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Research Article

Hybrid Structures Applied to Ideals in BCI-Algebras

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In this paper, the notion of hybrid structure is applied to the ideal theory in BCI-algebras. In fact, we introduce the notions of hybrid p-ideal, hybrid h-ideal, and hybrid a-ideal in BCI-algebras and investigate their related properties. Furthermore, we show that every hybrid p-ideal (or h-ideal or a-ideal) is a hybrid ideal in a BCI-algebra but converse need not be true in general and in support, and we exhibit counter examples for each case. Moreover, we consider characterizations of hybrid p-ideal, hybrid h-ideal, and hybrid a-ideal in BCI-algebras.

1. Introduction

Imai and Iséki [1, 2] initiated the study of "BCK/BCI-algebras" in 1966 as a generalization of the notions of set-theoretical difference and propositional calculus. A great deal of literature has been developed on the theory of (BCK/BCI)-algebras since then, in particular, more focus has been placed on the "ideal theory" of BCK/BCI-algebras. In (BCK/BCI)-algebras, different kinds of ideals on different aspects have been studied (see, for example, [3–6]).

Fuzzy sets, introduced by Zadeh [7], deal with potential uncertainties, related to the imprecision of states, perceptions, and preferences. Molodtsov [8] proposed the concept of a "soft set" as a new mathematical framework for dealing with uncertainties, free of the difficulties that have disrupted normal theoretical approaches. Molodtsov pointed out a number of ways to develop soft sets. Molodtsov applied soft set theory in a variety of ways, such as smoothness of functions, game theory, operational research, Riemann integration, Perron integration, probability, and measurement theory (see [8–10]). Algebraic structures such as (BCK/BCI)-algebra [11], d-algebras [12], group [13], semigroup [14], ring [15], semiring [16], and decision-making [17, 18] are theoretically applied by soft set theory. Muhiuddin et al. (see, for example, [19–23]) investigated the fuzzy set theoretical approach to the (BCK/BCI)-algebras on various aspects. Further concepts related to this analysis in different fields have also been studied in [24–34].

In a system of parameters, Jun et al. [35] proposed the concept of "hybrid structure" over the initial universe set by merging fuzzy sets and soft sets. The idea of a hybrid subalgebra, a hybrid field, and a hybrid linear space was introduced with this notion. The hybrid structure theory and its applications to (BCK/BCI)-algebras and semigroups have recently been studied in (see [36–40] and references).

The objective of this paper is to introduce the notions of hybrid p-ideal, hybrid h-ideal, and hybrid a-ideal in (BCI)-algebras and investigate their related properties. Furthermore, we show that hybrid (p-ideal, h-ideal, and a-ideal) are all hybrid ideals, but converse need not be true in general and in support, and we exhibit counter examples. Also, we provide conditions for a hybrid p-ideal (or hybrid h-ideal or hybrid a-ideal) to be a hybrid ideal.

2. Preliminaries

An algebra $Z = (Z; *, 0)$ of type $(2, 0)$ is a (BCI)-algebra if it satisfies for all $s, t, r \in Z$:

$$\begin{aligned} &(K_1)((s*t)*(s*r))*(r*t)=0,\\ &(K_2)(s*(s*t))*t=0,\\ &(K_3)s*s=0,\\ &(K_4)s*t=0 \text{ and } t*s=0 \Rightarrow s=t. \end{aligned} \tag{1}$$

If a (BCI)-algebra Z satisfies

$$(K_5)0*s=0, \tag{2}$$

then Z is a BCK − algebra.

Any (BCK/BCI)-algebra Z satisfies the following conditions:

$$\begin{aligned} &(P_1)s*0=s,\\ &(P_2)(s*t)*r=(s*r)*t,\\ &(P_3)s\leq t\Rightarrow s*r\leq t*r \text{ and } r*t\leq r*s,\\ &(P_4)(s*r)*(t*r)\leq(s*t), \end{aligned} \tag{3}$$

where $s\leq t$ if and only if $s*t=0$. Note that $(Z,\leq)$ is a partially ordered set.

Any (BCI)-algebra Z satisfies the following conditions [41, 42]:

$$\begin{aligned} &(P_5)0*(0*(s*t))=0*(t*s),\\ &(P_6)0*(s*t)=(0*s)*(0*t),\\ &(P_7)s*(s*(s*t))=s*t,\\ &(P_8)0*(0*(s*t))=(0*t)*(0*s). \end{aligned} \tag{4}$$

For more details on (BCK)-algebras and (BCI)-algebras, we refer the readers to [43–45].

A subset $(\varnothing\neq)T$ of a (BCK/BCI)-algebra Z is called a subalgebra if $t*s\in T\ \forall t,s\in T$ and is called an ideal of Z if $0\in T$ and $\forall t,s\in Z, t*s\in T, s\in T$ implies $t\in T$. Furthermore, a subset $(\varnothing\neq)T$ of BCI-algebra Z is called p-ideal (resp. q-ideal and a-ideal) if $0\in T$ and $\forall t,s,r\in Z$, $((t*r)*(s*r))\in T, s\in T$ implies $t\in T$ (resp. $(t*(s*r))\in T, s\in T$ implies $t*r\in T$, and $((t*r)*(0*s))\in T, r\in T$ implies $s*t\in T$).

Definition 1 (see [35]). For a set of parameters Z, an initial universe set U, a power set of the initial set $P(U)$, and the unit interval I, a hybrid structure (briefly, HS) in Z over U is defined to be a mapping $\tilde{g}_\delta\colon=(\tilde{g};t\tilde{\delta})\colon Z\longrightarrow P(U)\times I; z\mapsto(\tilde{g}(z)t;n\tilde{\delta}q(z))$, where $\tilde{g}\colon Z\longrightarrow P(U)$ and $\tilde{\delta}\colon Z\longrightarrow I$ are mappings.

