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Research Article

Nonhomogeneous Wavelet Dual Frames and Extension Principles in Reducing Subspaces

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It can be seen from the literature that nonhomogeneous wavelet frames are much simpler to characterize and construct than homogeneous ones. In this work, we address such problems in reducing subspaces of $L^2(\mathbb{R}^d)$. A characterization of nonhomogeneous wavelet dual frames is obtained, and by using the characterization, an MOEP and an MEP are derived under general assumptions for such wavelet dual frames.

1. Introduction

In view of the great design freedom and the potential applications in signal processing and many other fields, wavelet frames have been extensively investigated by many researchers (see [1–9] for details). In particular, the homogeneous wavelet dual frames (or called affine dual frames) in $L^2(\mathbb{R}^d)$ were original characterized by Han [10] and then characterized by Bownik [11], some of their variations can be found in [12–14]. Mixed extension principles (MEP) give us an important method to construct homogeneous wavelet dual frames from refinable functions, which were proposed by Ron and Shen [15, 16]. When the systems generated by the dilation and integer translation of two functions with refinable structure cannot form frames for their closed linear span, respectively, except two Bessel sequences, construction of homogeneous wavelet dual frames cannot be carried out like the multiresolution analysis (MRA) of Mallat. In such case, the mixed extension principles give an ideal answer to the problem and provide an effective design strategy for stable wavelet filters. Subsequently, mixed extension principles were developed by Daubechies et al. [17] in the form of mixed oblique extension principles (MOEP), which presented a more general method for constructing wavelet dual frames. From then on, the study of MEP and MOEP has interested many researchers [12, 13, 18–25]. Nonhomogeneous wavelet frames have natural connections with refinable structures and filter banks; for this type of wavelet frames, Han in [26–28] and Romero et al. in [29] extensively studied them both in theory and application. Also, notice that all of the above works are focused on the whole space $L^2(\mathbb{R}^d)$ or $L^2(\mathbb{R})$. In this work, for generality, we address nonhomogeneous wavelet dual frames and extension principles in reducing subspaces of $L^2(\mathbb{R}^d)$. A characterization of nonhomogeneous wavelet dual frames is obtained, and by using the characterization, an MOEP and an MEP are derived under general assumptions for such wavelet dual frames.

We begin with some notations and notions. We use $\mathbb{Z}$ to represent the set of integers, $\mathbb{N}$ to represent the set of positive integers, and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $d \in \mathbb{N}$. We use $\mathbb{Z}^d = [0,1)^d$ to represent a d-dimensional unit torus, and, given a set E on $\mathbb{R}^d$, $|E|$ to represent the Lebesgue measure of E, and χ_E to represent the characteristic function on E. A mapping $\tau\colon \mathbb{R}^d \mapsto \mathbb{T}^d$ is defined by

$$\tau(y) = y - k \text{ for } y \in \mathbb{T}^d + k \text{ and } k \in \mathbb{Z}^d. \tag{1}$$

For a function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, its Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\cdot) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle x, \cdot \rangle} \mathrm{d}x, \tag{2}$$

and is naturally extended to $L^2(\mathbb{R}^d)$, where $\langle \cdot, \cdot \rangle$ is the usual inner product on $\mathbb{R}^d$. Similarly, for a function $f \in L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$, its inverse Fourier transform $\check{f}$ is defined by

$$\check{f}(\cdot) = \int_{\mathbb{R}^d} f(x) e^{2\pi i \langle x, \cdot \rangle} \mathrm{d}x, \tag{3}$$

and is naturally extended to $L^2(\mathbb{R}^d)$. The spectrum of a function f is defined to be

$$\sigma(f) = \left\{ \xi \in \mathbb{T}^d \colon \sum_{k \in \mathbb{Z}^d} |\widehat{f}(\xi + k)|^2 > 0 \right\}, \tag{4}$$

for $f \in L^2(\mathbb{R}^d)$, and write δ as the Dirac sequences such that $\delta_{0,0} = 1$ and $\delta_{0,\beta} = 0$ for $0 \neq \beta \in \mathbb{R}^d$. We use A^* to represent the conjugate transpose for a matrix A and Γ_{A^*} to represent the full set of $A^{*-1}\mathbb{Z}^d/\mathbb{Z}^d$ containing the 0, i.e., a set of representatives distinct cosets of $A^{*-1}\mathbb{Z}^d/\mathbb{Z}^d$ containing the 0. A $d \times d$ matrix A is said to be expansive if the modulus of all its eigenvalues are greater than 1. In many literature studies of wavelet frame theory, A is required to be an expansive integer matrix include $A = 2I_d$ (I_d is the identity matrix of order d). The shift operator T_k with $k \in \mathbb{Z}^d$ and the dilation operator D related to A on $L^2(\mathbb{R}^d)$ are, respectively, defined by

$$T_k f(\cdot) = f(\cdot - k) \text{ and } Df(\cdot) = |\det A|^{1/2} f(A\cdot) \text{ for } f \in L^2\left(\mathbb{R}^d\right). \tag{5}$$

For a finite set $F \subset L^2(\mathbb{R}^d)$ and $h \in L^2(\mathbb{R}^d)$, the homogeneous wavelet system $X(F)$ and the nonhomogeneous wavelet system $X(h, F)$ are, respectively, defined to be

$$X(F) = \left\{ f_{j,k} = D^j T_k f \colon j \in \mathbb{Z}, k \in \mathbb{Z}^d, f \in F \right\}, \tag{6}$$

$$\begin{aligned} X(h, F) = &\left\{ h_{0,k} = T_k h \colon k \in \mathbb{Z}^d \right\} \cup \\ &\left\{ f_{j,k} = D^j T_k f \colon j \in \mathbb{N}_0, k \in \mathbb{Z}^d, f \in F \right\}. \end{aligned} \tag{7}$$

