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Research Article

Ordered Quasi(BI)-Γ-Ideals in Ordered Γ-Semirings

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In this paper, we have defined ordered quasi-Γ-ideals and ordered bi-Γ-ideals in ordered Γ-semirings by defining the relation "≤" in ordered Γ semiring S as $a \leq b$ if $a + x = b$ for any $a, b, x \in S$. By using this relation we have shown that ordered quasi-Γ-ideals and ordered bi-Γ-ideals in ordered Γ-semirings are generalization of quasi-ideals and bi-ideals in ordered semirings. Properties of many types of ordered Γ-ideals including (semi)prime, (strongly) irreducible, and maximal ordered quasi-Γ-ideals and ordered bi-Γ-ideals in ordered Γ-semirings S have been studied.

1. Introduction

The notion of semiring was introduced by Vandiver [1] in 1934 whereas Γ-semiring was introduced and studied by Rao [2] in 1995 as a generalization of the notion of Γ-rings as well as of semirings. It is well known that semiring is a generalization of a ring and Γ-semiring is a generalization of semiring but interestingly ideals of semirings do not coincide with ideals of rings. Applications of semirings are not limited to theoretical computer science, graph theory, optimization theory, automata, coding theory, and formal languages but in other branches of sciences and technologies as well.

Quasi-ideal was first introduced by O. Steinfeld for semigroups [3] and then for rings by the same author O. Steinfeld [4] where it is seen that quasi-ideal is a generalization of left and right ideals. Kiyoshi Iseki [5] introduced quasi-ideals for semirings without zero and showed results based on them. Donges studied quasi-ideals in semirings in [6] whereas Shabir et al. [7] characterized semirings by using quasi-ideals. Minimal quasi-ideals for Γ-semiring was studied by Iseki [5].

The concept of bi-ideal for semigroups was given by Good and Hughes [8] in 1952. Generalized bi-ideal was then introduced for rings in 1970 by Szasz [9, 10] and then for semigroups by Lajos [11]. Many types of ideals on the algebraic structures were characterized by several authors such as the following. In 2000, Dutta and Sardar studied the characterization of semiprime ideals and irreducible ideals of Γ-semirings [12]. In 2004, Sardar and Dasgupta [13] introduced the notions of primitive Γ-semirings and primitive ideals of Γ-semirings. In 2008, Kaushik et al. [14] introduced and studied bi-Γ-ideals in Γ-semirings. In 2008, Pianskool et al. [15] introduced and studied valuation Γ-semirings and valuation Γ-ideals of a Γ-semiring.

In 2008, Chinram studied properties of quasi-ideals in Γ-semirings [16], whereas, in 2009, quasi-ideals and minimal quasi-ideals in Γ-semiring were studied by Jagatap and Pawar [17]. In 2014, Jagatap [18] studied prime k-Bi-ideals in Γ-semirings. In 2015, Lavanya et al. [19] studied prime biternary Γ-ideals in ternary Γ-semirings. In 2016, Jagatap [20] defined the concept of bi-ideals in a Γ-semiring and studied properties of bi-ideals in a regular Γ-semiring. In 2018, Rao introduced ideals in ordered Γ-semirings [21]. It is natural to define and study properties of ordered quasi-Γ-ideals and ordered bi-Γ-ideals in ordered Γ-semirings. Therefore, we have extended the concept of ordered ideals in ordered Γ-semirings to that of ordered quasi-Γ-ideals and ordered bi-Γ-ideals in ordered Γ-semiring S by considering different order than the order defined in [21].

2. Preliminaries and Basic Definitions

In this section we will define important terminologies based on those concepts which are being used in this paper.

Definition 1. Let $(S, +)$ and $(\Gamma, +)$ be commutative semigroups. Then we call S a Γ-semiring, if there exists a mapping $S \times \Gamma \times S \longrightarrow S$ is written (x, α, y) as $x\alpha y$ such that it satisfies the following axioms for all $x, y, z \in S$ and $\alpha, \beta \in \Gamma$:

(i) $x\alpha(y+z) = x\alpha y + x\alpha z$

(ii) $(x+y)\alpha z = x\alpha z + y\alpha z$

(iii) $x(\alpha+\beta)y = x\alpha y + x\beta y$

(iv) $x\alpha(y\beta z) = (x\alpha y)\beta z$.

Example 2 (see [21]). Let S be a semiring and $M_{p,q}(S)$ denote the additive abelian semigroup of all $p \times q$ matrices with identity element whose entries are from S. Then $M_{p,q}(S)$ is a Γ-semiring with $\Gamma = M_{p,q}(S)$ ternary operation is defined by $x\alpha z = x(\alpha^t)z$ as the usual matrix multiplication, where α^t denotes the transpose of the matrix α, for all x, z and $\alpha \in M_{p,q}(S)$.

Definition 3. A Γ-semiring S is called an ordered Γ-semiring if it admits a compatible relation $\leq$; i.e., $\leq$ is a partial ordering on S satisfying the following conditions.

If $a \leq b$ and $c \leq d$ then

(i) $a + c \leq b + d$

(ii) $a\gamma c \leq b\gamma d$

(iii) $c\gamma a \leq d\gamma b, \forall a, b, c, d \in S$ and $\gamma \in \Gamma$.

Definition 4. An ordered Γ-semiring S is said to be commutative ordered Γ-semiring if $x\gamma y = y\gamma x$, for all $x, y \in S$ and $\gamma \in \Gamma$.

Definition 5 (see [21]). A nonempty subset A of an ordered Γ-semiring S is called a sub-Γ-semiring of S if $(A, +)$ is a subsemigroup of $(S, +)$ and $a\gamma b \in A$ for all $a, b \in A$ and $\gamma \in \Gamma$.

Definition 6 (see [21]). A nonempty subset A of an ordered Γ-semiring S is called a left (right) ideal if A is closed under addition, $S\Gamma A \subseteq A (A\Gamma S \subseteq A)$ and if, for any $\in S$, $b \in A$, $a \leq b$ then $a \in A$.

Definition 7. If A is both a left and a right ideal of ordered Γ-semirings S, then A is called an ideal of S.

Definition 8. A right ideal A of an ordered Γ-semiring S is called a right k ideal if $a \in A$ and $x \in S$ such that $a + x \in A$, and then $x \in A$.

A left k ideal of S is defined in a similar way.

Definition 9 (see [21]). A nonempty subset K of an ordered Γ-semiring S is called a k ideal if K is an ideal and $x \in S$, $x + y \in K$, $y \in K$; then $x \in K$.

Definition 10. An ordered Γ-semiring S is called totally ordered if any two elements of S are comparable.

Definition 11. Let K and L be ordered Γ-semirings. A mapping $f : K \longrightarrow L$ is called a homomorphism if

(i) $f(a+b) = f(a) + f(b)$,

(ii) $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in K, \alpha \in \Gamma$.

