Hindawi
Journal of Mathematics
Volume 2020, Article ID 1832589, 5 pages
https://doi.org/10.1155/2020/1832589

Research Article

A Note on the Inner-Outer Iterative Method for Solving the Linear Equation $Ax = b$

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Received 28 December 2019; Accepted 6 May 2020; Published 6 July 2020

Academic Editor: Kaleem R. Kazmi

Recently, Tian et al. [Computers and Mathematics with Applications, 75(2018): 2710-2722] came up with the inner-outer iterative method to solve the linear equation $Ax = b$ and studied the corresponding convergence of the method. In this paper, we improve the main results of the inner-outer method and get weaker convergence results. Moreover, the parameters can be adjusted suitably so that the convergence property of the method can be substantially improved.

1. Introduction

When it comes to solving the large sparse linear system

$$Ax = b, \tag{1}$$

where $A \in R^{N\times N}$ is a square nonsingular matrix and $x, b \in R^N$, an iterative method is commonly used. Multisplitting for parallel solutions was initially introduced and adopted by O'Leary and White [1] for solving the linear equations and further studied by many other authors [2–35]. Among them, Prof. Bai [2–9] did a mountain of great work and constructed the parallel nonlinear AOR method about matrix multisplitting, the parallel chaotic multisplitting method, the two-stage multisplitting method under suitable constraints about two-stage multisplitting, some new hybrid algebraic multilevel preconditioning algorithms, nonstationary multisplitting iterative algorithms, and the nonstationary multisplitting two-stage iterative algorithms. Apart from these methods, Gu et al. [13, 14], Cao et al. [16–18], Wang et al. [15, 24, 26, 27, 29, 30], and Zhang et al. [28, 29, 31] also constructed relaxed nonstationary two-stage multisplitting algorithms, nested stationary iterative algorithms, relaxed parallel multisplitting AOR, USAOR, and SSOR algorithms on an H-matrix, two relaxed multisplitting algorithms for different weighting types when A is a monotone matrix or M-matrix, the parallel multisplitting TOR algorithm, and the global relaxed parallel multisplitting USAOR (GUSAOR) algorithm. Recently, Tian et al. [22] studied the inner-outer iterative method for the linear equation $Ax = b$ and deduced the corresponding convergence of the inner-outer algorithm. In this paper, based the inner-outer iterative method, we conducted a further analysis and obtained some convergence results weaker than Tian et al.'s.

In this paper, we present the inner-outer iterative method in Section 2 and show the essentially procedural

deduction as well as its corresponding main results in Section 3. Compared with those earlier studies, our findings of convergence results are more applicable. Furthermore, our new convergent domain of the parameter α is wider than that in [22]. Therefore, the convergence property of the method can be substantially improved due to the suitable adjustability of the parameters we adopted.

2. The Inner-Outer Iterative Method

In 2018, Tian et al. [22] presented an inner-outer iterative algorithm for the linear system $Ax = b$. Let $A = M - N$ be a convergent splitting; then, the authors can obtain the following linear system equivalent to (1):

$$(I - R)x = c, \tag{2}$$

where $R = M^{-1}N$ and $c = M^{-1}b$.

Then, the inner-outer iterative method for (2) is expressed as follows:

$$(I - \alpha R)x_{k+1} = (1 - \alpha)Rx_k + c, \quad k = 0, 1, 2, \ldots, \tag{3}$$

with $\alpha > 0$. Here, we regard (3) as the outer iteration. Let $g = (1 - \alpha)Rx_k + c$, and define the inner linear system as

$$(I - \alpha R)z = g. \tag{4}$$

Then, applying the inner iteration, we can get

$$z_{s+1} = \alpha R z_s + g, \quad s = 0, 1, 2, \ldots, l - 1, \tag{5}$$

where we take $z_0 = z_k$ as the initial value and assign z_l as new x_{k+1}. Moreover, the parameters η and ϵ are the tolerances of the inner and outer iterations, respectively:

$$\begin{aligned} \left\| c - (I - R)x_{k+1} \right\|_2 &< \epsilon, \\ \left\| g - (I - \alpha R)z_{s+1} \right\|_2 &< \eta. \end{aligned} \tag{6}$$

Then, the inner-outer iterative algorithm is given in Algorithm 1.