Let M be a closed subspace of $L^2(\mathbb{R}^d)$. Let $\phi, \widetilde{\phi} \in M$, and two finite subsets $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_1, \widetilde{\psi}_2, \ldots, \widetilde{\psi}_L\}$ of M. If there are $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \left| \langle f, \psi_{l,j,k} \rangle \right|^2 \leq B\|f\|^2, \quad \forall f \in M, \tag{8}$$

then $X(\Psi)$ is said to be a homogeneous wavelet frame (HWF) for M; herein A and B are called frame bounds. In particular, if $A = B$ in (8), then $X(\Psi)$ is said to be a tight wavelet frames. If only the inequality on the right side of (8) is valid, then $X(\Psi)$ is said to be a Bessel sequence in M, herein B is called a Bessel bound. In addition, if the homogeneous wavelet systems $X(\Psi)$ and $X(\widetilde{\Psi})$ are both Bessel sequences in M and the identity

$$f = \sum_{l=1}^{L} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\psi}_{l,j,k} \rangle \psi_{l,j,k} \text{ for } f \in M, \tag{9}$$

holds in L^2-sense, then $(X(\Psi), X(\widetilde{\Psi}))$ is said to be a pair of homogeneous wavelet dual frames (HWDF) for M. Similarly, if there are $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{k \in \mathbb{Z}^d} \left| \langle f, \phi_{0,k} \rangle \right|^2 + \sum_{l=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \left| \langle f, \psi_{l,j,k} \rangle \right|^2 \leq B\|f\|^2, \quad \forall f \in M, \tag{10}$$

then $X(\phi, \Psi)$ is said to be a nonhomogeneous wavelet frames (NWF) for M, herein A and B are called frame bounds. In particular, if $A = B$ in (10), then $X(\phi, \Psi)$ is said to be a tight wavelet frames. If only the inequality on the right side of (10) is valid, then $X(\phi, \Psi)$ is said to be a Bessel sequence in M, herein B is called a Bessel bound. In addition, if the nonhomogeneous wavelet systems $X(\phi, \Psi)$ and $X(\widetilde{\phi}, \widetilde{\Psi})$ are both Bessel sequences in M and the identity

$$f = \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\phi}_{0,k} \rangle \phi_{0,k} + \sum_{l=1}^{L} \sum_{j=0}^{\infty} \sum_{k \in \mathbb{Z}^d} \langle f, \widetilde{\psi}_{l,j,k} \rangle \psi_{l,j,k} \quad \text{for } f \in M, \tag{11}$$

holds in L^2-sense, then

$$(X(\phi, \Psi), X(\widetilde{\phi}, \widetilde{\Psi})), \tag{12}$$

is said to be a pair of nonhomogeneous wavelet dual frames (NWDF) for M.

Since the nonhomogeneous wavelet frames have many desired properties, which can be studied in theory and application when they can be linked to homogeneous ones. Following this idea, Han [26] and Atreas et al. [18, 19] have discussed the relations between homogeneous wavelet dual frames and nonhomogeneous ones, and their connection with the refinable structures.

Definition 1. Given a nonzero closed subspace X of $L^2(\mathbb{R}^d)$. If $DX = X$ and $T_k X = X$ hold true for all $k \in \mathbb{Z}^d$, then X is said to be a reducing subspace.

A characterization of reducing subspace in Fourier domain was gained by Dai et al. (Theorem 1 in [30]), and it reads as follows.

Proposition 1. *Let A be a $d \times d$ expansive matrix, and $X \neq \{0\}$ a closed subspace of $L^2(\mathbb{R}^d)$. X is a reducing subspace of $L^2(\mathbb{R}^d)$ if and only if $X = FL^2(\Omega)$ for some $\Omega \subset \mathbb{R}^d$ and $|\Omega| \neq 0$ and satisfy $\Omega = A^*\Omega$, herein*

$$FL^2(\Omega) := \left\{ f \in L^2\left(\mathbb{R}^d\right) \colon \operatorname{supp}(\widehat{f}) \subset \Omega \right\}, \tag{13}$$

and $\operatorname{supp}(\widehat{f}) = \left\{ \xi \in \mathbb{R}^d \colon \widehat{f}(\xi) \neq 0 \right\}$.

According to Proposition 1, we often use $FL^2(\Omega)$ to represent a reducing subspace of $L^2(\mathbb{R}^d)$ rather than X. In particular, $FL^2(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and it is a reducing subspace of $L^2(\mathbb{R}^d)$ for a given expansive matrix, and Hardy space $FL^2([0, \infty))$ is a reducing subspace of $L^2(\mathbb{R})$ for a given expansive factor greater than 1. We refer the interested readers to [10, 31–34] for some related works on HWDFs in $FL^2(\Omega)$ for the details.

Definition 2. Let A be a $d \times d$ expansive matrix. A function $\kappa\colon \mathbb{Z}^d \longrightarrow \mathbb{N}_0$ is defined to be

$$\kappa(n) = \sup\left\{j \geq 0\colon A^{*^{-j}} n \in \mathbb{Z}^d\right\}, \tag{14}$$

and set $\kappa(0) = +\infty$.

Li and Zhang (Lemma 2.3 in [31]) have the following characterization for HWDF in $FL^2(\Omega)$.

Proposition 2. *Let $FL^2(\Omega)$ be a reducing subspace of $L^2(\mathbb{R}^d)$, and two finite subsets $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_1, \widetilde{\psi}_2, \ldots, \widetilde{\psi}_L\}$ of $FL^2(\Omega)$. Suppose that $X(\Psi)$ and $X(\widetilde{\Psi})$ are both Bessel sequences in $FL^2(\Omega)$. Then, $(X(\Psi), X(\widetilde{\Psi}))$ is a pair of HWDF in $FL^2(\Omega)$ if and only if, for each $n \in \mathbb{Z}^d$, we have*

$$\sum_{l=1}^{L} \sum_{j=-\infty}^{\kappa(n)} \widehat{\psi}_l\left(A^{*-j}\cdot\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*-j}(\cdot + n)\right)} = \delta_{0,n}\chi_\Omega(\cdot) \text{ a.e. on } \mathbb{R}^d. \tag{15}$$

Proposition 2 was also proved in [32] without any decay assumptions on the elements of Ψ and $\widetilde{\Psi}$. Some variations of Proposition 2 can be found in [35–37].

The rest of this work is arranged as below. In Section 2, we provide some necessary lemmas that will be applied in the following section. In Section 3, we give a characterization of NWDF in $FL^2(\Omega)$ by a pair of equations, and using this characterization we derive an MOEP and an MEP under general assumptions for such wavelet dual frames.

2. Some Auxiliary Lemmas

By Definition 2, it is easy to check the following lemmas.

Lemma 1. $\left\{A^{*-\kappa(n)-1}n\colon 0 \neq n \in \mathbb{Z}^d\right\} = \cup_{\gamma \in \Gamma_{A^*} \setminus \{0\}}(\mathbb{Z}^d + \gamma)$.

Lemma 2. *For $\phi, \widetilde{\phi} \in L^2(\mathbb{R}^d)$, we have*

$$\begin{aligned}&\sigma(\phi) \cap \tau(\sigma(\widetilde{\phi}) - \gamma)\\ &= \left\{\xi \in \mathbb{T}^d\colon \widehat{\phi}(\xi + k)\overline{\widehat{\widetilde{\phi}}(\xi + \gamma + n)} \neq 0 \text{ for some } k, n \in \mathbb{Z}^d\right\},\end{aligned} \tag{16}$$

for $\gamma \in \Gamma_{A^*}$.