3. Ordered Quasi-Γ-Ideals in Ordered-Γ-Semirings

In this section we define ordered quasi-Γ-ideal, ordered k quasi-Γ-ideal, ordered $m-k$ quasi-Γ-ideal, ordered prime quasi-Γ-ideal, ordered semiprime quasi-Γ-ideal, ordered maximal quasi-Γ-ideal, ordered irreducible, and ordered strongly irreducible quasi-Γ-ideal of an ordered Γ-semiring S. We define the relation "$\leq$" in b as $a \leq b$ if $a + x = b$ for any $a, b, x \in S$. For a nonempty subset X of S and $a, b \in S$, we say that $a \leq b$ in X if $a + x = b$ for any $x \in X$.

Definition 12. An ordered sub-Γ-semiring Q of an ordered Γ-semiring S is called an ordered quasi-Γ-ideal of S if $S\Gamma Q \cap Q\Gamma S \subseteq Q$ and if, for any $\in S$, $q \in Q$, $a \leq q$ then $a \in Q$ in Q.

Definition 13. An ordered quasi-Γ-ideals Q is called ordered k quasi-Γ-ideal of S if Q is an ordered sub-Γ-semiring of S, and if $q \in Q$ and $x \in S$ such that $x + q \in Q$, then $x \in Q$.

Clearly every ordered k quasi-Γ-ideal is an ordered quasi-Γ-ideal but converse is not true in general as shown in the following example.

Example 14. Let S be the set of nonnegative integers and $\Gamma = N$ be additive abelian semigroups. Tennary operation is defined as $(x, \gamma, y) \longrightarrow x\gamma y$, usual multiplication of integers. Then S is an ordered Γ-semirings. A subset $Q = 5S \setminus \{5\}$ of S is an ordered quasi-Γ-ideal of S but it is not an ordered k quasi-Γ-ideal of S.

Definition 15. An ordered quasi-Γ-ideal Q of an ordered Γ-semiring S is called ordered $m-k$ quasi-Γ-ideal of S if $q \in Q$, $x \in S$, $q\gamma x \in Q$ for $\gamma \in \Gamma$; then $x \in Q$.

Theorem 16. *Every ordered $m-k$ quasi-Γ-ideal of an ordered Γ-semiring S is an ordered k quasi-Γ-ideal of S.*

Proof. Let Q be an ordered $m-k$ quasi-Γ-ideal of an ordered Γ-semiring S. Consider $x + q \in Q$, $q \in Q$, $x \in S$, and $\gamma \in \Gamma$; then $(x+q)\gamma x \in Q$. Then $x \in Q$, since Q is an ordered $m-k$ quasi-Γ-ideal. Hence Q is an ordered k quasi-Γ-ideal of S. □

The following example shows that the converse of Theorem 16 is not true in general.

Example 17. Let S be the set of all natural numbers. Then S with usual ordering is an ordered semiring. If $\Gamma = S$, then S is an ordered Γ-semiring. If $Q = \{2, 4, 6, \ldots\}$ then Q forms an ordered k quasi-Γ-ideal but not ordered $m-k$ quasi-Γ-ideals of ordered Γ-semiring S.

Definition 18. An ordered Γ-semiring S is called band if every element of S is an idempotent.

Theorem 19. *Let Q be an ordered sub-Γ-semiring of an ordered Γ-semiring S in which semigroup $(S, +)$ is a band. Then Q is an ordered quasi-Γ-ideal of S if and only if Q is an ordered k quasi-Γ-ideals of S.*

Proof. Let Q be an ordered quasi-Γ-ideal of an ordered Γ-semiring S and $x+q \in Q$, $q \in Q$, and then $x+(x+q) = (x+x)+q = x+q \Longrightarrow x \leq x+q \Longrightarrow x \in Q$ as Q is an ordered quasi-Γ-ideal of S. Hence, Q is a an ordered k quasi-Γ-ideal of S.

Conversely suppose that Q is an ordered k quasi-Γ-ideal of an ordered Γ-semiring S. Let $x \in S$, $q \in Q$ and $x \leq q \Longrightarrow x+q' = q \Longrightarrow x+q' \in Q \Longrightarrow x \in Q$, since Q is an ordered k quasi-Γ-ideal of an ordered Γ-semiring S. Hence Q is an ordered quasi-Γ-ideal of S. □

Definition 20. An ordered quasi-Γ-ideal Q of S is called an irreducible ordered quasi-Γ-ideal if $Q_1 \cap Q_2 = Q$ implies $Q_1 = Q$ or $Q_2 = Q$, for any ordered quasi-Γ-ideal Q_1 and Q_2 of S.

Definition 21. An ordered quasi-Γ-ideal Q of S is called strongly irreducible ordered quasi-Γ-ideal if, for any ordered quasi-Γ-ideal Q_1 and Q_2 of S, $Q_1 \cap Q_2 \subseteq Q$ implies $Q_1 \subseteq Q$ or $Q_2 \subseteq Q$. It is obvious that every strongly irreducible ordered quasi-Γ-ideal is an irreducible ordered quasi-Γ-ideal.

Definition 22. An ordered quasi-Γ-ideal Q of S is called a prime ordered quasi-Γ-ideal if $Q_1\Gamma Q_2 \subseteq Q$ implies $Q_1 \subseteq Q$ or $Q_2 \subseteq Q$, for any ordered quasi-Γ-ideals Q_1, Q_2 of S.

Definition 23. An ordered k quasi-Γ-ideal Q of S is called a prime ordered k quasi-Γ-ideal if $Q_1\Gamma Q_2 \subseteq Q$ implies $Q_1 \subseteq Q$ or $Q_2 \subseteq Q$, for any ordered k quasi-Γ-ideals Q_1, Q_2 of S.

Definition 24. An ordered quasi-Γ-ideal Q of S is called a strongly prime ordered quasi-Γ-ideal if $(Q_1\Gamma Q_2) \cap (Q_2\Gamma Q_1) \subseteq Q$ implies that $Q_1 \subseteq Q$ or $Q_2 \subseteq Q$ for any ordered quasi-Γ-ideals Q_1, Q_2 of S.

Definition 25. An ordered quasi-Γ-ideal Q of S is called a semiprime ordered quasi-Γ-ideal if for any ordered quasi-Γ-ideals Q_1, of S $Q_1^2 = Q_1\Gamma Q_1 \subseteq Q$ implies $Q_1 \subseteq Q$.

It is clear that every strongly prime ordered quasi-Γ-ideal in S is a prime ordered quasi-Γ-ideal and every prime ordered quasi-Γ-ideal in S is a semiprime ordered quasi-Γ-ideal.

Definition 26. An ordered quasi-Γ-ideal Q of an ordered Γ-semiring S is said to be maximal ordered quasi-Γ-ideal if $Q \neq S$ and for every ordered quasi-Γ-ideal C of S with $Q \subseteq C \subseteq S$, then either $Q = C$ or $Q = S$.

