Research Article

Strong Law of Large Numbers of Pettis-Integrable Multifunctions

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Received 11 February 2019; Accepted 26 March 2019; Published 15 April 2019

Academic Editor: Tepper LGill

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Using reversed martingale techniques, we prove the strong law of large numbers for independent Pettis-integrable multifunctions with convex weakly compact values in a Banach space. The Mosco convergence of reversed Pettis-integrable martingale of the form \((E^n \times X)_{n \geq 1}\), where \((B_n)_{n \geq 1}\) is a decreasing sequence of the sub-σ-algebra of \(\mathcal{B}\) is provided.

1. Introduction

The strong law of large numbers (SLLN) is used in a variety of fields including statistics, probability theory, and areas of economics and insurance. In recent years, SLLN has been extensively studied by several researchers. Let us mention Artstein and Hart [1], Castaing and Ezzaki [2], Etemadi [3], Ezzaki [4], Hess [5], and Hiai [6].

In the theory of integration in infinite-dimensional spaces, Pettis-integrability is a more general concept than that of Bochner-integrability. The purpose of this paper is to prove the SLLN for measurable and Pettis-integrable multifunctions by using the techniques of reversed martingale. The proof is based on the recent properties of Pettis-integrable multifunctions. See for example Akhiat et al. [7], Chowdhury [8], El Amri and Hess [9], Geitz et al. [10], Thobie and Satco [11], and Musial [12].

The paper is organized as follows.

In Section 2, we recall some definitions and results that will be used after. In Section 3, we prove the SLLN for Pettis-integrable multifunctions with convex weakly compact values.

2. Notations and Definitions

Throughout this paper, we assume that \((\Omega, \mathcal{F}, P)\) is a complete probability space and \((\mathcal{B}_n)_{n \geq 1}\) is a decreasing sequence of sub-σ-algebra of \(\mathcal{B}\), such that \(\mathcal{B}_\infty = \bigcap_{n \geq 1} \mathcal{B}_n\). E is a separable Banach space with the dual space \(E^*\).

Let \((e_k^*)_{k \in \mathbb{N}}\) be a dense sequence in \(E^*\) with respect to the Mackey topology \(\tau(E^*, E)\), \(\mathcal{B}_{E*}\) (resp., \(\mathcal{B}_E\)), and the closed unit ball of \(E\) (resp., \(E^*\)). Let \(cc(E)\) (resp., \(c_{wk}(E)\)) be the family of all nonempty convex and closed (resp., convex weakly compact) subsets of \(E\).

Given \(B\) in \(cc(E)\), the distance function and the support function associated with \(B\) are defined by

\[
d(x, B) = \inf \{ \|x - y\|, y \in B\}, \quad (x \in E),
\]

\[
\delta^*(x^*, B) = \sup \{ \langle x^*, y \rangle, y \in B\}, \quad (x^* \in E^*).
\]

For any \(C\) in \(c_{wk}(E)\), we get

\[ |C| = \sup \{ \|x\|, x \in C\}. \]

A \(cc(E)\)-valued sequence \((Y_n)_{n \geq 1}\) is called Mosco convergent to a closed convex set \(Y_\infty\) if \(Y_\infty = s - \text{lin} Y_n = w - \text{ls} Y_n\), where

\[
s - \text{lin} Y_n = \{ y \in E, \exists y_n \to y \ a.s. \ y_n \in Y_n, \forall n \geq 1 \},
\]

\[
w - \text{ls} Y_n = \{ y \in E, \exists y_k \to y \ weakly, \ y_k \in Y_n, \forall k \geq 1 \}.
\]

If \((Y_n)_{n \geq 1}\) Mosco converges to \(Y_\infty\) in \(cc(E)\), we write \(M - \lim Y_n = Y_\infty\).
A measurable function $g : \Omega \rightarrow E$ is Pettis-integrable, if $g$ is scalarly integrable
(i.e. $(x^*, g)$ is integrable), and, for each $A \in \mathcal{B}$, there exists $x_A$ in $E$, such that
\[
\int_A \langle x^*, g \rangle \, dP = \langle x^*, x_A \rangle, \quad \forall x^* \in E^*.
\] (4)
x_A is called the Pettis-integral of $g$ over $A$. We will denote by $P^1_E(\mathcal{B})$ the space of all measurable and Pettis-integrable $E$-valued function defined on $(\Omega, \mathcal{B}, P)$. We consider the space $P^1_E(\mathcal{B})$ provided with the following topologies.
(i) The topology of the usual Pettis norm is as follows:
\[
\|g\|_P = \sup_{x^* \in \mathcal{B}^*} \int_{\Omega} |\langle x^*, g \rangle| \, dP.
\] (5)
(ii) The topology is induced by the duality $(P^1_E(X), L^\infty \otimes E^*)$. Recall that a sequence $(g_n)_{n \geq 1}$ in $P^1_E(X)$ converges to $g$ in this topology if, for each $h \in L^\infty(\mathcal{B})$ and for all $x^* \in E^*$, one has
\[
\lim_{n \rightarrow +\infty} \int_{\Omega} h(\omega) \langle x^*, g_n(\omega) \rangle \, dP = \int_{\Omega} h(\omega) \langle x^*, g(\omega) \rangle \, dP.
\] (6)
This topology is known as the weak topology and is denoted by $\omega$-$P_P$.