Proof. For a.e. $\xi \in \sigma(\phi) \cap \tau(\sigma(\widetilde{\phi}) - \gamma)$ with $\gamma \in \Gamma_{A^*}$, there exist $k_\xi, m_\xi \in \mathbb{Z}^d$ and $\eta \in \sigma(\widetilde{\phi})$ such that

$$\begin{gathered}\widehat{\phi}\left(\xi + k_\xi\right) \neq 0,\\ \xi = \eta - \gamma + m_\xi.\end{gathered} \tag{17}$$

Since $\eta \in \sigma(\widetilde{\phi})$, there exists $k_\eta \in \mathbb{Z}^d$ such that

$$\widehat{\widetilde{\phi}}\left(\eta + k_\eta\right) \neq 0, \tag{18}$$

or equivalently,

$$\widehat{\widetilde{\phi}}\left(\xi + k_\xi + \gamma + n_\xi\right) \neq 0, \tag{19}$$

where $n_\xi = k_\eta - m_\xi - k_\xi$. It follows that $\widehat{\phi}(\xi + k_\xi)\overline{\widehat{\widetilde{\phi}}(\xi + \gamma + k_\xi + n_\xi)} \neq 0$, and thus

$$\begin{aligned}&\sigma(\phi) \cap \tau(\sigma(\widetilde{\phi}) - \gamma)\\ &\subset \left\{\xi \in \mathbb{T}^d\colon \widehat{\phi}(\xi + k)\overline{\widehat{\widetilde{\phi}}(\xi + \gamma + n)} \neq 0 \text{ for some } k, n \in \mathbb{Z}^d\right\}.\end{aligned} \tag{20}$$

Now, let us prove the converse inclusion. Suppose $\xi \in \mathbb{T}^d$ satisfies that

$$\widehat{\phi}(\xi + k)\overline{\widehat{\widetilde{\phi}}(\xi + \gamma + n)} \neq 0, \tag{21}$$

for some $\gamma \in \Gamma_{A^*}$ and $k, n \in \mathbb{Z}^d$. Then, there exists $n_0 \in \mathbb{Z}^d$ such that $\eta = \xi + \gamma - n_0 \in \sigma(\widetilde{\phi})$. This implies that $\xi = \eta + n_0 - \gamma$, and thus $\xi \in \tau(\sigma(\widetilde{\phi}) - \gamma)$. The converse inclusion therefore follows. This proof has been completed. □

Lemma 3. *(see Lemma 2.4 in [10]). Let $\phi \in L^2(\mathbb{R}^d)$. If $\left\{T_k\phi\colon k \in \mathbb{Z}^d\right\}$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ and bound is B, then we have*

$$\sum_{k \in \mathbb{Z}^d} |\widehat{\phi}(\cdot + k)|^2 \leq B \text{ a.e. on } \mathbb{Z}^d. \tag{22}$$

Lemma 4. *Let $j \in \mathbb{Z}$ and $\phi \in L^2(\mathbb{R}^d)$. Then, for $f \in L^2(\mathbb{R}^d)$ and $k \in \mathbb{Z}^d$, the kth Fourier coefficient of $\sum_{l \in \mathbb{Z}^d} |\det A|^{j/2}\widehat{f}(A^{j*}(\cdot + l))\widehat{\phi}(\cdot + l)$ is $\langle f, \phi_{j,k}\rangle$. In particular,*

$$\sum_{l \in \mathbb{Z}^d} |\det A|^{j/2}\widehat{f}\left(A^{*j}(\cdot + l)\right)\overline{\widehat{\phi}(\cdot + l)} = \sum_{k \in \mathbb{Z}^d} \langle f, \phi_{j,k}\rangle e^{2\pi i\langle k, \cdot\rangle}, \tag{23}$$

if $\left\{\phi_{j,k}\colon k \in \mathbb{Z}^d\right\}$ is a Bessel sequence.

Proof. Since $f, \phi \in L^2(\mathbb{R}^d)$, we have $\widehat{f}(A^{*j}\cdot)\overline{\widehat{\phi}(\cdot)} \in L^1(\mathbb{R}^d)$, and thus

$$\begin{aligned}\int_{\mathbb{Z}^d} \sum_{l \in \mathbb{Z}^d} |\det A|^{j/2}\widehat{f}\left(A^{*j}(\xi + l)\right)\overline{\widehat{\phi}(\xi + l)}e^{-2\pi i\langle k, \xi\rangle}\,\mathrm{d}\xi &= |\det A|^{j/2}\int_{\mathbb{R}^d} \widehat{f}\left(A^{*j}\xi\right)\overline{\widehat{\phi}(\xi)}e^{-2\pi i\langle k, \xi\rangle}\,\mathrm{d}\xi\\ &= |\det A|^{-j/2}\int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\widehat{\phi}\left(A^{*-j}\xi\right)}e^{-2\pi i\langle k, A^{*-j}\xi\rangle}\,\mathrm{d}\xi\\ &= \int_{\mathbb{R}^d} \widehat{f}(\xi)\overline{\left[\phi_{j,k}(\cdot)\right]^\wedge(\xi)}\,\mathrm{d}\xi\\ &= \langle f, \phi_{j,k}\rangle,\end{aligned} \tag{24}$$

by the Plancherel theorem.

If $\{\phi_{j,k}: k \in \mathbb{Z}^d\}$ is a Bessel sequence, then $\{\langle f, \phi_{j,k}\rangle\}_{k\in\mathbb{Z}^d} \in \ell^2(\mathbb{Z}^d)$, and thus, we obtain (23) directly from (24). This proof has been completed. □

Lemma 5. *If $X(\phi, \Psi)$ is a Bessel sequence in $L^2(\mathbb{R}^d)$ and bound is B. Then, we have*

$$|\widehat{\phi}(\cdot)|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\left|\widehat{\psi}_l\left(A^{*-j}\cdot\right)\right|^2 \le B \text{ a.e. on } \mathbb{R}^d. \tag{25}$$

Proof. By the Bessel sequence assumption of $X(\phi, \Psi)$, we have

$$\sum_{k\in\mathbb{Z}^d}\left|\langle f, \phi_{0,k}\rangle\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \psi_{l,j,k}\rangle\right|^2 \le B\|f\|^2, \quad \text{for } \forall f \in L^2\left(\mathbb{R}^d\right). \tag{26}$$