Theorem 27. *In an ordered Γ-semiring S, every maximal ordered quasi-Γ-ideal of S is irreducible ordered quasi-Γ-ideal of S.*

Proof. Let Q_m be a maximal ordered quasi-Γ-ideal of an ordered Γ-semiring S. Suppose Q_m is not irreducible and $Q_m = U \cap V \Longrightarrow Q_m \neq U$ and $Q_m \neq V \Longrightarrow Q_m \subset U \subset S$ and $Q_m \subset V \subset S$ a contradiction. Hence Q_m is irreducible ordered quasi-Γ-deal of an ordered Γ-semiring S. □

Theorem 28. *Let Q be an ordered quasi-Γ-ideal of an ordered Γ-semiring S.*

(i) *If Q is a prime ordered quasi-Γ-ideal, then Q is a strongly irreducible ordered quasi-Γ-ideal.*

(ii) *If Q is a strongly irreducible ordered quasi-Γ-ideal, then Q is an irreducible ordered quasi-Γ-ideal.*

Proof. Let Q be an ordered quasi-Γ-ideal of an ordered Γ-semiring S.

(i) Suppose Q is a prime ordered quasi-Γ-ideal, J and K are ordered quasi-Γ-ideal of an ordered Γ-semiring S such that $J \cap K \subseteq B$. Then $J\Gamma K \subseteq Q \Longrightarrow J \subseteq Q$ or $K \subseteq Q$. Hence, Q is a strongly irreducible ordered quasi-Γ-ideal.

(ii) Suppose Q is a strongly irreducible ordered quasi-Γ-ideal; J and K are ordered quasi-Γ-ideal of an ordered Γ-semiring S such that $J \cap K = Q$. Then certainly $J \cap K \subseteq Q \Longrightarrow J \subseteq Q$ or $K \subseteq Q$. Hence $J = Q$ or $K = Q$.

Therefore, Q is an irreducible ordered quasi-Γ-ideal of S. □

Corollary 29. *Let S be an ordered Γ-semiring. If Q is a prime ordered quasi-Γ-ideal of S, then Q is an irreducible ordered quasi-Γ-ideal of S.*

Theorem 30. *Let $f : K \longrightarrow L$ be a homomorphism of ordered Γ-semirings. If J is an ordered quasi-Γ-ideal of L, then $f^{-1}(J)$ is an ordered quasi-Γ-ideal of an ordered Γ-semiring K.*

Proof. Suppose J is an ordered quasi-Γ-ideal of L, $f : K \longrightarrow L$ is a homomorphism of ordered Γ-semirings, and $x, y \in f^{-1}(J) \Longrightarrow f(x), f(y) \in J \Longrightarrow f(x)+f(y) = f(x+y) \in J \Longrightarrow x+y \in f^{-1}(J)$. If $x, y \in f^{-1}(J) \Longrightarrow f(x), f(y) \in J \Longrightarrow f(x)\gamma f(y) \in J \Longrightarrow f(x\gamma y) \in J \Longrightarrow x\gamma y \in f^{-1}(J)$. We need to show here that $f^{-1}(J)$ is an ordered quasi-Γ-ideal of K; i.e., $K\Gamma f^{-1}(J) \cap f^{-1}(J)\Gamma K \subseteq f^{-1}(J)$. For that we suppose $j \in K\Gamma f^{-1}(J) \cap f^{-1}(J)\Gamma K \Longrightarrow j = s\alpha x$ and $j = y\beta s'$ for $s, s' \in K, x, y \in f^{-1}(J)$ and $\alpha, \beta \in \Gamma \Longrightarrow f(x), f(y) \in J$, which is a quasi-ideal. Now $f(j) = f(s\alpha x) = f(s)\alpha f(x)$ and $f(j) = f(y)\beta f(s') \Longrightarrow f(j) \in L\Gamma J \cap J\Gamma L$, as $f(s), f(s') \in L \Longrightarrow f(j) \in J \Longrightarrow j \in f^{-1}(J)$.

Let $x \in K, y \in f^{-1}(J)$ such that $x \leq y \Longrightarrow x+y = y \Longrightarrow f(x+y) = f(y) \Longrightarrow f(x)+f(y) = f(y) \in J \Longrightarrow f(x) \leq f(y) \Longrightarrow f(x) \in J \Longrightarrow x \in f^{-1}(J)$. Hence $f^{-1}(J)$ is an ordered quasi-Γ-ideal of an ordered Γ-semiring K. □

Theorem 31. *Let S be an ordered Γ-semiring. If Q is an ordered $m-k$ quasi-Γ-ideal of S, then Q is a maximal ordered quasi-Γ-ideal of S.*

Proof. Let Q be an ordered $m-k$ quasi-Γ-ideal of an ordered Γ-semiring S. Suppose J is an ordered quasi-Γ-ideal of S such that $Q \subseteq J, x \in J, y \in Q$ and $\gamma \in \Gamma$. We have $y\gamma x \in Q \Longrightarrow x \in Q$, since Q is an ordered $m-k$ quasi-Γ-ideal of S. Therefore $Q = J$. Hence ordered $m-k$ quasi-Γ-ideal Q of S is maximal ordered quasi-Γ-ideal of S. □

Theorem 32. *Let S be an ordered Γ-semiring in which $(S,+)$ is cancellative semigroup and $H = \{x \in S \mid x+x = x\}$. If $H \neq \emptyset$, then H is an ordered $m-k$ quasi-Γ-ideal of an ordered Γ-semiring S.*

Proof. Let $x \in H, y \in S$ and $\beta \in \Gamma$. Then $x = x + x \Longrightarrow x\beta y = (x + x)\beta y = x\beta y + x\beta y \Longrightarrow x\beta y \in H$. Similarly $y\beta x \in H$. Suppose $x, y \in H$. Then $x + x = x,\ y + y = y \Longrightarrow (x + y) + (x + y) = (x + x) + (y + y) = x + y \Longrightarrow x + y \in H$. Suppose $x \leq y,\ y \in H$. Then $x + y = y \Longrightarrow x + x + y = x + y \Longrightarrow x + x = x$. Let $s\alpha h \in S\Gamma H$ for $s \in S, \alpha \in \Gamma$ and $h \in H$. Next, $s\alpha h + s\alpha h = s\alpha(h + h) = s\alpha h \Longrightarrow s\alpha h \in H \Longrightarrow S\Gamma H \subseteq H$. In a similar way, we can show that $H\Gamma S \subseteq H$. Therefore, $S\Gamma H \cap H\Gamma S \subseteq H$. Hence H is a quasi-ideal.