A multifunction $X : \Omega \rightarrow \mathcal{C}(E)$ is said to be measurable, if for every open set $U$ of $E$ the set $X(U) = \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$ is an element of $\mathcal{B}$. The Effros $\sigma$-field $\mathcal{B}$ on $\mathcal{C}(E)$ is generated by the subsets $U^* = \{F \in \mathcal{C}(E) / F \cap U \neq \emptyset\}$, so $X$ is measurable if, for any $B \in \mathcal{B}$, one has $X^{-1}(B) \in \mathcal{B}$.

Two measurable multifunctions $X$ and $Y$ are said to be equal scalarly almost surely, if the following equality holds:
\[
\delta^* (x^*, X(.)) = \delta^* (x^*, Y(.)) \quad \text{a.s.} \quad \forall x^* \in E^*.
\] (7)
For any convex and weakly compact measurable multifunction $X$, the measurability of $X$ is equivalent to that of its support functional $\delta^* (x^*, X)$. (See [13].)

The tribe trace of $\mathcal{B}$ on $cwk(E)$ is defined by the set $\{cwk(E) \cap B/B \in \mathcal{B}\}$ and is denoted by $\Xi$. A measurable function $f : \Omega \rightarrow E$ is said to be a selection of $X$, if, for any $\omega \in \Omega$, $f(\omega) \in X(\omega)$. We denote by $S^1_E(\mathcal{B})$ the set of all measurable selection of $X$. It is known that a convex and closed valued multifunction $X$ in separable Banach space is measurable if dom$(X) \in \mathcal{B}$ and $X$ has a Castaing representation (i.e., there exists a sequence $(f_n)_{n \geq 1}$ of measurable selections of $X$ such that for all $\omega \in \Omega, X(\omega) = cl(f_n(\omega), n \geq 1)$).

The distribution $P_t$ of the measurable multifunction $\Gamma : \Omega \rightarrow cc(E)$ on the measurable space $(cc(E), \mathcal{B})$ is defined by
\[
P_t(B) = P\left(\Gamma^{-1}(B)\right), \quad \forall B \in \mathcal{B}.
\] (8)
Two measurable multifunctions $\Gamma$ and $\Delta$ are said to be independent, if the following equality holds:
\[
P_{(\Gamma, \Delta)} = P_\Gamma \otimes P_\Delta.
\] (9)
The measurable multifunction $X : \Omega \rightarrow cwk(E)$ is scalarly integrable, if, for any $x^* \in E^*$, the real function $\delta^* (x^*, X(.))$ is integrable.

We say that the measurable multifunction $X$ is Pettis-integrable, if it is scalarly integrable and, for each $A \in \mathcal{B}$, there exists $K_A \in cwk(E)$ such that
\[
\int_A \delta^* (x^*, X) \, dP = \delta^* (x^*, K_A), \quad \forall x^* \in E^*.
\] (10)
$K_A$ is called the Pettis-integral of $X$ over $A$. Let $S^1_E(\mathcal{B})$ be the set of all $\mathcal{B}$-measurable and Pettis-integrable selections of $X$.

The multivalued Pettis-integral of a $cwk(E)$-valued multifunction $X$ is defined by
\[
\int_X dP = \left\{ \int f dP, f \in S^1_E(\mathcal{B}) \right\},
\] (11)
and $\int_A X dP$ is convex and $\sigma(E, E^*)$ compact and, for each $A \in \mathcal{B}$, we have
\[
\int_A \delta^* (x^*, X) \, dP = \delta^* \left(\int_A X dP\right), \quad \forall x^* \in E^*.
\] (12)
We will denote by $P^1_{cwk(E)}(\mathcal{B})$ the set of all Pettis-integrable $cwk(E)$-valued multifunctions.

Given a sub $\sigma$-algebra $\mathcal{B}$ and a $cwk(E)$-valued Pettis-integrable multifunction, the Pettis conditional expectation of $X$ with respect to $\mathcal{B}$ is a $\mathcal{B}$-measurable $cwk(E)$-valued Pettis-integrable multifunction denoted by $E^\mathcal{B} X$ which satisfies
\[
\int_A E^\mathcal{B} X dP = \int_A X dP, \quad \forall A \in \mathcal{B}.
\] (13)
The two following propositions (see [7]) give a sufficient condition of the existence of the conditional expectation for a Pettis integrable $E$-valued function and for a $cwk(E)$-valued Pettis-integrable multifunction.