By Lemma 4, we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \phi_{0,k}\rangle\right|^2 &+ \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \psi_{l,j,k}\rangle\right|^2 \\ &= \int_{\mathbb{Z}^d}\left|\sum_{k\in\mathbb{Z}^d}\widehat{f}(\xi+k)\overline{\widehat{\phi}(\xi+k)}\right|^2 d\xi \\ &\quad + \sum_{l=1}^{L}\sum_{j=0}^{\infty}|\det A|^j\int_{\mathbb{Z}^d}\left|\sum_{k\in\mathbb{Z}^d}\widehat{f}\left(A^{*j}(\xi+k)\right)\overline{\widehat{\psi}_l(\xi+k)}\right|^2 d\xi,\end{aligned} \tag{27}$$

and the above equality can be written as

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \phi_{0,k}\rangle\right|^2 &+ \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \psi_{l,j,k}\rangle\right|^2 = \int_{\mathbb{T}^d}\left(\sum_{k\in\mathbb{Z}^d}\widehat{f}(\xi+k)\overline{\widehat{\phi}(\xi+k)}\right)\left(\sum_{k\in\mathbb{Z}^d}\overline{\widehat{f}(\xi+k)}\widehat{\phi}(\xi+k)\right)d\xi \\ &+ \sum_{l=1}^{L}\sum_{j=0}^{\infty}|\det A|^j\int_{\mathbb{T}^d}\left(\sum_{k\in\mathbb{Z}^d}\widehat{f}\left(A^{*j}(\xi+k)\right)\overline{\widehat{\psi}_l(\xi+k)}\right)\left(\sum_{k\in\mathbb{Z}^d}\overline{\widehat{f}(A^{*j}(\xi+k))}\widehat{\psi}_l(\xi+k)\right)d\xi, \\ &= \int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}\widehat{\phi}(\xi)\left(\sum_{k\in\mathbb{Z}^d}\widehat{f}(\xi+k)\overline{\widehat{\phi}(\xi+k)}\right)d\xi + \sum_{l=1}^{L}\sum_{j=0}^{\infty}|\det A|^j\int_{\mathbb{R}^d}\overline{\widehat{f}(A^{*j}\xi)}\widehat{\psi}_l(\xi)\left(\sum_{k\in\mathbb{Z}^d}\widehat{f}\left(A^{*j}(\xi+k)\right)\overline{\widehat{\psi}_l(\xi+k)}\right)d\xi.\end{aligned} \tag{28}$$

Replacing $A^{*j}\xi$ by ξ in the second part of (28) and then separating the series into two parts: $k = 0$ and $k \neq 0$, we obtain

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \phi_{0,k}\rangle\right|^2 &+ \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle f, \psi_{l,j,k}\rangle\right|^2 = \int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}\widehat{\phi}(\xi)\left(\sum_{k\in\mathbb{Z}^d}\widehat{f}(\xi+k)\overline{\widehat{\phi}(\xi+k)}\right)d\xi \\ &+ \sum_{l=1}^{L}\sum_{j=0}^{\infty}\int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}\widehat{\psi}_l\left(A^{*-j}\xi\right)\left(\sum_{k\in\mathbb{Z}^d}\widehat{f}\left(\xi+A^{*j}k\right)\overline{\widehat{\psi}_l(A^{*-j}\xi+k)}\right)d\xi \\ &= \int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2\left(|\widehat{\phi}(\xi)|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\left|\widehat{\psi}_l\left(A^{*-j}\xi\right)\right|^2\right)d\xi + \int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}\widehat{\phi}(\xi)\left(\sum_{0\neq k\in\mathbb{Z}^d}\widehat{f}(\xi+k)\overline{\widehat{\phi}(\xi+k)}\right)d\xi \\ &+ \int_{\mathbb{R}^d}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\overline{\widehat{f}(\xi)}\widehat{\psi}_l\left(A^{*-j}\xi\right)\left(\sum_{0\neq k\in\mathbb{Z}^d}\widehat{f}\left(\xi+A^{*j}k\right)\overline{\widehat{\psi}_l(A^{*-j}\xi+k)}\right)d\xi \\ &= \int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2\left(|\widehat{\phi}(\xi)|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\left|\widehat{\psi}_l\left(A^{*-j}\xi\right)\right|^2\right)d\xi + \int_{\mathbb{R}^d}\overline{\widehat{f}(\xi)}\sum_{0\neq k\in\mathbb{Z}^d}\widehat{f}(\xi+k) \\ &\cdot\left(\widehat{\phi}(\xi)\overline{\widehat{\phi}(\xi+k)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\widehat{\psi}_l\left(A^{*-j}\xi\right)\overline{\widehat{\psi}_l(A^{*-j}(\xi+k))}\right)d\xi,\end{aligned} \tag{29}$$

by the definition of κ in (14).

Suppose that (25) is not valid, then there exists $E \subset \mathbb{R}^d$ with $|E| > 0$ such that $|\widehat{\phi}(\cdot)|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}|\widehat{\psi}_l(A^{*-j}\cdot)|^2 > B$ on E. As a result, we have $|\widehat{\phi}(\cdot)|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}|\widehat{\psi}_l(A^{*-j}\cdot)|^2 > B$ on some $E' = E \cap ([0,1)^d + k_0)$ with $|E'| > 0$ and $k_0 \in \mathbb{Z}^d$. Take f such that $\widehat{f} = \chi_{E'}$ in (29); then, we obtain

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\left|\langle f,\phi_{0,k}\rangle\right|^2 + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\left|\langle f,\psi_{l,j,k}\rangle\right|^2 \\ &\quad > B\left|E'\right| = B\|\widehat{f}\|^2 = B\|f\|^2,\end{aligned} \tag{30}$$

by the Plancherel theorem, which contradict with (26). Therefore, we prove that (25) holds. This proof has been completed. □

Lemma 6. *(see Lemma 3.2 in [36]). Let $\mathbb{V}$ be dense in a Hilbert space $\mathscr{H}$. If $\{f_i\}_{i\in I}$ and $\{g_i\}_{i\in I}$ are both Bessel sequences in $\mathscr{H}$, then $(\{f_i\}_{i\in I},\{g_i\}_{i\in I})$ is a pair of dual frames for $\mathscr{H}$ if and only if*

$$\langle f,g\rangle = \sum_{i\in I}\langle f,g_i\rangle\langle f_i,g\rangle, \tag{31}$$

for $f,g \in \mathbb{V}$.