Therefore, H is an ordered quasi-Γ-ideal of an ordered Γ-semiring. Suppose $x\gamma y \in H, x \in H$, and $\gamma \in \Gamma$. Then $x\gamma y + x\gamma y = x\gamma y \Longrightarrow x\gamma(y + y) = x\gamma y \Longrightarrow y + y = y \Longrightarrow y \in H$. Hence H is an ordered $m - k$ quasi-Γ-ideal of S. □

Theorem 33. *The intersection of family of prime (or semiprime) ordered quasi-Γ-ideal of S is a semiprime ordered quasi-Γ-ideal.*

Proof. Let $\{P_i \mid i \in \Lambda\}$ be the family of prime ordered quasi-Γ-ideal of S and $P = \bigcap_{i=1}^{k} P_i$. Since $S\Gamma P = S\Gamma(\bigcap_{i=1}^{k} P_i) \subseteq \bigcap_{i=1}^{k}(S\Gamma P_i)$ and $P\Gamma S = (\bigcap_{i=1}^{k} P_i)\Gamma S \subseteq \bigcap_{i=1}^{k}(P_i\Gamma S) \Longrightarrow S\Gamma P \bigcap P\Gamma S \subseteq ((\bigcap_{i=1}^{k}(S\Gamma P_i)) \cap (\bigcap_{i=1}^{k}(P_i)\Gamma S) = \bigcap_{i=1}^{k}(S\Gamma P_i \cap P_i\Gamma S) \subseteq \bigcap_{i=1}^{k} P_i = P$. Hence P is a quasi-Γ-ideal.

Now for any $y \in P$ such that $x \leq y \Longrightarrow x \in P_i\ \forall i \Longrightarrow x \in P$. Hence P is an ordered quasi-Γ-ideal of S. For any ordered quasi-Γ-ideal Q of S, $Q\Gamma Q \subseteq P$ implies $Q\Gamma Q \subseteq P_i$ for all $i \in \Lambda$. As P_i are semiprime ordered quasi-Γ-ideals, $Q \subseteq P_i$ for all $i \in \Lambda$. Hence $Q \subseteq P$. □

Theorem 34. *Every strongly irreducible, semiprime ordered quasi-Γ-ideal of S is a strongly prime ordered quasi-Γ-ideal.*

Proof. Let Q be a strongly irreducible and semiprime ordered quasi-Γ-ideal of S. For any ordered quasi-Γ-ideal Q_1 and Q_2 of S, let $(Q_1\Gamma Q_2) \cap (Q_2\Gamma Q_1) \subseteq Q$. $Q_1 \cap Q_2$ is a quasi-Γ-ideal of S, since $(Q_1 \cap Q_2)^2 = (Q_1 \cap Q_2)\Gamma(Q_1 \cap Q_2) \subseteq Q_1\Gamma Q_2$. Similarly we get $(Q_1 \cap Q_2)^2 = (Q_1 \cap Q_2)\Gamma(Q_1 \cap Q_2) \subseteq Q_2\Gamma Q_1$. Therefore $(Q_1 \cap Q_2)^2 \subseteq Q_1\Gamma Q_2 \cap Q_2\Gamma Q_1 \subseteq Q$. As Q is a semiprime ordered quasi-Γ-ideal, $Q_1 \cap Q_2 \subseteq Q$. But Q is a strongly irreducible ordered quasi-Γ-ideal. Therefore, $Q_1 \subseteq Q$ or $Q_2 \subseteq Q$. Thus Q is a strongly prime ordered quasi-Γ-ideal of S. □

Theorem 35. *If Q is an ordered k quasi-Γ-ideal of S and $q \in S$ such that $q \notin Q$, then there exists an irreducible ordered k quasi-Γ-ideal Q_1 of S such that $Q \subseteq Q_1$ and $q \notin Q_1$.*

Proof. Let F be the family of ordered k quasi-Γ-ideals of S which contain Q but do not contain element q. Then F is nonempty as $Q \in F$. This family of ordered k quasi-Γ-ideals of S forms a partially ordered set under the set inclusion. Hence by Zorn's lemma there exists a maximal ordered k quasi-Γ-ideal say Q_1 in F. Therefore $Q \subseteq Q_1$ and $q \notin Q_1$. Now to show that Q_1 is an irreducible ordered k quasi-Γ-ideal of S, let K and L be any two ordered k quasi-Γ-ideals of S such that $K \cap L = Q_1$. Suppose that K and L both contain Q_1 properly. But Q_1 is a maximal ordered k quasi-Γ-ideal in F. Hence we get $q \in K$ and $q \in L$. Therefore $q \in K \cap L = Q_1$ which is absurd. Thus either $K = Q_1$ or $L = Q_1$. Therefore, Q_1 is an irreducible ordered k quasi-Γ-ideal of S. □

Theorem 36. *The following statements are equivalent in S:*

(1) *The set of ordered k quasi-Γ-ideals of S is totally ordered set under inclusion of sets.*

(2) *Each ordered k quasi-Γ-ideal of S is strongly irreducible.*

(3) *Each ordered k quasi-Γ-ideal of S is irreducible.*

Proof. (1) $\Longrightarrow$ (2) Suppose that the set of ordered k quasi-Γ-ideals of S is a totally ordered set under inclusion of sets. Let B be any ordered k quasi-Γ-ideal of S. Then B is a strongly irreducible ordered k quasi-Γ-ideal of S for that let Q_1 and Q_2 be any two ordered k quasi-Γ-ideal of S such that $Q_1 \cap Q_2 \subseteq Q$. But, by the hypothesis, we have either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. Therefore, $Q_1 \cap Q_2 = Q_1$ or $Q_1 \cap Q_2 = Q_2$. Hence $Q_1 \subseteq Q$ or $Q_2 \subseteq Q$. Thus Q is a strongly irreducible ordered k quasi-Γ-ideal of S.

(2) $\Longrightarrow$ (3) Suppose that each ordered k quasi-Γ-ideal of S is strongly irreducible. Let Q be any ordered k quasi-Γ-ideal of S such that $Q = Q_1 \cap Q_2$ for any ordered k quasi-Γ-ideal Q_1 and Q_2 of S. But by hypothesis $Q_1 \subseteq Q$ or $Q_2 \subseteq Q$. As $Q \subseteq Q_1$ and $Q \subseteq Q_2$, we get $Q_1 = Q$ or $Q_2 = Q$. Hence Q is an irreducible ordered k quasi-Γ-ideal of S.

(3) $\Longrightarrow$ (1) Suppose that each ordered k quasi-Γ-ideal of S is an irreducible ordered k quasi-Γ-ideal. Let Q_1 and Q_2 be any two ordered k quasi-Γ-ideal of S. Then $Q_1 \cap Q_2$ is also ordered k quasi-Γ-ideal of S. Hence $Q_1 \cap Q_2 = Q_1 \cap Q_2$ will imply $Q_1 \cap Q_2 = Q_1$ or $Q_1 \cap Q_2 = Q_2$ by assumption. Therefore either $Q_1 \subseteq Q_2$ or $Q_2 \subseteq Q_1$. This shows that the set of ordered k quasi-Γ-ideal of S is totally ordered set under inclusion of sets. □

Theorem 37. *A prime ordered k quasi-Γ-ideal Q of S is a prime one sided ordered k ideal of S.*

