Phi : X \times X \to \mathbb{R} \) defined by

\[
\Phi(x, y) = \ln^2 \left( \frac{y + 1}{x + 1} \right) + y - x \quad (14)
\]
and the set-valued mapping $T : X \to 2^X$ defined by

$$
T(0) = \{1\}; \\
T\left(1 - \frac{1}{2^n}\right) = \left\{1 - \frac{1}{2^n} : k \in \mathbb{N}\right\}; \\
T(1) = X.
$$

(15)

One can see that

(i) $x \in [1]_G$ for all $x \in X$.

(ii) For all $n, m \in \mathbb{N}^*$ with $n > m$, we have $1 - 1/2^n \in [1 - 1/2^m]_G$.

(iii) $(X, d)$ is a complete left $K$-quasimetric space.

(iv) $\Phi$ is obviously $K$-function.

(v) For all $x \in X$, there exists $y \in Tx \cap [x]_G$ such that

$$
d(x, y) \leq \ln^2 \left(\frac{y + 1}{x + 1}\right) + y - x
$$

(16)

All assumptions of Theorem 9 are satisfied and $1/2 \in T(1/2)$.

**Corollary 11.** Under assumptions of Theorem 9, with $T : X \to X$, $T$ has a fixed point in $X$.

Using Remark 7, we have immediately the following.

**Corollary 12.** Under assumptions of Theorem 9, if $\Phi : X \times X \to \mathbb{R}$ is an $M$-function, then $T$ has a fixed point in $X$.

The following theorem improves the results of [2, 11, 12] and generalizes the main theorem of Chaira et al. [4]

**Theorem 13.** Let $(X, \delta, G)$ be a left $K$-complete quasimetric space with a reflexive digraph satisfying the (OSC) property and $\varphi : X \to [0, \infty)$ a lower semicontinuous function. If the mapping $T : X \to X$ satisfies, for all $x \in X$,

$$
Tx \in [x]_G \Longrightarrow \delta(x, Tx) \leq \varphi(x) - \varphi(Tx),
$$

(18)

then $T$ has a fixed point in $X$.

**Proof.** We consider the function $\phi : X \times X \to \mathbb{R}$ defined by

$$
\phi(x, y) = \varphi(x) - \varphi(y), \quad \forall x, y \in X
$$

(19)

It is clear that $\phi$ is a K-function. Applying Theorem 9, the proof is complete.□

**Data Availability**

There were no data used to support this study.

**Conflicts of Interest**

The authors declare that there are no conflicts of interest regarding the publication of this paper.

**References**