Lemma 7. *(see Lemma 2.5 in [37]). Let ϕ, $\widetilde{\phi}$, ψ, $\widetilde{\psi} \in L^2(\mathbb{R}^d)$, we have*

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\phi}_{0,k}\rangle\langle\phi_{0,k},g\rangle + \sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle f,\widetilde{\psi}_{j,k}\rangle\langle\psi_{j,k},g\rangle \\ &\quad = S_1(f,g) + S_2(f,g),\end{aligned} \tag{32}$$

for f, $g \in \mathscr{D}$, where

$$\begin{aligned}S_1(f,g) &= \int_{\mathbb{R}^d}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\left(\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)} + \sum_{j=0}^{\infty}\widehat{\psi}(A^{*-j}\xi)\overline{\widehat{\widetilde{\psi}}(A^{*-j}\xi)}\right)\mathrm{d}\xi, \\ S_2(f,g) &= \int_{\mathbb{R}^d}\overline{\widehat{g}(\xi)}\sum_{0\neq k\in\mathbb{Z}^d}\widehat{f}(\xi+k)\left(\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+k)} + \sum_{j=0}^{\kappa(k)}\widehat{\psi}(A^{*-j}\xi)\overline{\widehat{\widetilde{\psi}}(A^{*-j}(\xi+k))}\right)\mathrm{d}\xi, \\ \mathscr{D} &= \left\{f\in L^2(\mathbb{R}^d):\ \widehat{f}\in L^\infty(\mathbb{R}^d)\text{ and }\operatorname{supp}(\widehat{f})\subset\left\{\xi\in\mathbb{R}^d:\ C^{-1}\le|\xi|\le C\right\}\text{ for some }C>1\right\}.\end{aligned} \tag{33}$$

3. Main Conclusions

The following theorem generalizes Proposition 2. A characterization of NWDF of form (12) in $FL^2(\Omega)$ is obtained.

Theorem 1. *Let $FL^2(\Omega)$ be a reducing subspace of $L^2(R^d)$. Let ϕ, $\widetilde{\phi} \in FL^2(\Omega)$, and two finite subsets $\Psi = \{\psi_1,\psi_2,\ldots,\psi_L\}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_1,\widetilde{\psi}_2,\ldots,\widetilde{\psi}_L\}$ of $FL^2(\Omega)$. Suppose that $X(\phi,\Psi)$ and $X(\widetilde{\phi},\widetilde{\Psi})$ are both Bessel sequences in $FL^2(\Omega)$. Then, $(X(\phi,\Psi),X(\widetilde{\phi},\widetilde{\Psi}))$ is a pair of NWDF for $FL^2(\Omega)$ if and only if, for each $n\in\mathbb{Z}^d$, we have*

$$\begin{aligned}&\widehat{\phi}(\cdot)\overline{\widehat{\widetilde{\phi}}(\cdot+n)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\cdot\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*-j}(\cdot+n))} \\ &\quad = \delta_{0,n}\chi_\Omega(\cdot)\text{ a.e. on }\mathbb{R}^d.\end{aligned} \tag{34}$$

Proof. Let $\mathscr{D}$ be defined as in (33). Since $\mathscr{D}\cap FL^2(\Omega)$ is dense in $FL^2(\Omega)$, we can see from Lemma 6 that $(X(\phi,\Psi),X(\widetilde{\phi},\widetilde{\Psi}))$ is a pair of NWDF for $FL^2(\Omega)$ if and only if

$$\begin{aligned}&\sum_{k\in\mathbb{Z}^d}\langle(\widehat{f}\chi_\Omega)^\vee,\widetilde{\phi}_{0,k}\rangle\langle\phi_{0,k},(\widehat{g}\chi_\Omega)^\vee\rangle \\ &\quad + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}\langle(\widehat{f}\chi_\Omega)^\vee,\widetilde{\psi}_{l,j,k}\rangle\langle\psi_{l,j,k},(\widehat{g}\chi_\Omega)^\vee\rangle \\ &\quad = \langle(\widehat{f}\chi_\Omega)^\vee,(\widehat{g}\chi_\Omega)^\vee\rangle,\end{aligned} \tag{35}$$

for $f,g\in\mathscr{D}$. By Lemma 7, (35) can be written as

$$I_1(f,g) + I_2(f,g) = \int_{\mathbb{R}^d}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\chi_\Omega(\xi)\,\mathrm{d}\xi,\quad\text{for } f,g\in\mathscr{D}, \tag{36}$$

where

$$\begin{aligned}I_1(f,g) &= \int_{\mathbb{R}^d}\widehat{f}(\xi)\overline{\widehat{g}(\xi)}\chi_\Omega(\xi) \\ &\quad\cdot\left(\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*-j}\xi)}\right)\mathrm{d}\xi, \\ I_2(f,g) &= \int_{\mathbb{R}^d}\overline{\widehat{g}(\xi)}\chi_\Omega(\xi)\sum_{0\neq k\in\mathbb{Z}^d}\left(\widehat{f}\chi_\Omega\right)(\xi+k) \\ &\quad\cdot\left(\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+k)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(k)}\widehat{\psi}_l\left(A^{*-j}\xi\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*-j}(\xi+k))}\right)\mathrm{d}\xi.\end{aligned} \tag{37}$$

Clearly, (34) implies (36). Now, we need to show that (36) also implies (34) to complete the proof.

Suppose (36) holds. Since ϕ, $\widetilde{\phi}\in FL^2(\Omega)$ and Ψ, $\widetilde{\Psi}\subset FL^2(\Omega)$, (34) always holds on $\mathbb{R}^d\setminus\Omega$, so we only need to prove that (34) holds a.e. on Ω. By applying Lemma 5 and the Cauchy–Schwarz inequality, we know the series $\widehat{\phi}(\cdot)\overline{\widehat{\widetilde{\phi}}(\cdot+n)} + \sum_{l=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_l(A^{*^{-j}}\cdot)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}(\cdot+n))}$ with $n\in\mathbb{Z}^d$ converges absolutely a.e. on $\mathbb{R}^d$, as a result of this, belongs to $L^\infty(\mathbb{R}^d)$. Therefore, almost every point in $\mathbb{R}^d$ is a Lebesgue point. Let $\xi_0\in\Omega$ be such a point. For $0<\epsilon<1/2$, take f and g such that

$$\widehat{f}(\cdot)=\frac{\chi_{B(\xi_0,\epsilon)}(\cdot)}{\sqrt{|B(\xi_0,\epsilon)|}}=\overline{\widehat{g}(\cdot)} \tag{38}$$

in ((36), where $B(\xi_0,\epsilon)=\{\xi\in\mathbb{R}^d\colon |\xi-\xi_0|<\epsilon\}$. Then, we have

$$\begin{aligned}&\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\chi_\Omega(\xi)\\&\cdot\left(\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)}+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}\xi)}\right)\mathrm{d}\xi\\&=\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\chi_\Omega(\xi)\mathrm{d}\xi,\end{aligned} \tag{39}$$

and letting $\epsilon\longrightarrow 0$, we obtain

$$\widehat{\phi}(\xi_0)\overline{\widehat{\widetilde{\phi}}(\xi_0)}+\sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi_0\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}\xi_0)}=1,\quad \xi_0\in\Omega. \tag{40}$$

For $0\neq k_0\in\mathbb{Z}^d$, take f and g such that

$$\widehat{f}(\cdot+k_0)=\frac{\chi_{B(\xi_0,\epsilon)}(\cdot)}{\sqrt{|B(\xi_0,\epsilon)|}}=\overline{\widehat{g}(\cdot)}, \tag{41}$$

in (36), where $0<\epsilon<1/2$. Then, we have

$$\begin{aligned}&\frac{1}{|B(\xi_0,\epsilon)|}\int_{B(\xi_0,\epsilon)}\chi_\Omega(\xi)\\&\cdot\left(\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+k_0)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k_0)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}(\xi+k_0))}\right)\mathrm{d}\xi=0,\end{aligned} \tag{42}$$

and letting $\epsilon\longrightarrow 0$, we obtain

$$\widehat{\phi}(\xi_0)\overline{\widehat{\widetilde{\phi}}(\xi_0+k_0)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(k_0)}\widehat{\psi}_l\left(A^{*^{-j}}\xi_0\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}(\xi_0+k_0))}=0,\quad \xi_0\in\Omega. \tag{43}$$

By the arbitrariness of ξ_0 and $0\neq k_0\in\mathbb{Z}^d$, then we obtain (34). This proof has been completed.