Proof. Let Q be a prime k quasi-Γ-ideal of S. Suppose Q is not a one sided k ideal of S. Therefore, $Q\Gamma S \nsubseteq Q$ and $S\Gamma Q \nsubseteq Q$. As Q is a prime k quasi-Γ-ideal, $(Q\Gamma S)\Gamma(S\Gamma Q) \nsubseteq Q$. $(Q\Gamma S)\Gamma(S\Gamma Q) \subseteq (Q\Gamma S) \cap (S\Gamma Q) \subseteq Q$ which is a contradiction. Therefore, $Q\Gamma S \subseteq Q$ or $S\Gamma Q \subseteq Q$. Thus Q is a prime one sided k ideal of S. □

Theorem 38. *An ordered k quasi-Γ-ideal Q of S is prime if and only if, for a right ordered k ideal R and a left ordered k ideal L of S, $R\Gamma L \subseteq Q$ implies $R \subseteq Q$ or $L \subseteq Q$.*

Proof. Suppose that an ordered k quasi-Γ-ideal of S is a prime ordered k quasi-Γ-ideal of S. Let R be a right ordered k ideal and L be a left ordered k ideal of S such that $R\Gamma L \subseteq Q$. R and L are ordered k quasi-Γ-ideal of S. Hence $R \subseteq Q$ or $L \subseteq Q$. Conversely, we have to show that an ordered k quasi-Γ-ideal Q of S is a prime ordered k quasi-Γ-ideal of S. Let A and C be any two ordered k quasi-Γ-ideals of S such that $A\Gamma C \subseteq Q$. For any $a \in A$ and $c \in C$, $(a)_r \subseteq A$ and $(c)_l \subseteq C$, where $(a)_r$ and $(c)_l$ denote the right ordered k ideal and left ordered k ideal generated by a and c, respectively. Thus $(a)_r\Gamma(c)_l \subseteq A\Gamma C \subseteq Q$. Hence by the assumption $(a)_r \subseteq B$ or $(c)_l \subseteq Q$ which further

gives that $a \in Q$ or $c \in Q$. In this way we get that $A \subseteq Q$ or $C \subseteq Q$. Hence Q is a prime ordered k quasi-Γ-ideal of S. □

4. Ordered Bi-Γ-Ideals in Ordered-Γ-Semirings

In this section we define ordered bi-Γ-ideal, ordered k bi-Γ-ideal, ordered $m-k$ bi-Γ-ideal, ordered prime bi-Γ-ideal, ordered semiprime bi-Γ-ideal, ordered maximal bi-Γ-ideal, ordered irreducible, and ordered strongly irreducible bi-Γ-ideal of an ordered Γ-semiring S. Recall the relation in the previous section "$\leq$" in b as $a \leq b$ if $a+x = b$ for any $a, b, x \in S$. Further for a nonempty subset X of S and $a, b \in S$, $a \leq b$ in X if $a + x = b$ for any $x \in X$.

Definition 39. An ordered sub-Γ-semiring B of an ordered Γ-semiring S is called an ordered bi-Γ-ideal of S if $B\Gamma S\Gamma B \subseteq B$ and if, for any $\in S$, $b \in B$, $a \leq b$ in B, then $a \in B$.

Definition 40. An ordered bi-Γ-ideals B is called ordered k bi-Γ-ideal of S if B is an ordered sub-Γ-semiring of S, and if $b \in B$ and $x \in S$ such that $x + b \in B$, then $x \in B$.

Clearly every ordered k bi-Γ-ideal is an ordered bi-Γ-ideal but converse is not true in general as shown in the following example.

Example 41. Let S be the set of nonnegative integers and $\Gamma = N$ be additive abelian semigroups. Tennary operation is defined as $(x, \gamma, y) \longrightarrow x\gamma y$, usual multiplication of integers. Then S is an ordered Γ-semirings. A subset $B = 7S \setminus \{7\}$ of S is an ordered bi-Γ-ideal of S but it is not an ordered k bi-Γ-ideal of S.

Definition 42. An ordered bi-Γ-ideal B of an ordered Γ-semiring S is called ordered $m-k$ bi-Γ-ideal of S if $b \in B$, $x \in S$, $b\gamma x \in B$ for $\gamma \in \Gamma$, and then $x \in B$.

Theorem 43. *Every ordered $m-k$ bi-Γ-ideal of an ordered Γ-semiring S is an ordered k bi-Γ-ideal of S.*

Proof. Let B be an ordered $m-k$ bi-Γ-ideal of an ordered Γ-semiring S. Consider $x + b \in B$, $b \in B$, $x \in S$ and $\gamma \in \Gamma$; then $(x + b)\gamma x \in B$. Then $x \in B$, since B is an ordered $m-k$ bi-Γ-ideal. Hence B is an ordered k bi-Γ-ideal of S. □

The following example shows that the converse of Theorem 43 is not true in general.

Example 44. Let S be the set of all natural numbers. Then S with usual ordering is an ordered semiring. If $\Gamma = S$, then S is an ordered Γ-semiring. If $B = \{2, 4, 6, \ldots\}$ then B forms an ordered k bi-Γ-ideal but not ordered $m-k$ bi-Γ-ideals of ordered Γ-semiring S.

Definition 45. An ordered Γ-semiring S is called band if every element of S is a γ-idempotent.

Theorem 46. *Let B be an ordered sub-Γ-semiring of an ordered Γ-semiring S in which semigroup $(S, +)$ is a band. Then B is an ordered bi-Γ-ideal of S if and only if B is an ordered k bi-Γ-ideals of S.*

Proof. Let B be an ordered bi-Γ-ideal of an ordered Γ-semiring S and $x+b \in B$, $x \in B$. Then $x+(x+b) = (x+x)+b = x+b \Longrightarrow x \leq x+b \Longrightarrow x \in B$ as B is an ordered bi-Γ-ideal of S. Hence B is a an ordered k bi-Γ-ideal of S.

Conversely suppose that B is an ordered k bi-Γ-ideal of an ordered Γ-semiring S. Let $x \in S, b \in B$ and $x \leq b$ in $B \Longrightarrow x + b' = b$ for any $b' \in B \Longrightarrow x + b' \in B \Longrightarrow x \in B$, since B is an ordered k bi-Γ-ideal of an ordered Γ-semiring S. Hence B is an ordered bi-Γ-ideal of S. □

Definition 47. An ordered bi-Γ-ideal B of S is called an irreducible ordered bi-Γ-ideal if $B_1 \cap B_2 = B$ implies $B_1 = B$ or $B_2 = B$, for any ordered bi-Γ-ideal B_1 and B_2 of S.

Definition 48. An ordered bi-Γ-ideal B of S is called strongly irreducible ordered bi-Γ-ideal if, for any ordered bi-Γ-ideal B_1 and B_2 of S, $B_1 \cap B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$. It is obviously that every strongly irreducible ordered bi-Γ-ideal is an irreducible ordered bi-Γ-ideal.

Definition 49. An ordered bi-Γ-ideal B of S is called a prime ordered bi-Γ-ideal if $B_1\Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any ordered bi-Γ-ideals B_1, B_2 of S.