**Proposition 1.** Assume that $E$ is separable. Let $\mathcal{B}$ be a sub $\sigma$-algebra of $\mathcal{B}$ and let $X$ be a Pettis integrable $E$-valued function such that $E^\mathcal{B}[X] \in [0, +\infty]$. Then, there exists a unique $\mathcal{B}$-measurable Pettis-integrable $E$-valued function denote by $E^\mathcal{B} X$, which enjoys the following property; for every $h \in L^\infty(\mathcal{B})$, one has
\[
\int h E^\mathcal{B} X dP = \int h X dP.
\] (14)

**Proposition 2.** Assume that $E^*$ is separable. Let $\mathcal{B}$ be a sub-$\sigma$-algebra of $\mathcal{B}$ and let $X$ be a $cwk(E)$-valued Pettis integrable multifunction such that $E^\mathcal{B}[X] \in [0, +\infty]$. Then, there exists a unique $\mathcal{B}$-measurable $cwk(E)$-valued Pettis-integrable multifunction, which enjoys the following property; for every $h \in L^\infty(\mathcal{B})$, one has
\[
\int h E^\mathcal{B} X dP = \int h X dP.
\] (15)

Akhlat et al. [14] extended the previous theorem in a $cc(E)$-valued Pettis-integrable multifunction.

We close this section by the following useful corollaries.
Corollary 3 (see [15]). Let \((\mathcal{B}_n)_{n \geq 1}\) be a decreasing sequence of a sub-\(\sigma\)-algebra of \(\mathfrak{B}\) and let \(f \in L^1_\delta(\mathfrak{B})\); set \(\mathcal{B}_\infty = \bigcap_{n \geq 1} \mathcal{B}_n\). Then,
\[
\lim_{n \to \infty} E^{\mathcal{B}_n} f = E^{\mathcal{B}_\infty} f \quad \text{a.s.} \tag{16}
\]

Corollary 4. Let \(\mathcal{B}\) be a sub-\(\sigma\)-algebra of \(\mathfrak{B}\) and let \(X\) be a \(\text{cwk}(E)\)-valued Pettis-integrable multifunction such that \(E^\mathcal{B}|X| \in [0, +\infty[.\) Then, for all \(f \in S^\text{P}_X(\mathfrak{B})\), the following properties hold:

1. \(E^\mathcal{B} f(\cdot) \in E^\mathcal{B} X(\cdot)\) almost surely.
2. \(\delta^\mathcal{B}(x^*, E^\mathcal{B} X(\cdot)) = E^\mathcal{B} \delta^\mathcal{B}(x^*, X(\cdot))\) almost surely.

Proof. (1) Let \(f \in S^\text{P}_X(\mathfrak{B})\) and \((e_k^*)_{k \geq 1}\) be a dense sequence in \(E^*\) with respect to the Mackey topology \(\tau(E^*, E)\), then
\[
\int_A \langle e_k^* f \rangle dP \leq \int_A \delta^\mathcal{B} (e_k^*, X) dP, \quad \forall A \in \mathcal{B}, \tag{17}
\]
so
\[
\langle e_k^* \int_A f dP \rangle \leq \delta^\mathcal{B} (e_k^*, \int_A X dP),
\]
and hence
\[
\langle e_k^* \int_A E^\mathcal{B} f dP \rangle \leq \delta^\mathcal{B} (e_k^*, \int_A E^\mathcal{B} X) dP,
\]
and hence
\[
\langle e_k^* \int_A E^\mathcal{B} f(\cdot) \rangle \leq \delta^\mathcal{B} (e_k^*, E^\mathcal{B} X(\cdot)) \quad \text{a.s.} \tag{19}
\]
We conclude that
\[
E^\mathcal{B} f(\cdot) \in E^\mathcal{B} X(\cdot) \quad \text{a.s.} \tag{20}
\]
(2) For any \(A \in \mathcal{B}\), we have \(\delta^\mathcal{B}(x^*, \int_A X dP) = \int_A \delta^\mathcal{B}(x^*, X) dP\), then, by Proposition 2, we have
\[
\delta^\mathcal{B}(x^*, \int_A E^\mathcal{B} X dP) = \int_A \delta^\mathcal{B}(x^*, X) dP.
\]
Since \(E^\mathcal{B} X\) is a Pettis-integrable multifunction,
\[
\int_A \delta^\mathcal{B}(x^*, E^\mathcal{B} X) dP = \int_A \delta^\mathcal{B}(x^*, X) dP = \int_A E^\mathcal{B} \delta^\mathcal{B}(x^*, X) dP.
\]
And hence by uniqueness of the conditional expectation of \(\delta^\mathcal{B}(x^*, X)\), we obtain
\[
\delta^\mathcal{B}(x^*, E^\mathcal{B} X(\cdot)) = E^\mathcal{B} \delta^\mathcal{B}(x^*, X(\cdot)) \quad \text{a.s.} \tag{22}
\]

3. Strong Law of Large Numbers for Pettis-Integrable Multifunctions

Our first result is the following theorem.