Next, by using Theorem 1, we derive an MOEP and an MEP for constructing NWDF in $FL^2(\Omega)$ under the following setup: $\phi,\widetilde{\phi}\in FL^2(\Omega)$, and satisfy. □

Assumption 1. $\lim_{j\longrightarrow\infty}\widehat{\phi}(A^{*^{-j}}\cdot)\overline{\widehat{\widetilde{\phi}}(A^{*^{-j}}\cdot)}=1$ a.e. on Ω.

Assumption 2. ϕ and $\widetilde{\phi}$ are both A-refinable functions, that is, there are two $\mathbb{Z}^d$-periodic measurable functions H_0 and $\widetilde{H}_0$ such that

$$\begin{aligned}\widehat{\phi}(A^*\cdot)&=H_0(\cdot)\widehat{\phi}(\cdot),\\\widehat{\widetilde{\phi}}(A^*\cdot)&=\widetilde{H}_0(\cdot)\widehat{\widetilde{\phi}}(\cdot)\text{ a.e. on }\mathbb{R}^d.\end{aligned} \tag{44}$$

Under this setup, define $\Psi=\{\psi_1,\psi_2,\ldots,\psi_L\}$ and $\widetilde{\Psi}=\{\widetilde{\psi}_1,\widetilde{\psi}_2,\ldots,\widetilde{\psi}_L\}\subset FL^2(\Omega)$ by

$$\begin{aligned}\widehat{\psi}_l(A^*\cdot)&=H_l(\cdot)\widehat{\phi}(\cdot),\\\widehat{\widetilde{\psi}}_l(A^*\cdot)&=\widetilde{H}_l(\cdot)\widehat{\widetilde{\phi}}(\cdot),\end{aligned} \tag{45}$$

with $1\le l\le L$, and H_l and $\widetilde{H}_l$ being $\mathbb{Z}^d$-periodic measurable functions.

Theorem 2. *Let $FL^2(\Omega)$ be a reducing subspace of $L^2(R^d)$. Let $\phi,\widetilde{\phi}\in FL^2(\Omega)$ satisfy Assumptions 1 and 2, and two finite subsets $\Psi=\{\psi_1,\psi_2,\ldots,\psi_L\}$ and $\widetilde{\Psi}=\{\widetilde{\psi}_1,\widetilde{\psi}_2,\ldots,\widetilde{\psi}_L\}$ of $FL^2(\Omega)$ defined as in (45). Suppose that $X(\phi,\Psi)$ and $X(\widetilde{\phi},\widetilde{\Psi})$ are both Bessel sequences in $FL^2(\Omega)$. Also, suppose there exists a $\mathbb{Z}^d$-periodic function $\Theta\in L^\infty(\mathbb{T}^d)$, and define η by $\widehat{\eta}(\cdot)=\Theta(\cdot)\widehat{\widetilde{\phi}}(\cdot)$ a.e. on $\mathbb{R}^d$. Then, $(X(\eta,\Psi),X(\widetilde{\phi},\widetilde{\Psi}))$ is a pair of NWDF for $FL^2(\Omega)$ if and only if*

$$\lim_{j\longrightarrow\infty}\Theta\left(A^{*^{-j}}\cdot\right)=1,\text{ a.e. on }\Omega, \tag{46}$$

$$\begin{aligned}&\Theta(A^*\cdot)H_0(\cdot)\overline{\widetilde{H}_0(\cdot+\gamma)}+\sum_{l=1}^{L}H_l(\cdot)\overline{\widetilde{H}_l(\cdot+\gamma)}\\&=\Theta(\cdot)\delta_{0,\gamma}\text{ a.e. on }\sigma(\phi)\cap\tau(\sigma(\widetilde{\phi})-\gamma)\text{ with }\gamma\in\Gamma_{A^*}.\end{aligned} \tag{47}$$

Proof. Since $X(\phi,\Psi)$ is a Bessel sequence in $FL^2(\Omega)$, especially, we have $\{\phi_{0,k}\colon k\in\mathbb{Z}^d\}$ and $\{\psi_{l,j,k}\colon j\in\mathbb{N}_0,k\in\mathbb{Z}^d,1\le l\le L\}$ are both Bessel sequences in $FL^2(\Omega)$, respectively. By the definition of η and Lemma 3, we know $\{\eta_{0,k}\colon k\in\mathbb{Z}^d\}$ is a Bessel sequence in $FL^2(\Omega)$. As a result, $X(\eta,\Psi)$ is also a Bessel sequence in $FL^2(\Omega)$. So, by Theorem 1, $(X(\eta,\Psi),X(\widetilde{\phi},\widetilde{\Psi}))$ is a pair of NWDF in $FL^2(\Omega)$ if and only if, for every $n\in\mathbb{Z}^d$, we have

$$\begin{aligned}&\Theta(\cdot)\widehat{\phi}(\cdot)\overline{\widehat{\widetilde{\phi}}(\cdot+n)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\cdot\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}(\cdot+n))}\\&=\delta_{0,n}\chi_\Omega(\cdot)\text{ a.e. on }\mathbb{R}^d.\end{aligned} \tag{48}$$

Next, we will show that (48) is equivalent to (46) and (47).