(i) Input $R, c, \alpha, \epsilon, \eta$
Output: x
Step 1: $x \longleftarrow c$
Step 2: $z \longleftarrow Rx$
Step 3: while $\|c - x + x\|_2 \geq \epsilon$
Step 4: $g \longleftarrow (1 - \alpha)z + c$
Step 5: repeat
Step 6: $x \longleftarrow g + \alpha z$
Step 7: $z \longleftarrow Rx$
Step 8: until $\|g + \alpha z - x\|_2 < \eta$
Step 9: end while
Step 10: $x \longleftarrow z + c$

Algorithm 1: KThe inner-outer iterative algorithm [22].

3. Main Results

In Algorithm 1, the authors discussed the convergence of the inner-outer iterative algorithm and studied the convergence of the iterations (3) and (5), respectively.

Lemma 1 (see [12]). *The iterative sequence $x_{m+1} = Rx_m + c$ converges to the corresponding solution of $Ax = b$ for all starting vectors x_0 and for all b if and only if (iff) $\rho(R) < 1$.*

Theorem 1 (see [22]). *Let $\rho(R) < 1$ and $0 < \alpha < 1$; then, the outer iteration (3) is convergent, where $\rho(R)$ denotes the spectral radius of matrix A.*

Theorem 2 (see [22]). *Let $\rho(R) < 1$ and $0 < \alpha < 1$; then, the inner iteration (5) is convergent, where $\rho(R)$ denotes the spectral radius of matrix A.*

Remark 1. Through careful analysis of the proving process of Theorems 1 and 2, we found that the convergence parameter α may actually be weaker and weaker. In addition, parameter α can be adjusted suitably so that the convergence of this method could be improved obviously. Thus, we may get the following convergence results.

Through further analysis, the outer iteration (3) is associated with the matrix splitting:

$$\begin{aligned} I - R &= M_1 - N_1, \\ M_1 &= I - \alpha R, \\ N_1 &= (1 - \alpha)R, \end{aligned} \tag{7}$$

and the corresponding outer iteration matrix is

$$R_1 = M_1^{-1}N_1 = (1 - \alpha)(I - \alpha R)^{-1}R. \tag{8}$$

The inner iteration for (4) has the following splitting:

$$\begin{aligned} M_1 &= I - \alpha R = M_2 - N_2, \\ M_2 &= I, \\ N_2 &= \alpha R, \end{aligned} \tag{9}$$

and the inner iteration matrix is

$$R_2 = \alpha R. \tag{10}$$

Theorem 3. *Let $\rho(R) < 1$ and $0 < \alpha < ((1 + \rho)/2\rho)$; then, the outer iteration (3) is convergent, where $\rho = \rho(R)$ denotes the spectral radius of matrix A.*

Proof. Let λ_i be an eigenvalue of R. Then, $|\lambda_i| \leq \rho(R) < 1$ from the assumption. Assume that μ_i is an eigenvalue of R_1 in (8); then,

$$\mu_i = \frac{(1 - \alpha)\lambda_i}{1 - \alpha\lambda_i}. \tag{11}$$

□

Case 1. When $0 < \alpha \leq 1$ and $|\lambda_i| \leq \rho(R) < 1$:

$$|\mu_i| = \left| \frac{(1 - \alpha)\lambda_i}{1 - \alpha\lambda_i} \right| \leq \frac{(1 - \alpha)\rho}{1 - \alpha\rho} < 1. \tag{12}$$

Case 2. When $1 < \alpha < ((1 + \rho)/2\rho)$ and $|\lambda_i| \leq \rho(R) < 1$:

$$|\mu_i| = \left|\frac{(1-\alpha)\lambda_i}{1-\alpha\lambda_i}\right| = \left|\frac{(\alpha-1)\lambda_i}{1-\alpha\lambda_i}\right|. \tag{13}$$

Since $1 < \alpha < ((1+\rho)/2\rho)$, we may obtain

$$\begin{aligned} &2\alpha\rho < 1+\rho \Longrightarrow \alpha\rho < \frac{1+\rho}{2} < 1 \Longrightarrow 1-\alpha\rho > 0, \\ &2\alpha\rho < 1+\rho \Longrightarrow \alpha\rho - \rho < 1-\alpha\rho \Longrightarrow \frac{\alpha\rho-\rho}{1-\alpha\rho} < 1. \end{aligned} \tag{14}$$

So, we can immediately get

$$|\mu_i| = \left|\frac{(1-\alpha)\lambda_i}{1-\alpha\lambda_i}\right| = \left|\frac{(\alpha-1)\lambda_i}{1-\alpha\lambda_i}\right| \le \frac{(\alpha-1)\rho}{1-\alpha\rho} < 1. \tag{15}$$

From Lemma 1 and equations (12) and (15), the outer iteration (3) is convergent.