Theorem 5. Let \((\mathcal{B}_n)_{n \geq 1}\) be a decreasing sequence of sub-\(\sigma\)-algebra of \(\mathfrak{B}\) and set \(\mathcal{B}_\infty = \bigcap_{n \geq 1} \mathcal{B}_n\). Let \(E\) be a separable Banach space and \(f \in P^1_E(\mathfrak{B})\) such that \(E^{\mathcal{B}_\infty} f(\cdot) \in [0, +\infty[.\) Then
\[
\lim_{n \to \infty} E^{\mathcal{B}_n} f(\cdot) = E^{\mathcal{B}_\infty} f(\cdot) \quad \text{a.s.} \tag{23}
\]

Proof. Since \(E^{\mathcal{B}_\infty} f(\cdot) \in [0, +\infty[\), so, by Proposition 1, we have that \(E^{\mathcal{B}_n} f(\cdot)\) exists and is in \(P^1_E(\mathcal{B}_n)\) and provides a \(\mathcal{B}_\infty\)-measurable partition \((\mathcal{B}_k)_{k \geq 1}\) of \(\Omega\) such that \(f_k = f 1_{\mathcal{B}_k} \in L^1_E(\mathfrak{B})\) (see [14]). Using Corollary 3, we obtain
\[
\lim_{n \to \infty} E^{\mathcal{B}_n} f(\cdot) = E^{\mathcal{B}_\infty} f(\cdot) \quad \text{a.s.} \tag{24}
\]
As \(\mathcal{B}_k \subset \mathcal{B}_\infty \subset \mathcal{B}_n\), for every \(n \geq 1\) and for every \(k \geq 1\), then
\[
E^{\mathcal{B}_n} f_k = E^{\mathcal{B}_\infty} f_1 \mathcal{B}_k f = 1_{\mathcal{B}_k} E^{\mathcal{B}_n} f.
\]
On the other hand,
\[
E^{\mathcal{B}_\infty} f = \sum_{k=1}^{+\infty} E^{\mathcal{B}_n} f_k = \sum_{k=1}^{+\infty} \lim_{n \to \infty} E^{\mathcal{B}_n} f_k
\]
\[
= \sum_{k=1}^{+\infty} 1_{\mathcal{B}_k} \lim_{n \to \infty} E^{\mathcal{B}_n} f = \lim_{n \to \infty} E^{\mathcal{B}_n} f.
\]
Therefore,
\[
\lim_{n \to \infty} E^{\mathcal{B}_n} f(\cdot) = E^{\mathcal{B}_\infty} f(\cdot) \quad \text{a.s.} \tag{27}
\]

Let us prove the following results which will be used after.

Proposition 6. Assume that \(E^*\) is separable. Let \(X\) and \(Y\) be two \(\text{cwk}(E)\)-valued Pettis-integrable multifunctions and \(\mathfrak{B}\) be a sub-\(\sigma\)-algebra of \(\mathfrak{B}\) such that \(E^\mathfrak{B}|X| \in [0, +\infty[\) and \(E^\mathfrak{B}|Y| \in [0, +\infty[\). Then
\[
E^\mathfrak{B} (X + Y) = E^\mathfrak{B} X + E^\mathfrak{B} Y \quad \text{a.s.} \tag{28}
\]

Proof. Let \((e_k^*)_{k \geq 1}\) be a dense sequence in \(E^*\) with respect to the Mackey topology \(\tau(E^*, E)\). Since \(X\) and \(Y\) are \(\text{cwk}(E)\)-valued Pettis-integrable multifunctions, then, for any \(A\) in \(\mathfrak{B}\), there exist two sets \(K_A(X)\) and \(K_A(Y)\) in \(\text{cwk}(E)\), such that for all \(k \geq 1\),
\[
\delta^\mathfrak{B}(e_k^*, A) = \int_A \delta^\mathfrak{B}(e_k^*, X) dP \quad \text{and} \quad \delta^\mathfrak{B}(e_k^*, A) = \int_A \delta^\mathfrak{B}(e_k^*, Y) dP.
\]
By Theorem 5.1.6 in [8], \(X + Y\) is \(\text{cwk}(E)\)-valued Pettis-integrable multifunction, then, for any \(A\) in \(\mathfrak{B}\), there exists a set \(K_A(X + Y)\) \(\text{cwk}(E)\) such that
\[
\delta^\mathfrak{B}(e_k^*, A) = \int_A \delta^\mathfrak{B}(e_k^*, X + Y) dP. \tag{29}
\]
On the other hand,
\[
\delta^\mathfrak{B}(e_k^*, A) = \delta^\mathfrak{B}(e_k^*, K_A(X) + K_A(Y)) = \delta^\mathfrak{B}(e_k^*, K_A(X)) + \delta^\mathfrak{B}(e_k^*, K_A(Y)) =
\]

\[
\delta^\mathfrak{B}(e_k^*, A).
\]
\[ = \int_{A} \delta^{*}(e_{k}^{*}, X) \, dP + \int_{A} \delta^{*}(e_{k}^{*}, Y) \, dP \]
\[ = \int_{A} \delta^{*}(e_{k}^{*}, X) + \delta^{*}(e_{k}^{*}, Y) \, dP = \int_{A} \delta^{*}(e_{k}^{*}, X + Y) \, dP = \delta^{*}(e_{k}^{*}, K_{A}(X + Y)). \] 

(30)

Then \( K_{A}(X + Y) = K_{A}(X) + K_{A}(Y) \) a.s.