First, we suppose that (46) and (47) hold. For $0\neq n\in\mathbb{Z}^d$, by Assumption 2 and (45) and the $\mathbb{Z}^d$-periodicity of $\widetilde{H}_0$ and $\widetilde{H}_l$, $1\le l\le L$, we have

$$\begin{aligned}&\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+n)}+\sum_{l=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}(\xi+n))}\\&=\widehat{\phi}\left(A^{*^{-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}(A^{*^{-1}}(\xi+n))}\\&\quad\cdot\left(\Theta(\xi)H_0\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_0(A^{*^{-1}}\xi)}+\sum_{l=1}^{L}H_l\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_l(A^{*^{-1}}\xi)}\right)\\&\quad+\sum_{l=1}^{L}\sum_{j=1}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l(A^{*^{-j}}(\xi+n))},\end{aligned} \tag{49}$$

due to $A^{*^{-j}}n \in \mathbb{Z}^d$ for $j = 1, 2, \ldots, \kappa(n)$. By using (47), (49) can be written as

$$\begin{aligned}\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+n)} &+ \sum_{l=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}(\xi+n)\right)} = \Theta\left(A^{*^{-1}}\xi\right)\widehat{\phi}\left(A^{*^{-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-1}}(\xi+n)\right)} + \sum_{l=1}^{L}\sum_{j=1}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}(\xi+n)\right)}\\
&= \widehat{\phi}\left(A^{*^{-2}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-2}}(\xi+n)\right)}\left(\Theta\left(A^{*^{-1}}\xi\right)H_0\left(A^{*^{-2}}\xi\right)\overline{\widetilde{H}_0\left(A^{*^{-2}}\xi\right)} + \sum_{l=1}^{L}H_l\left(A^{*^{-2}}\xi\right)\overline{\widetilde{H}_l\left(A^{*^{-2}}\xi\right)}\right) + \sum_{l=1}^{L}\sum_{j=2}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}(\xi+n)\right)},\end{aligned} \tag{50}$$

and the last equality is deduced in the same procedure as (49). Continuing this way $\kappa(n) + 1$ times, we conclude that

$$\begin{aligned}\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+n)} &+ \sum_{l=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}(\xi+n)\right)}\\
&= \widehat{\phi}\left(A^{*^{-\kappa(n)-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-\kappa(n)-1}}(\xi+n)\right)}\\
&\quad\times\left(\Theta\left(A^{*^{-\kappa(n)}}\xi\right)H_0\left(A^{*^{-\kappa(n)-1}}\xi\right)\overline{\widetilde{H}_0\left(A^{*^{-\kappa(n)-1}}(\xi+n)\right)}\right.\\
&\quad\left.+\sum_{l=1}^{L}H_l\left(A^{*^{-\kappa(n)-1}}\xi\right)\overline{\widetilde{H}_l\left(A^{*^{-\kappa(n)-1}}(\xi+n)\right)}\right).\end{aligned} \tag{51}$$

Observe that $A^{*^{-\kappa(n)-1}}n = n_0 + \gamma$ for some $n_0 \in \mathbb{Z}^d$ and $\gamma \in \Gamma_{A^*}\setminus\{0\}$ by Lemma 1, and (51) therefore follows

$$\begin{aligned}\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+n)} &+ \sum_{l=1}^{L}\sum_{j=0}^{\kappa(n)}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}(\xi+n)\right)}\\
&= \widehat{\phi}\left(A^{*^{-\kappa(n)-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-\kappa(n)-1}}\xi+n_0+\gamma\right)}\\
&\quad\times\left(\Theta\left(A^{*^{-\kappa(n)}}\xi\right)H_0\left(A^{*^{-\kappa(n)-1}}\xi\right)\overline{\widetilde{H}_0\left(A^{*^{-\kappa(n)-1}}\xi+\gamma\right)}\right.\\
&\quad\left.+\sum_{l=1}^{L}H_l\left(A^{*^{-\kappa(n)-1}}\xi\right)\overline{\widetilde{H}_l\left(A^{*^{-\kappa(n)-1}}\xi+\gamma\right)}\right) = 0\end{aligned} \tag{52}$$

by (47) and the $\mathbb{Z}^d$-periodicity of $\widetilde{H}_0$ and $\widetilde{H}_l, 1 \le l \le L$.

Similarly, for $n = 0$, we have $\kappa(0) = +\infty$, and then

$$\begin{aligned}\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)} &+ \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)}\\
&= \widehat{\phi}\left(A^{*^{-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-1}}\xi\right)}\left(\Theta(\xi)H_0\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_0\left(A^{*^{-1}}\xi\right)} + \sum_{l=1}^{L}H_l\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_l\left(A^{*^{-1}}\xi\right)}\right) + \sum_{l=1}^{L}\sum_{j=1}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)}\\
&= \Theta\left(A^{*^{-1}}\xi\right)\widehat{\phi}\left(A^{*^{-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-1}}\xi\right)} + \sum_{l=1}^{L}\sum_{j=1}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)}\\
&= \cdots\\
&= \Theta\left(A^{*^{-N}}\xi\right)\widehat{\phi}\left(A^{*^{-N}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-N}}\xi\right)} + \sum_{l=1}^{L}\sum_{j=N}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)},\end{aligned} \tag{53}$$

for $N \in \mathbb{N}$. By applying Lemma 5 and the Cauchy–Schwarz inequality, we know the series

$$\sum_{l=1}^{L}\sum_{j=N}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)}, \tag{54}$$

converges absolutely for a.e. $\xi \in \mathbb{R}^d$, and therefore, we have

$$\lim_{N\longrightarrow\infty}\sum_{l=1}^{L}\sum_{j=N}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right) = 0, \quad \text{for a.e.}\,\xi \in \mathbb{R}^d. \tag{55}$$

Letting $N \longrightarrow \infty$ in (53), we obtain

$$\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)} = 1, \quad \text{for a.e.}\,\xi \in \Omega, \tag{56}$$

by Assumption 1 and (46).

Next, we prove the converse implication, i.e., (48) implies (46) and (47). Suppose that (48) holds. First, pick $\gamma \in \Gamma_{A^*}\setminus\{0\}$, and we have $A^*\gamma \in \mathbb{Z}^d$. For any $p \in \mathbb{Z}^d$, set $n = A^*(\gamma + p)$, for the choice of n, we have $\kappa(n) = 0$ because