Remark 2. Since $\rho(R) < 1$, then

$$\frac{1+\rho}{2\rho} > 1. \tag{16}$$

So, our new convergent domain about parameter α in Theorem 3 is wider than that in Theorem 1 [22].

Theorem 4. *Let $\rho(R) < 1$ and $0 < \alpha < (1/\rho)$; then, the inner iteration (5) is convergent, where $\rho = \rho(R)$ denotes the spectral radius of matrix A.*

Proof. Let ϕ_i be an eigenvalue of R_2 in (10); then,

$$\phi_i = \alpha\lambda_i, \tag{17}$$

where λ_i is an eigenvalue of R. Then, from (17), we can get

$$|\phi_i| = |\alpha\lambda_i| \le \alpha\rho < 1. \tag{18}$$

From Lemma 1 and equation (18), we complete the proof.

Remark 3. Since $\rho(R) < 1$, then

$$\frac{1}{\rho} > 1. \tag{19}$$

So, the new convergent domain about parameter α in Theorem 4 is wider than that in Theorem 2 [22].

Next, the authors gave the overall convergence for the inner-outer iterative algorithm without considering the parameters ϵ and η, which shows that the inner-outer iterative algorithm converges linearly to the exact solution x^* of linear system (2).

Lemma 2 (see [12]). *For all operator norms, $\rho(R) \le \|r\|$. For all R and for all $\epsilon > 0$, there is an operator norm $\|R\|_* \le \rho(R) + \epsilon$. The norm $\|\bullet\|_*$ depends on both R and ϵ, where $\rho(C)$ denotes the spectral radius of matrix C.*

Lemma 3 (see [12]). *Assume $\|\bullet\|$ satisfies $\|AB\| \le \|A\| \cdot \|B\|$. Then, $\|X\| < 1$ implies that $I - X$ is invertible, $(I-X)^{-1} = \sum_{i=0}^{\infty} X^i$, and $\|(I-X)^{-1}\| \le (1/(1-\|X\|))$.*

Now, we rewrite the inner-outer iterative algorithm as a two-stage iteration framework [21, 22]:

$$\begin{cases} x_{k,0} = x_k, x_0 = c, x_{k+1} = x_{k,m_k}, \\ x_{k,j+1} = \alpha R x_{k,j} + (1-\alpha) R x_k + c, k = 0,1,2,\ldots, \\ \qquad\qquad j = 0,1,2,\ldots,m_k - 1. \end{cases} \tag{20}$$

Theorem 5 (see [22]). *Let $A = M - N$ be a convergence splitting, $0 < \alpha < 1$, and m_k be the number of inner iteration steps at the k-th outer iteration. Then, the iteration sequence $\{x_k\}_{k=0}^{\infty}$ generated by (20) converges to the exact solution x^* of (2), faster than the iteration sequence derived from (2) for the same initial value x_0.*

Through careful analysis, we found that the iteration sequence $\{x_k\}_{k=0}^{\infty}$ generated by (20) still converges to the exact solution x^* of (2) when $\alpha > 1$. So, by similar proving process, we can get the following convergence theorem.

Theorem 6. *Assume $A = M - N$ to be a convergence splitting, $0 < \alpha < (2/(1+\rho))$, and m_k be the number of the inner iteration steps at the k-th outer iteration. Then, the iteration sequence $\{x_k\}_{k=0}^{\infty}$ generated by (20) converges to the exact solution x^* of (2), faster than the iteration sequence derived from (2) for the same initial value x_0.*

Proof. From equation (20), we can obtain

$$x_{k,j+1} = \left[(\alpha R)^{j+1} + (1-\alpha)\sum_{s=0}^{j}(\alpha R)^s R\right] x_k + \sum_{s=0}^{j}(\alpha R)^s c. \tag{21}$$

Then, we can get

$$x_{k+1} = H_k x_k + E_k c, \quad k = 0,1,2,\ldots, \tag{22}$$

where $H_k = (\alpha R)^{m_k} + (1-\alpha)\sum_{s=0}^{m_k-1}(\alpha R)^s R$ and $E_k = \sum_{s=0}^{m_k-1}(\alpha R)^s$. Since x^* is the exact solution of (1), then from (22), we have

$$x_* = H_k x_* + E_k c, \quad k = 0,1,2,\ldots \tag{23}$$

Subtracting equation (23) from equation (22), we can get

$$x_{k+1} - x_* = H_k(x_k - x_*) = \cdots = H_k H_{k-1}\cdots H_0(x_0 - x^*), \quad k = 0,1,2,\ldots, \tag{24}$$

$$\begin{aligned} H_k &= (\alpha R)^{m_k} + (1-\alpha)\sum_{s=0}^{m_k-1}(\alpha R)^s R \\ &= (\alpha R)^{m_k} - \sum_{s=0}^{m_k-1}(\alpha R)^s[(I-R) - (1-\alpha R)] \\ &= (\alpha R)^{m_k} + \left(I - (\alpha R)^{m_k}\right) - \sum_{s=0}^{m_k-1}(\alpha R)^s(I-R) \\ &= I - \sum_{s=0}^{m_k-1}(\alpha R)^s(I-R). \end{aligned} \tag{25}$$