Therefore,
\[ \int_{A} (X + Y) \, dP = \int_{A} X dP + \int_{A} Y dP \text{ a.s.} \] 

(31)

On the other hand,
\[ \int_{A} \delta^{*}(e_{k}^{*}, E_{\Sigma}(X + Y)) \, dP \]
\[ = \delta^{*}(e_{k}^{*}, \int_{A} E_{\Sigma}(X + Y) \, dP) \]
\[ = \delta^{*}(e_{k}^{*}, \int_{A} X + YdP) \]

By (31),
\[ \delta^{*}(e_{k}^{*}, \int_{A} X + YdP) = \delta^{*}(e_{k}^{*}, \int_{A} XdP + \int_{A} YdP) \]
\[ = \delta^{*}(e_{k}^{*}, \int_{A} E_{\Sigma}XdP + \int_{A} E_{\Sigma}YdP) = \]
\[ = \int_{A} \delta^{*}(e_{k}^{*}, E_{\Sigma}X + E_{\Sigma}Y) \, dP, \]

then
\[ \int_{A} \delta^{*}(e_{k}^{*}, E_{\Sigma}(X + Y)) \, dP \]
\[ = \int_{A} \delta^{*}(e_{k}^{*}, E_{\Sigma}X + E_{\Sigma}Y) \, dP, \] 

(34)

and therefore
\[ E_{\Sigma}(X + Y) = E_{\Sigma}X + E_{\Sigma}Y \text{ a.s.} \] 

(35)

\[ \square \]

**Theorem 7.** Let \( X \) and \( Y \) be in \( P^{1}_{\text{cwk}(E)}(\Sigma) \); let \( Z \) be a random variable with values in a measurable space \((\Sigma, \Gamma)\) such that \((Z, X)\) and \((Z, Y)\) have the same distribution. If \( E^{(Z)}|X| \in [0, +\infty]\) and \( E^{(Z)}|Y| \in [0, +\infty]\). Then
\[ E^{(Z)}(X) = E^{(Z)}(X) \text{ a.s.} \] 

(36)

**Proof.** Since \( X \) is Pettis-integrable with values in \( \text{cwk}(E) \), \( S_{\infty}^{c}(\sigma(X)) \neq \emptyset \). So by [11], there exists \( \{f_{n}\}_{n \geq 1} \) selection Pettis-integrable of \( X, \sigma(X) \)-measurable such that
\[ X(\omega) = \text{cl} \{f_{n}(\omega), \ n \geq 1 \}. \] 

(37)

Let \( f : \Omega \rightarrow E \) be a fixed element in \( S_{\infty}^{c}(\sigma(X)) \), \( h \in L^{\infty}(\sigma(Z)) \), and \( x^{*} \in E^{*} \). Then, by Doob’s factorisation lemma, we can find a \( \xi \)-measurable mapping \( u \) from \( \text{cwk}(E) \) into \( E \) (i.e., \( (\text{cwk}(E), \xi) \stackrel{\Lambda}{\rightarrow} (E, \mathcal{B}(E)) \)) and a \( \Gamma \)-measurable function \( v \) from \( \Sigma \) into \( \mathbb{R} \) (i.e., \((\Sigma, \Gamma) \stackrel{\nu}{\rightarrow} (\mathbb{R}, \mathcal{B}(\mathbb{R}))\)), which satisfy
\[ f : (\Omega, \mathcal{F}, P) \stackrel{X}{\rightarrow} (\text{cwk}(E), \xi) \stackrel{u}{\rightarrow} (E, \mathcal{B}(E)), \]
\[ h : (\Omega, \mathcal{F}, P) \stackrel{Z}{\rightarrow} (\Sigma, \Gamma) \stackrel{\nu}{\rightarrow} (\mathbb{R}, \mathcal{B}(\mathbb{R})), \]
\[ f(\omega) = (u \circ X)(\omega). \] 

(38)

On the other hand,
\[ \delta^{*}(h \circ x^{*}, E_{\Sigma}(\sigma(X))) = \int_{\Omega} \delta^{*}(h \circ x^{*}, X) \, dP \]
\[ = \sup \left\{ \int_{\Omega} h(\omega)(x^{*}, f) \, dP \mid f \in E_{\Sigma}(\sigma(X)) \right\}. \]

(39)

And
\[ \int_{\Omega} h(\omega)(x^{*}, f) \, dP \]
\[ = \int_{\Omega} (v \circ Z)(\omega)(x^{*}, u \circ X)(\omega) \, dP. \] 

(40)

By using the application
\[ (\Omega, \mathcal{F}, P) \stackrel{g}{\rightarrow} \]
\[ (\Sigma \times \text{cwk}(E), \Gamma \circ \xi) \stackrel{g}{\rightarrow} \]
\[ (\mathbb{R}, \mathcal{B}(\mathbb{R})) \omega \rightarrow \]
\[ (Z, X) \rightarrow \]
\[ v \circ Z(\omega)(x^{*}, u \circ X(\omega)) = h(\omega)(x^{*}, f(\omega)). \] 