$\gamma + p \notin \mathbb{Z}^d$. Since (48) is valid for all $0 \neq n \in \mathbb{Z}^d$, by applying Assumption 2 and (45), we have

$$\begin{aligned} 0 &= \Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi+n)} + \sum_{l=1}^{L}\widehat{\psi}_l(\xi)\overline{\widehat{\widetilde{\psi}}_l(\xi+n)} \\ &= \widehat{\phi}\left(A^{*^{-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-1}}(\xi+n)\right)}\left(\Theta(\xi)H_0\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_0\left(A^{*^{-1}}(\xi+n)\right)} + \sum_{l=1}^{L}H_l\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_l\left(A^{*^{-1}}(\xi+n)\right)}\right) \\ &= \widehat{\phi}\left(A^{*^{-1}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-1}}\xi+\gamma+p\right)}\left(\Theta(\xi)H_0\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_0\left(A^{*^{-1}}\xi+\gamma\right)} + \sum_{l=1}^{L}H_l\left(A^{*^{-1}}\xi\right)\overline{\widetilde{H}_l\left(A^{*^{-1}}\xi+\gamma\right)}\right), \end{aligned} \tag{57}$$

by the $\mathbb{Z}^d$-periodicity of $\widetilde{H}_0$ and $\widetilde{H}_l, 1 \le l \le L$. Take $\xi \in \sigma(\phi) \cap \tau(\sigma(\widetilde{\phi}) - \gamma)$ with $\gamma \in \Gamma_{A^*} \setminus \{0\}$ and replace ξ with $A^*(\xi+\lambda)$ in (57), where $\lambda \in \mathbb{Z}^d$, and it follows that

$$\begin{aligned} &\widehat{\phi}(\xi+\lambda)\overline{\widehat{\widetilde{\phi}}(\xi+\gamma+p)}\left(\Theta(A^*\xi)H_0(\xi)\overline{\widetilde{H}_0(\xi+\gamma)}\right. \\ &\quad\left. + \sum_{l=1}^{L}H_l(\xi)\overline{\widetilde{H}_l(\xi+\gamma)}\right) = 0, \end{aligned} \tag{58}$$

due to the $\mathbb{Z}^d$-periodicity of $\Theta, H_0, \widetilde{H}_0, H_l$ and $\widetilde{H}_l, 1 \le l \le L$. Since λ and p are arbitrary integers and $\xi \in \sigma(\phi) \cap \tau(\sigma(\widetilde{\phi}) - \gamma)$ with $\gamma \in \Gamma_{A^*} \setminus \{0\}$, by Lemma 2, there exist some λ_0 and p_0 such that

$$\widehat{\phi}(\xi+\lambda_0)\overline{\widehat{\widetilde{\phi}}(\xi+\gamma+p_0)} \neq 0. \tag{59}$$

Then, we obtain

$$\begin{aligned} &\Theta(A^*\xi)H_0(\xi)\overline{\widetilde{H}_0(\xi+\gamma)} + \sum_{l=1}^{L}H_l(\xi)\overline{\widetilde{H}_l(\xi+\gamma)} = 0 \\ &\quad \text{a.e. } \xi \in \sigma(\phi) \cap \tau(\sigma(\widetilde{\phi}) - \gamma) \text{ with } \gamma \in \Gamma_{A^*} \setminus \{0\}. \end{aligned} \tag{60}$$

Finally, we need to prove (46) holds, and (47) holds for $\gamma = 0$ to finish the proof. Taking $n = 0$ in (48), we have $\kappa(0) = +\infty$ and

$$\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)} = \chi_\Omega(\xi). \tag{61}$$

Replacing ξ with $A^*\xi$ in (61), we have

$$\Theta(A^*\xi)\widehat{\phi}(A^*\xi)\overline{\widehat{\widetilde{\phi}}(A^*\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j+1}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j+1}}\xi\right)} = \chi_\Omega(\xi), \tag{62}$$

due to $\Omega = A^*\Omega$. By applying Assumption 2 and (45), we have

$$\begin{aligned} &\Theta(A^*\xi)\widehat{\phi}(A^*\xi)\overline{\widehat{\widetilde{\phi}}(A^*\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j+1}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j+1}}\xi\right)} \\ &= \widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)}\left(\Theta(A^*\xi)H_0(\xi)\overline{\widetilde{H}_0(\xi)} + \sum_{l=1}^{L}H_l(\xi)\overline{\widetilde{H}_l(\xi)}\right) \\ &\quad + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)}, \end{aligned} \tag{63}$$

and thus,

$$\begin{aligned} &\Theta(\xi)\widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)} + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)} \\ &= \widehat{\phi}(\xi)\overline{\widehat{\widetilde{\phi}}(\xi)}\left(\Theta(A^*\xi)H_0(\xi)\overline{\widetilde{H}_0(\xi)} + \sum_{l=1}^{L}H_l(\xi)\overline{\widetilde{H}_l(\xi)}\right) \\ &\quad + \sum_{l=1}^{L}\sum_{j=0}^{\infty}\widehat{\psi}_l\left(A^{*^{-j}}\xi\right)\overline{\widehat{\widetilde{\psi}}_l\left(A^{*^{-j}}\xi\right)}. \end{aligned} \tag{64}$$

Therefore, we obtain

$$\begin{aligned} &\Theta(A^*\xi)H_0(\xi)\overline{\widetilde{H}_0(\xi)} + \sum_{l=1}^{L}H_l(\xi)\overline{\widetilde{H}_l(\xi)} = \Theta(\xi), \\ &\qquad \text{for a.e. } \xi \in \sigma(\phi) \cap \sigma(\widetilde{\phi}). \end{aligned} \tag{65}$$

On the contrary, collecting (53), (55), and (61), we conclude that

$$\lim_{N\longrightarrow\infty}\Theta\left(A^{*^{-N}}\xi\right)\widehat{\phi}\left(A^{*^{-N}}\xi\right)\overline{\widehat{\widetilde{\phi}}\left(A^{*^{-N}}\xi\right)} = \chi_\Omega(\xi). \tag{66}$$

By Assumption 1, then we obtain (46). This proof has been completed. □

Remark 1. In the literature, the function Θ is related to the notion of mixed fundamental function, which plays a significant role in MOEP.

Let $\Theta = 1$ a.e. on Ω in the previous theorem, an immediate consequence is generalization of the MEP, which reads as follows.

Corollary 1. *Let $FL^2(\Omega)$ be a reducing subspace of $L^2(R^d)$. Let $\phi, \widetilde{\phi} \in FL^2(\Omega)$ satisfy Assumptions 1 and 2, and two finite subsets $\Psi = \{\psi_1, \psi_2, \ldots, \psi_L\}$ and $\widetilde{\Psi} = \{\widetilde{\psi}_1, \widetilde{\psi}_2, \ldots, \widetilde{\psi}_L\}$ of $FL^2(\Omega)$ defined as in (45). Suppose that $X(\phi, \Psi)$ and $X(\widetilde{\phi}, \widetilde{\Psi})$ are both Bessel sequences in $FL^2(\Omega)$. Then, $(X(\phi, \Psi), X(\widetilde{\phi}, \widetilde{\Psi}))$ is a pair of NWDF for $FL^2(\Omega)$ if and only if*

$$H_0(\cdot)\overline{\widetilde{H}_0(\cdot+\gamma)} + \sum_{l=1}^{L} H_l(\cdot)\overline{\widetilde{H}_l(\cdot+\gamma)} = \delta_{0,\gamma} \quad \text{a.e. on } \sigma(\phi) \cap \tau(\sigma(\widetilde{\phi}) - \gamma) \text{ with } \gamma \in \Gamma_{A^*}. \tag{67}$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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