Let λ_i be an eigenvalue of R; then, from equation (25), we can obtain that

$$\phi_i^k = 1 - \frac{(1-\lambda_i)(1-(\alpha\lambda_i)^{m_k})}{1-\alpha\lambda_i}, \tag{26}$$

is an eigenvalue of H_k.

$$\begin{aligned}\left|\phi_i^k\right| &= \left|1 - \frac{(1-\lambda_i)(1-(\alpha\lambda_i)^{m_k})}{1-\alpha\lambda_i}\right| \\ &= \left|\frac{\lambda_i\left(1-\alpha+\alpha^{m_k}\lambda_i^{m_k-1}-(\alpha\lambda_i)^{m_k}\right)}{1-\alpha\lambda_i}\right| \\ &< \frac{|1-\alpha| + |\alpha\lambda_i|^{m_k-1}(1-\lambda_i)\alpha}{|1-\alpha\lambda_i|} \\ &= \frac{|1-\alpha|}{|1-\alpha\lambda_i|},\end{aligned} \tag{27}$$

as $m_k \longrightarrow \infty$.

Case 3. When $0 < \alpha \le 1$ and $|\lambda_i| \le \rho(R) < 1$: from equation (27), we have

$$\begin{aligned}\left|\phi_i^k\right| &< \frac{|1-\alpha|}{|1-\alpha\lambda_i|} \\ &< \frac{1-\alpha}{1-\alpha\lambda_i} \\ &< 1.\end{aligned} \tag{28}$$

Case 4. When $1 < \alpha < 2/(1+\rho)$ and $|\lambda_i| \le \rho(R) < 1$: since $1 < \alpha < (2/(1+\rho))$, we can obtain

$$\begin{aligned}&1 < \alpha < \frac{2}{1+\rho} \Longrightarrow \alpha < \frac{2}{1+\rho} < \frac{1}{\rho} \Longrightarrow \alpha\rho < 1, \\ &1 < \alpha < \frac{2}{1+\rho} \Longrightarrow \alpha\rho + \alpha < 2 \Longrightarrow \alpha - 1 < 1 - \alpha\rho \Longrightarrow \frac{\alpha-1}{1-\alpha\rho} < 1.\end{aligned} \tag{29}$$

So, from equation (27), we can immediately get

$$\begin{aligned}\left|\phi_i^k\right| &< \frac{|1-\alpha|}{|1-\alpha\lambda_i|} \\ &< \frac{\alpha-1}{1-\alpha\rho} \\ &< 1.\end{aligned} \tag{30}$$

Then, we can obtain $\rho(H_k) < 1$. Next, from the proving process of Theorem 5, we can get $\rho(H_k) < \rho(R)$ for $k = 0, 1, 2, \cdots$.

Remark 4. Since $\rho(R) < 1$, then

$$\frac{2}{1+\rho} > 1. \tag{31}$$

Consequently, our new convergent domain of the parameter α in Theorem 6 is wider than the convergent domain in Theorem 5 [22].

4. Conclusions

In this paper, based on the convergence of the inner-outer iteration, we obtain more applicable convergence results. Furthermore, our new convergent domain of the parameter α is wider than that in [22]. Therefore, the convergence property of the method can be substantially improved due to the suitable adjustability of the parameters we adopted [34, 35].

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This research was supported by the National Natural Science Foundation of China (11226337, 11501525, 11961082, 11801528, 61571104, and 41906003), the Excellent Youth Foundation of Science Technology Innovation of Henan Province (184100510001 and 184100510004), the Aeronautical Science Foundation of China (2016ZG55019 and 2017ZD55014), Basic Research Projects of Key Scientific Research Projects Plan in Henan Higher Education Institutions (20zx003), Science and Technological Research of Key Projects of Henan Province (182102210242 and 182102110065), Project of Youth Backbone Teachers of Colleges and Universities of Henan Province (2019GGJS100 and 2019GGJS176), and Sichuan Science and Technology Program (2019YJ0357).

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