(41)

And by the classical transfer theorem, we have
\[ \int_{\Omega} \phi \circ g(\omega) \, dP = \int_{\Omega} h(\omega)(x^{*}, f(\omega)) \, dP \]
\[ = \int_{\Omega} v \circ Z(\omega)(x^{*}, u \circ X(\omega)) \, dP = \]
\[ = \int_{\Sigma \times \text{cwk}(E)} \phi(z, B) \, dP_{(Z,X)}(z, B) = \]
\[ = \int_{\Sigma \times \text{cwk}(E)} v(z)(x^{*}, u(B)) \, dP_{(Z,X)}(z, B) = \]
\[ = \int_{\Sigma \times \text{cwk}(E)} v(z)(x^{*}, u(B)) \, dP_{(Z,Y)}(z, B). \]
Then
\[ \int \phi(z, B) dP(z, B) = \int_{\mathcal{F}(E)} \int_{\mathcal{F}(E)} \phi(z, B) dP(Z, Y) (z, B) = \int \Omega \int \phi(z, B) dP(z, B). \]

In particular, if \( \phi(z, B) = Z(\omega) \langle x^*, u \circ Y(\omega) \rangle dP \), then
\[ \int h(\omega) \langle x^*, u \circ Y(\omega) \rangle dP. \]

By combining relation (39), (43) and the fact that \( \forall B \in cwk(E), u(B) \in B, P_Y \ a.s., \) we obtain
\[ \int \delta^*(h \otimes x^*, X) dP = \sup \left\{ \int h(\omega) \langle x^*, f(\omega) \rangle dP, f \in S^p_Y(\sigma(Y)) \right\} \]
and therefore
\[ \int \delta^*(h \otimes x^*, X) dP = \int \delta^*(h \otimes x^*, Y) = \int \delta^*(h \otimes x^*, \sigma(Y)). \]

In particular, if \( h = 1_A, \) for any \( A \in \sigma(Z), \) we have
\[ \int 1_A \delta^*(x^*, X) dP = \int 1_A \delta^*(x^*, X) dP, \]
then
\[ \delta^*(x^*, \int A X dP) = \delta^*(x^*, \int A Y dP). \]

Now, we give the main result of this work.

**Theorem 9.** Assume that \( E^* \) is separable. Let \( (X_n)_{n \geq 1} \) be a sequence of independent measurable multifunctions in \( F_{cwk(E)}(\mathcal{F}). \)

Let \( S_n = X_1 + X_2 + X_3 + \cdots + X_n, \) \( \mathcal{B}_n = \sigma(S_n, X_{n+1}, X_{n+2}, \ldots), \) \( \mathcal{B}_\infty = \bigcap_{n \geq 1} \mathcal{B}_n \) and assume that
(i) \( \forall n, \forall j \in \{1, 2, \ldots, n\}, (S_n, X_1) \) and \( (S_n, X_j) \) have the same distribution.
(ii) \( \forall n, \forall j \in \{1, 2, \ldots, n\}, E^{\mathcal{B}_n}[X_j] \in [0, +\infty]. \)

Then, we have the following assertions:
(1) \( \forall n, \forall j \in \{1, 2, \ldots, n\}, E^{\mathcal{B}_n}[X_j] = E^{\mathcal{B}_n}[X_1] \ a.s. \)
(2) \( E^{\mathcal{B}_n}[X_1] = E^{\mathcal{B}_n}[X_1] \ a.s. \)
(3) \( M - \lim_{n \to \infty} S_n/n = M - \lim_{n \to \infty} E^{\mathcal{B}_n}[X_1] = E^{\mathcal{B}_\infty}[X_1] \ a.s. \)

**Proof.** The first equality follows from the Theorem 7.

Now let us prove the second equality. Let \( (e_k^*)_{k \geq 1} \) be a dense sequence in \( E^* \) for the Mackey topology. Set \( \mathcal{B}_1 = \sigma(S_1), \) \( \mathcal{B}_2 = \sigma(S_2, S_3, \ldots, S_n) \) and \( f_1 = \delta^*(e_k^*, X_1). \)

Since \( X_1 \) and \( S_n \) are independent of \( \mathcal{B}_1, f_1 \) and \( S_n \) are independent of \( \mathcal{B}_2, \) then, by applying Theorem 8, we have
\[ E^{\mathcal{B}_1}[f_1] = E^{\mathcal{B}_2}[f_1] \ a.s. \]

So, \( E^{\mathcal{B}_1}[e_k^*, X_1] = E^{\mathcal{B}_2}[e_k^*, X_1], \) then, by Corollary 4, we obtain
\[ \delta^*(e_k^*, E^{\mathcal{B}_1}X_1) = \delta^*(e_k^*, E^{\mathcal{B}_2}X_1), \]
and since \( E^{\mathcal{B}_1}X_1(.) \) and \( E^{\mathcal{B}_2}X_1(.) \) are convex and weakly compact, then
\[ E^{\mathcal{B}_1}X_1(.) = E^{\mathcal{B}_2}X_1(.) \ a.s. \]

(i) Now, we show the last assertion.