Definition 50. An ordered k bi-Γ-ideal B of S is called a prime ordered k bi-Γ-ideal if $B_1\Gamma B_2 \subseteq B$ implies $B_1 \subseteq B$ or $B_2 \subseteq B$, for any ordered k bi-Γ-ideals B_1, B_2 of S.

Definition 51. An ordered bi-Γ-ideal B of S is called a strongly prime ordered bi-Γ-ideal if $(B_1\Gamma B_2)\cap(B_2\Gamma B_1) \subseteq B$ implies that $B_1 \subseteq B$ or $B_2 \subseteq B$ for any ordered bi-Γ-ideals B_1, B_2 of S.

Definition 52. An ordered bi-Γ-ideal B of S is called a semiprime ordered bi-Γ-ideal if for any ordered bi-Γ-ideals B_1 of S $B_1^2 = B_1\Gamma B_1 \subseteq B$ implies $B_1 \subseteq B$.

It is clear that every strongly prime ordered bi-Γ-ideal in S is a prime ordered bi-Γ-ideal and every prime ordered bi-Γ-ideal in S is a semiprime ordered bi-Γ-ideal.

Definition 53. An ordered bi-Γ-ideal B of an ordered Γ-semiring S is said to be maximal ordered bi-Γ-ideal if $B \neq S$ and for every ordered bi-Γ-ideal C of S with $B \subseteq C \subseteq S$, then either $B = C$ or $B = S$.

Theorem 54. *In an ordered Γ-semiring S, every maximal ordered bi-Γ-ideal of S is irreducible ordered bi-Γ-ideal of S.*

Proof. Let B_m be a maximal ordered bi-Γ-ideal of an ordered Γ-semiring S. Suppose B_m is not irreducible and $B_m = U \cap V \Longrightarrow B_m \neq U$ and $B_m \neq V \Longrightarrow B_m \subset U \subset S$ and $B_m \subset V \subset S$ a contradiction. Hence B_m is irreducible ordered bi-Γ-deal of an ordered Γ-semiring S. □

Theorem 55. *Let B be an ordered bi-Γ-ideal of an ordered Γ-semiring S.*

(i) *If B is a prime ordered bi-Γ-ideal, then B is a strongly irreducible ordered bi-Γ-ideal.*

(ii) *If B is a strongly irreducible ordered bi-Γ-ideal, then B is an irreducible ordered bi-Γ-ideal.*

Proof. Let B be an ordered bi-Γ-ideal of an ordered Γ-semiring S.

(i) Suppose B is a prime ordered bi-Γ-ideal; J and K are ordered bi-Γ-ideal of an ordered Γ-semiring S such that $J \cap K \subseteq B$. Then $J\Gamma K \subseteq B \Longrightarrow J \subseteq B$ or $K \subseteq B$. Hence B is a strongly irreducible ordered bi-Γ-ideal.

(ii) Suppose B is a strongly irreducible ordered bi-Γ-ideal; J and K are ordered bi-Γ-ideal of an ordered Γ-semiring S such that $J \cap K = B$. Then certainly $J \cap K \subseteq B \Longrightarrow J \subseteq B$ or $K \subseteq B$. Hence $J = B$ or $K = B$.

Therefore, B is an irreducible ordered bi-Γ-ideal of S. □

Corollary 56. *Let S be an ordered Γ-semiring. If B is a prime ordered bi-Γ-ideal of S, then B is an irreducible ordered bi-Γ-ideal of S.*

Theorem 57. *Let $f : K \longrightarrow L$ be a homomorphism of ordered Γ-semirings. If J is an ordered bi-Γ-ideal of L then $f^{-1}(J)$ is an ordered bi-Γ-ideal of an ordered Γ-semiring K.*

Proof. Suppose J is an ordered bi-Γ-ideal of L, $f : K \longrightarrow L$ be a homomorphism of ordered Γ-semirings and $x, y \in f^{-1}(J) \Longrightarrow f(x), f(y) \in J \Longrightarrow f(x) + f(y) = f(x + y) \in J \Longrightarrow x + y \in f^{-1}(J)$. If $x, y \in f^{-1}(J) \Longrightarrow f(x), f(y) \in J \Longrightarrow f(x)\gamma f(y) \in J \Longrightarrow f(x\gamma y) \in J \Longrightarrow x\gamma y \in f^{-1}(J)$. Now for $x, y \in f^{-1}(J), \alpha, \beta \in \Gamma, s \in K \Longrightarrow f(x), f(y) \in J$ and $f(s) \in L$. Since J is an ordered bi-Γ-ideal, so we have $f(x)\alpha f(s)\beta f(y) \in J \Longrightarrow f(x\alpha s\beta y) \in J \Longrightarrow x\alpha s\beta y \in f^{-1}(J) \Longrightarrow f^{-1}(J)$ that is a bi-Γ-ideal of S.

Let $x \in K, y \in f^{-1}(J)$ such that $x \leq y$ in $f^{-1}(J) \Longrightarrow x + y' = y$ for any $y' \in f^{-1}(J) \Longrightarrow f(x + y') = f(y) \Longrightarrow f(x) + f(y') = f(y) \in J \Longrightarrow f(x) \leq f(y)$ in $J \Longrightarrow f(x) \in J \Longrightarrow x \in f^{-1}(J)$. Hence $f^{-1}(J)$ is an ordered bi-Γ-ideal of an ordered Γ-semiring K. □

Theorem 58. *Let S be an ordered Γ-semiring. If B is an ordered $m-k$ bi-Γ-ideal of S, then B is a maximal ordered bi-Γ-ideal of S.*

Proof. Let B be an ordered $m-k$ bi-Γ-ideal of an ordered Γ-semiring S. Suppose J is an ordered bi-Γ-ideal of S such that $B \subseteq J$, $x \in J, y \in B$ and $\gamma \in \Gamma$. We have $y\gamma x \in B \Longrightarrow x \in B$, since B is an ordered $m-k$ bi-Γ-ideal of S. Therefore, $B = J$. Hence ordered $m-k$ bi-Γ-ideal B of S is maximal ordered bi-Γ-ideal of S. □

Theorem 59. *Let S be an ordered Γ-semiring in which $(S, +)$ is cancellative semigroup and $H = \{x \in S \mid x + x = x\}$. If $H \neq \emptyset$, then H is an ordered k and $m-k$ bi-Γ-ideal of an ordered Γ-semiring S.*

Proof. Let $x \in H, y \in S$ and $\beta \in \Gamma$. Then $x = x + x \Longrightarrow x\beta y = (x+x)\beta y = x\beta y + x\beta y$. Therefore, $x\beta y \in H$. Similarly $y\beta x \in H$. Suppose $x, y \in H$. Then $x + x = x, y + y = y \Longrightarrow (x + y) + (x + y) = (x + x) + (y + y) = x + y \Longrightarrow x + y \in H$. Suppose $x \leq y$, $y \in H$. Then $x + y = y \Longrightarrow x + x + y = x + y \Longrightarrow x + x = x$. Further $x, y \in H, \alpha, \beta \in \Gamma$ and $s \in S$, and then $x\alpha s\beta y = (x + x)\alpha s\beta y = x\alpha s\beta y + a\alpha s\beta y \Longrightarrow x\alpha s\beta y \in H \Longrightarrow H\Gamma S\Gamma H \subseteq H$.