**Step 1.** We claim that \( S_n/n = E^{\mathcal{B}_1}X_1 \ a.s. \)

We have \( S_n = \sum_{i=1}^n X_i = E^{\mathcal{B}_1}(\sum_{i=1}^n X_i), \) so, by Proposition 6 and the first and the second assertions of the theorem, we obtain
\[ E^{\mathcal{B}_1}(\sum_{i=1}^n X_i) = \sum_{i=1}^n E^{\mathcal{B}_1}(X_i) = \sum_{i=1}^n E^{\mathcal{B}_2}(X_i) \]
and therefore
\[ E^{\mathcal{B}_1}X_1(.) = nE^{\mathcal{B}_2}X_1(.) \ a.s. \]

Then
\[ \frac{S_n}{n} = E^{\mathcal{B}_1}X_1(.) \ a.s. \]

**Step 2.** We show that \( M - \lim_{n \to \infty} E^{\mathcal{B}_1}X_1 \ a.s. \)

(ii) We begin by proving that \( E^{\mathcal{B}_1}X_1 \subset \mathcal{B}_\infty \)

Since \( X_1 \) is Pettis-integrable, then there exists \( (g_k)_{k \geq 1} \) Pettis-integrable selection of \( X_1 \) such that
\[ X_1 = \sigma(g_k, k \geq 1) \].
On the other hand \( E^{\mathcal{B}}_{\omega}[X_1] \in [0, +\infty) \), then \( E^{\mathcal{B}}_\omega X_1 \) exists and is in \( P_{\omega}(\mathcal{B}_n) \). Let \( g \in S^E_{\omega}\mathcal{B}_n \); by Corollary 4, \( E^{\mathcal{B}}_\omega g \) exists and \( E^{\mathcal{B}_n} g(\omega) \in E^{\mathcal{B}}_\omega X_1(\omega) \) a.s.

Using Theorem 5, we have \( \lim_{n \to \infty} E^{\mathcal{B}}_\omega g = E^{\mathcal{B}}_\omega g \) a.s. Since by [17]

\[
s^E_{\omega} = \{ E^{\mathcal{B}}_\omega f : f \in S^E \}
\]

Then \( E^{\mathcal{B}}_\omega g \in S^E_{\omega}\mathcal{B}_n \) and by (56), \( S^E_{\omega}\mathcal{B}_n \subset S^E_{\omega}\mathcal{B}_n \) a.s.

Then

\[
E^{\mathcal{B}}_\omega X_1(\omega) \subset s - l E^{\mathcal{B}}_\omega X_1(\omega) \ a.s. (58)
\]

(iii) Now, we show that \( w - l E^{\mathcal{B}}_\omega X_1(\omega) \subset E^{\mathcal{B}}_\omega X_1(\omega) \) a.s.

Let \( (e_k^*)_{k \geq 1} \) be a dense sequence in \( E^* \) for the Mackey topology; we have

\[
\delta^* (e_k^*, E^{\mathcal{B}}_\omega X_1) = E^{\mathcal{B}}\delta^* (e_k^*, X_1),
\]

then, by Corollary 3,

\[
\lim_{n \to \infty} E^{\mathcal{B}}\delta^* (e_k^*, X_1) = E^{\mathcal{B}}\delta^* (e_k^*, X_1) \ a.s. (60)
\]

Hence, there exists a negligible set \( N = \bigcup_{k \geq 1} N_k \); for all \( k \geq 1 \) and for every \( \omega \in \Omega \setminus N \), we have

\[
\lim_{n \to \infty} E^{\mathcal{B}}\delta^* (e_k^*, X_1(\omega)) = E^{\mathcal{B}}\delta^* (e_k^*, X_1(\omega)) = \delta^* (e_k^*, E^{\mathcal{B}}_\omega X_1(\omega)) a.s. (61)
\]

So, for all \( x^* \in E^* \) and for all \( \omega \in \Omega \setminus N \),

\[
\lim_{n \to \infty} \delta^* (x^*, E^{\mathcal{B}}_\omega X_1(\omega)) = \delta^* (x^*, E^{\mathcal{B}}_\omega X_1(\omega)) a.s. (62)
\]

Let \( \omega \in \Omega \setminus N \) and \( x \in w - l E^{\mathcal{B}}_\omega X_1(\omega) \), then there exists \( (x_m)_{m \geq 1} \) in \( E^{\mathcal{B}}_\omega X_1(\omega) \) such that \( x_m \xrightarrow{\omega} x \), which implies

\[
\lim_{m \to \infty} (x^*, x_m) = (x^*, x).
\]

Then

\[
\langle x^*, x \rangle = \lim_{m \to \infty} \langle x^*, x_m \rangle \leq \limsup_{m \to \infty} \delta^* (x^*, E^{\mathcal{B}}_\omega X_1) = \limsup_{m \to \infty} E^{\mathcal{B}}\delta^* (x^*, X_1) = \delta^* (x^*, E^{\mathcal{B}}_\omega X_1(\omega)) a.s. (63)
\]

Consequently, \( x \in E^{\mathcal{B}}_\omega X_1(\omega) \), then

\[
w - l s E^{\mathcal{B}}_\omega X_1(\omega) \subset E^{\mathcal{B}}_\omega X_1(\omega) \ a.s. (64)
\]

This yields

\[
M - \lim_{n \to \infty} \frac{S_n}{n} = M - \lim_{n \to \infty} E^{\mathcal{B}}_\omega X_1(\omega) = E^{\mathcal{B}}_\omega X_1(\omega) a.s. (65)
\]

\[
\square
\]

**Corollary 10.** Under the same hypothesis of Theorem 9, we have

\[
M - \lim_{n \to \infty} \frac{S_n}{n} = E^{\mathcal{B}}_\omega X_1(\omega) = \int X_1(\omega) dP(\omega) a.s. (66)
\]

**Proof.** By the previous theorem, we need only to check that

\[
E^{\mathcal{B}}_\omega X_1(\omega) = \int_{\Omega} X_1(\omega) dP(\omega) a.s. (67)
\]

Since \( \int_{\Omega} X_1(\omega) dP(\omega) \) is convex and weakly compact. Now let \( (e_k^*)_{k \geq 1} \) be a dense sequence in \( E^* \) for the Mackey topology; we have

\[
\lim_{n \to \infty} \delta^* (e_k^*, E^{\mathcal{B}}_\omega X_1) = \lim_{n \to \infty} E^{\mathcal{B}}\delta^* (e_k^*, X_1) = E^{\mathcal{B}}\delta^* (e_k^*, X_1) = \delta^* (e_k^*, E^{\mathcal{B}}_\omega X_1) a.s. (68)
\]

On the other hand, \( (E^{\mathcal{B}}\delta^* (e_k^*, X_1)) \) is a Pettis reversed martingale, so, for each positive integer \( m \), set \( \mathcal{F}_m = \sigma(X_m, X_{m+1}, X_{m+2}, \ldots) \) and \( \mathcal{F}_\infty = \bigcap_{m=1}^{\infty} \mathcal{F}_m \). Hence, for all any \( n \geq m \), the multifunction \( \sum_{j=m}^{n} X_j \) is \( \mathcal{F}_m \)-measurable. Moreover, by the previous theorem, we have

\[
\frac{1}{n} \sum_{j=1}^{m-1} \delta^* (e_k^*, X_j) + \frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j) = \frac{1}{n} \delta^* (e_k^*, \sum_{j=1}^{m-1} X_j) + \frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j)
\]

(69)

\[
\frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j) + \frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j) = \frac{m-1}{n} E^{\mathcal{B}}\delta^* (e_k^*, X_1) + \frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j) a.s.
\]

Then

\[
\frac{1}{n} \sum_{j=1}^{m-1} \delta^* (e_k^*, X_j) + \frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j) = \frac{m-1}{n} E^{\mathcal{B}}\delta^* (e_k^*, X_1) + \frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j) a.s.
\]

(70)

Since, for every fixed positive integer \( m \), we have \( \lim_{n \to \infty} ((m-1)/n) E^{\mathcal{B}}\delta^* (e_k^*, X_1) = 0 a.s. \) Then by (70)

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{j=m}^{m-1} \delta^* (e_k^*, X_j) = \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{m-1} \delta^* (e_k^*, X_j)
\]

(71)

Since, for all fixed integer \( m \), the multifunction \( (1/n) \sum_{j=m}^{n} X_j \) is \( \mathcal{F}_m \)-measurable and by (71) \( E^{\mathcal{B}}_\omega X_1 \)
is $\mathcal{F}_m$-measurable and so is $\mathcal{F}_\infty$-measurable. Then by the independence of $(\sigma(X_n))_{n\in\mathbb{N}}$ and the Kolmogorov’s Zero-One law (see [18]), we conclude that for all $k \in \mathbb{N}^*$

\[
\delta^* \left(e_k^*, E^{\mathcal{B}_\infty} X_1(.)\right) = \int_\Omega \delta^* \left(e_k^*, X_1(\omega)\right) dP(\omega)
\]

(72)

\[
= \delta^* \left(e_k^*, \int_\Omega X_1(\omega) dP(\omega)\right) \ a.s.
\]

Since $E^{\mathcal{B}_\infty} X_1$ and $\int_\Omega X_1(.) dP$ are $cwk(E)$-valued multifunctions and (72) is true for all $k \geq 1$, then $E^{\mathcal{B}_\infty} X_1(.) = \int_\Omega X_1(\omega) dP(\omega) \ a.s.$

Therefore,

\[
M - \lim_{n \to \infty} \frac{S_n}{n} = E^{\mathcal{B}_\infty} X_1(.) = \int_\Omega X_1(\omega) dP(\omega) \ a.s. \quad (73)
\]

\[\square\]

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References