Therefore, H is an ordered bi-Γ-ideal of an ordered Γ-semiring. Suppose $y, x + y \in H$. Then $x + x = x, (x + y) + (x + y) = x + y \Longrightarrow (x+x)+(y+y) = x+y \Longrightarrow x+(y+y) = x + y \Longrightarrow y + y = y \Longrightarrow y \in H$. Hence H is an ordered k bi-Γ-ideal of an ordered Γ-semiring S. Suppose $x\gamma y \in H, x \in H$ and $\gamma \in \Gamma$. Then $x\gamma y + x\gamma y = x\gamma y \Longrightarrow x\gamma(y + y) = x\gamma y \Longrightarrow y + y = y \Longrightarrow y \in H$. Hence H is an ordered $m-k$ bi-Γ-ideal of S. □

Theorem 60. *The intersection of family of prime (or semiprime) ordered bi-Γ-ideal of S is a semiprime ordered bi-Γ-ideal.*

Proof. Let $\{P_i \mid i \in \Lambda\}$ be the family of prime ordered bi-Γ-ideal of S and $P = \cap_{i=1}^{k} P_i$. For $x, y \in P \Longrightarrow x, y \in P_i\ \forall i \Longrightarrow x\alpha s\beta y \in P_i\ \forall \alpha, \beta \in \Gamma, s \in S \Longrightarrow x\alpha s\beta y \in P$. Now for any $y \in P$ such that $x \leq y \Longrightarrow x \in P_i\ \forall i \Longrightarrow x \in P$. Hence P is an ordered bi-Γ-ideal of S. For any ordered bi-Γ-ideal B of S, $B\Gamma B \subseteq P$ implies $B\Gamma B \subseteq P_i$ for all $i \in \Lambda$. As P_i are semiprime ordered bi-Γ-ideals, $B \subseteq P_i$ for all $i \in \Lambda$. Hence $B \subseteq P$. □

Theorem 61. *Every strongly irreducible, semiprime ordered bi-Γ-ideal of S is a strongly prime ordered bi-Γ-ideal.*

Proof. Let B be a strongly irreducible and semiprime ordered bi-Γ-ideal of S. For any ordered bi-Γ-ideal B_1 and B_2 of S, let $(B_1\Gamma B_2) \cap (B_2\Gamma B_1) \subseteq B$. $B_1 \cap B_2$ is a bi-Γ-ideal of S. Since $(B_1 \cap B_2)^2 = (B_1 \cap B_2)\Gamma(B_1 \cap B_2) \subseteq B_1\Gamma B_2$. Similarly we get $(B_1 \cap B_2)^2 = (B_1 \cap B_2)\Gamma(B_1 \cap B_2) \subseteq B_2\Gamma B_1$. Therefore, $(B_1 \cap B_2)^2 \subseteq B_1\Gamma B_2 \cap B_2\Gamma B_1 \subseteq B$. As B is a semiprime ordered bi-Γ-ideal, $B_1 \cap B_2 \subseteq B$. But B is a strongly irreducible ordered bi-Γ-ideal. Therefore $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly prime ordered bi-Γ-ideal of S. □

Theorem 62. *If B is an ordered k bi-Γ-ideal of S and $b \in S$ such that $b \notin B$, then there exists an irreducible ordered k bi-Γ-ideal B_1 of S such that $B \subseteq B_1$ and $b \notin B_1$.*

Proof. Let F be the family of ordered k bi-Γ-ideals of S which contain B but do not contain element b. Then F is nonempty as $B \in F$. This family of ordered k bi-Γ-ideals of S forms a partially ordered set under the set inclusion. Hence by Zorn's lemma there exists a maximal ordered k bi-Γ-ideal say B_1 in F. Therefore $B \subseteq B_1$ and $b \notin B_1$. Now to show that B_1 is an irreducible ordered k bi-Γ-ideal of S. Let K and L be any two ordered k bi-Γ-ideals of S such that $K \cap L = B_1$. Suppose that K and L both contain B_1 properly. But B_1 is a maximal ordered k bi-Γ-ideal in F. Hence we get $b \in K$ and $b \in L$. Therefore $b \in K \cap L = B_1$ which is absurd. Thus either $K = B_1$ or $L = B_1$. Therefore B_1 is an irreducible ordered k bi-Γ-ideal of S. □

Theorem 63. *Following statements are equivalent in S:*

(1) *The set of ordered k bi-Γ-ideals of S is totally ordered set under inclusion of sets.*

(2) *Each ordered k bi-Γ-ideal of S is strongly irreducible.*

(3) *Each ordered k bi-Γ-ideal of S is irreducible.*

Proof. (1) $\Longrightarrow$ (2) Suppose that the set of ordered k bi-Γ-ideals of S is a totally ordered set under inclusion of sets. Let B be any ordered k bi-Γ-ideal of S. Then B is a strongly irreducible ordered k bi-Γ-ideal of S for that let B_1 and B_2 be any two ordered k bi-Γ-ideal of S such that $B_1 \cap B_2 \subseteq B$. But by the hypothesis, we have either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. Therefore, $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$. Hence $B_1 \subseteq B$ or $B_2 \subseteq B$. Thus B is a strongly irreducible ordered k bi-Γ-ideal of S.

(2) $\Longrightarrow$ (3) Suppose that each ordered k bi-Γ-ideal of S is strongly irreducible. Let B be any ordered k bi-Γ-ideal of S such that $B = B_1 \cap B_2$ for any ordered k bi-Γ-ideal B_1 and B_2 of S. But by hypothesis $B_1 \subseteq B$ or $B_2 \subseteq B$. As $B \subseteq B_1$ and $B \subseteq B_2$, we get $B_1 = B$ or $B_2 = B$. Hence B is an irreducible ordered k bi-Γ-ideal of S.

(3) $\Longrightarrow$ (1) Suppose that each ordered k bi-Γ-ideal of S is an irreducible ordered k bi-Γ-ideal. Let B_1 and B_2 be any two ordered k bi-Γ-ideal of S. Then $B_1 \cap B_2$ is also ordered k bi-Γ-ideal of S. Hence $B_1 \cap B_2 = B_1 \cap B_2$ will imply $B_1 \cap B_2 = B_1$ or $B_1 \cap B_2 = B_2$ by assumption. Therefore either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. This shows that the set of ordered k bi-Γ-ideal of S is totally ordered set under inclusion of sets. □

Theorem 64. *A prime ordered k bi-Γ-ideal B of S is a prime one sided ordered k ideal of S.*

Proof. Let B be a prime k bi-Γ-ideal of S. Suppose B is not a one sided k ideal of S. Therefore, $B\Gamma S \nsubseteq B$ and $S\Gamma B \nsubseteq B$. As B is a prime k bi-Γ-ideal, $(B\Gamma S)\Gamma(S\Gamma B) \nsubseteq B$. $(B\Gamma S)\Gamma(S\Gamma B) = B\Gamma(S\Gamma S)\Gamma B \subseteq B\Gamma S\Gamma B \subseteq B$, which is a contradiction. Therefore, $B\Gamma S \subseteq B$ or $S\Gamma B \subseteq B$. Thus B is a prime one sided k ideal of S. □

Theorem 65. *An ordered k bi-Γ-ideal B of S is prime if and only if, for a right ordered k ideal R and a left ordered k ideal L of S, $R\Gamma L \subseteq B$ implies $R \subseteq B$ or $L \subseteq B$.*

Proof. Suppose that an ordered k bi-Γ-ideal of S is a prime ordered k bi-Γ-ideal of S. Let R be a right ordered k ideal and L be a left ordered k ideal of S such that $R\Gamma L \subseteq B$. R and L are ordered k bi-Γ-ideal of S. Hence $R \subseteq B$ or $L \subseteq B$. Conversely, we have to show that an ordered k bi-Γ-ideal B of S is a prime ordered k bi-Γ-ideal of S. Let A and C be any two ordered k bi-Γ-ideals of S such that $A\Gamma C \subseteq B$. For any $a \in A$ and $c \in C$, $(a)_r \subseteq A$ and $(c)_l \subseteq C$, where $(a)_r$ and $(c)_l$ denote the right ordered k ideal and left ordered k ideal generated by a and c, respectively. Thus $(a)_r\Gamma(c)_l \subseteq A\Gamma C \subseteq B$. Here by the assumption we get that $(a)_r \subseteq B$ or $(c)_l \subseteq B$ which gives that $a \in B$ or $c \in B$. Thus $A \subseteq B$ or $C \subseteq B$. Hence B is a prime ordered k bi-Γ-ideal of S. □

5. Conclusion

In this work we have given many types of ordered (k), $(m-k)$ $-$ Γ-ideals mainly ordered k-(quasi, bi)-Γ-ideal, ordered $m-k$ (quasi, bi)-Γ-ideal, ordered (semi)prime, (k)-(semi)prime (quasi, bi)-Γ-ideals, ordered maximal(k)-(quasi, bi)-Γ-ideals, ordered (strongly) irreducible (quasi, bi)-Γ-ideals. We have studied properties of these different ideals and some of their relations in ordered Γ-semirings. Let $f : K \longrightarrow L$ be a homomorphism of ordered Γ-semirings. We have defined homomorphism and prove that if J is an ordered quasi-Γ-ideal of L, then $f^{-1}(J)$ is an ordered quasi-Γ-ideal of an ordered Γ-semiring K. The same is true in case of ordered bi-Γ-ideals. For further research work the extended identical study can be taken under consider for other algebraic structures.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] H. S. Vandiver, "Note on a simple type of algebra in which the cancellation law of addition does not hold," *Bulletin of the American Mathematical Society*, vol. 40, no. 12, pp. 914–920, 1934.

[2] M. M. Rao, "Γ-semirings. I," *Southeast Asian Bulletin of Mathematics*, vol. 19, no. 1, pp. 49–54, 1995.

[3] O. Steinfeld, "On ideal-quotients and prime ideals," *Acta Mathematica Hungarica*, vol. 4, pp. 289–298, 1953.

[4] O. Steinfeld, "Uher die quasi-ideals von halbgruppen," *Publicationes Mathematicae*, vol. 4, pp. 262–275, 1956.

[5] K. Iséki, "Quasi-ideals in semirings without zero," *Proceedings of the Japan Academy, Series A: Mathematical Sciences*, vol. 34, pp. 79–81, 1958.

[6] C. Donges, *On Quasi-Ideals of Semirings*, vol. 17, no. 1, Germany, DW-3392 Clausthal-Zellerfeld, Germany, 1994.

[7] M. Shabir, A. Ali, and S. Batool, "A note on quasi-ideals in semirings," *Southeast Asian Bulletin of Mathematics*, vol. 27, no. 5, pp. 923–928, 2004.

[8] R. A. Good and D. R. Hughes, "Associated groups for a semigroup," *Bulletin of the American Mathematical Society*, vol. 58, pp. 624-625, 1952.

[9] F. A. Szasz, "Generalized biideals of rings I," *Mathematische Nachrichten*, vol. 47, pp. 355–360, 1970.

[10] F. A. Szasz, "Generalized biideals of rings II," *Mathematische Nachrichten*, vol. 47, pp. 361–364, 1970.

[11] S. Lajos, "Notes on generalized bi-ideals in semigroups," *Soochow Journal of Mathematics*, vol. 10, pp. 55–59, 1984.

[12] T. K. Dutta and S. K. Sardar, "Semiprime ideals and irreducible ideals of Γ-semirings," *Novi Sad Journal of Mathematics*, vol. 30, no. 1, pp. 97–108, 2000.

[13] S. K. Sardar and U. Dasgupta, "On primitive Γ-semirings," *Novi Sad Journal of Mathematics*, vol. 34, no. 1, pp. 1–12, 2004.

[14] J. P. Kaushik, M. A. Ansari, and M. Khan, "On bi-Γ-ideal in Γ-semirings," *International Journal of Contemporary Mathematical Sciences*, vol. 3, no. 25-28, pp. 1255–1260, 2008.

[15] S. Pianskool, S. Sangwirotjanapat, and S. Tipyota, "Valuation Γ-semirings and valuation Γ-ideals," *Thai Journal of Mathematics Special Issue (Annual Meeting in Mathematics)*, vol. 6, no. 3, pp. 93–102, 2008.

[16] R. Chinram, "A note on quasi-ideals in Γ-semirings," *International Mathematical Forum. Journal for Theory and Applications*, vol. 3, no. 25-28, pp. 1253–1259, 2008.

[17] R. D. Jagatap and Y. S. Pawar, "Quasi-ideals and minimal quasi-ideals in Γ-semirings," *Novi Sad Journal of Mathematics*, vol. 39, no. 2, pp. 79–87, 2009.

[18] R. D. Jagatap, "Prime k-Bi-ideals in Γ-semirings," *Palestine Journal of Mathematics*, vol. 3 (Spec 1), pp. 489–494, 2014.

[19] M. S. Lavanya, D. M. Rao, and V. S. J. Rajendra, "Prime bi-ternary G-ideals in ternary G-semirings," *British Journal of Research*, vol. 2, no. 6, pp. 156–166, 2015.

[20] R. D. Jagatap and Y. S. Pawar, "Bi-ideals in Γ-semirings," *Bulletin of International Mathematical Virtual Institute*, vol. 6, no. 2, pp. 169–179, 2016.

[21] M. M. Rao, "Ideals in ordered Γ-semirings," *Discussiones Mathematicae. General Algebra and Applications*, vol. 38, no. 1, pp. 47–68, 2018.