Definition 2 (see [35]). Let Z be a (BCK/BCI)-algebra. For a HS $\tilde{g}_\delta$ in Z over U, $\tilde{g}_\delta$ is said to be a hybrid subalgebra of Z if the following statements are valid:

$$(\forall s,t\in Z)\quad \begin{pmatrix} \tilde{g}(s*t)\supseteq\tilde{g}(s)\cap\tilde{g}(t),\\ \tilde{\delta}(s*t)\leq\vee\left\{\tilde{\delta}(s),\tilde{\delta}(t)\right\} \end{pmatrix}. \tag{5}$$

Definition 3 (see [39]). A HS $\tilde{g}_\delta$ in Z over U is said to be a hybrid ideal of Z over U if

$$\begin{aligned} &(\mathrm{HI}_1)(\forall s\in Z)\quad \begin{pmatrix} \tilde{g}(0)\supseteq\tilde{g}(s),\\ \tilde{\delta}(0)\leq\tilde{\delta}(s) \end{pmatrix},\\ &(\mathrm{HI}_2)(\forall s,t\in Z)\quad \begin{pmatrix} \tilde{g}(s)\supseteq\tilde{g}(s*t)\cap\tilde{g}(t),\\ \tilde{\delta}(s)\leq\vee\left\{\tilde{\delta}(s*t),\tilde{\delta}(t)\right\} \end{pmatrix}. \end{aligned} \tag{6}$$

Proposition 1 (see [39]). *Let $\tilde{g}_\delta=(\tilde{g},t\tilde{\delta})$ be a hybrid ideal of Z. If the inequality $s\leq t$ holds in Z, then $\tilde{g}(s)\supseteq\tilde{g}(t)$ and $\tilde{\delta}(s)\leq\tilde{\delta}(t)$.*

3. Hybrid Ideals in BCI-Algebras

Throughout the following sections Z denotes BCI-algebra unless stated otherwise.

Definition 4. A HS $\tilde{g}_\delta$ in Z over U is said to be a hybrid p-ideal of Z over U if it satisfies (HI_1) and $(\mathrm{HI}_3)(\forall s,t,r\in Z)(\tilde{g}(s)\supseteq\tilde{g}((s*r)*(t*r))\cap\tilde{g}\ (t),\tilde{\delta}(s)\leq\vee\{\tilde{\delta}((s*r)*(t*r)),\tilde{\delta}(t)\})$.

Example 1. Let $U=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$ be the initial universe set. On a set of parameters $Z=\{f,s,n,t\}$, we define the operation $*$ in Table 1.

Then, $(Z,*,f)$ is a (BCI)-algebra. Let $\tilde{g}_\delta$ be a HS in Z over U which is given by Table 2.

By routine verification $\tilde{g}_\delta$ is a hybrid p-ideal of Z over U.

Theorem 1. *For any (BCI)-algebra, every hybrid p-ideal is a hybrid ideal.*

Proof. Suppose that $\tilde{g}_\delta$ is a hybrid p-ideal of Z over U. Since $s*0=s$ for all $s\in Z$, we have

$$\begin{aligned} \tilde{g}(s)&\supseteq\tilde{g}((s*0)*(t*0))\cap\tilde{g}(t)\\ &\supseteq\tilde{g}(s*t)\cap\tilde{g}(t),\\ \tilde{\delta}(s)&\leq\cup\left\{\tilde{\delta}((s*0)*(t*0)),\tilde{\delta}(t)\right\},\\ &\leq\cup\left\{\tilde{\delta}(s*t),\tilde{\delta}(t)\right\}. \end{aligned} \tag{7}$$

for all $s,t\in Z$. Hence, $\tilde{g}_\delta$ is a hybrid ideal of Z over U.

As shown in the following example, the converse of Theorem 1 is not generally valid. □

Example 2. Let $U=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5\}$ be the initial universe set and $Z=\{f,s,n,t,e\}$ be the set of parameters. We define the binary operation $*$ on Z by Table 3.

Then, $(Z,*,f)$ is a (BCI)-algebra. Now, define a HS $\tilde{g}_\delta$ in Z over U which is given by Table 4.

It is easy to check that $\tilde{g}_\delta$ is a hybrid ideal of Z over U but not hybrid p-ideal as

$$\begin{aligned} \tilde{g}(n)=\{\gamma_2,\gamma_4\}\supseteq\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}&=\tilde{g}((n*n)*(s*n))\cap\tilde{g}(s)\\ &=\tilde{g}(f*s)\cap\tilde{g}(s)=\tilde{g}(f)\cap\tilde{g}(s). \end{aligned} \tag{8}$$

Theorem 2. *If $\tilde{g}_\delta$ is a hybrid p-ideal of Z over U, then*

$$\begin{aligned} &\tilde{g}(s)\supseteq\tilde{g}(0*(0*s)),\\ &\tilde{\delta}(s)\leq\tilde{\delta}(0*(0*s)), \end{aligned} \tag{9}$$

Table 1: Cayley table of the binary operation*.

$*$	f	s	n	t
f	f	s	n	t
s	s	f	t	n
n	n	t	f	s
t	t	n	s	f

Table 2: Table representation of hybrid structure $\widetilde{g}_{\delta}$.

Z	$\widetilde{g}$	$\widetilde{\delta}$
f	U	0.3
s	$\{\gamma_1, \gamma_2, \gamma_3\}$	0.4
n	$\{\gamma_1, \gamma_2\}$	0.5
t	$\{\gamma_1, \gamma_2\}$	0.5

Table 3: Cayley table of the binary operation*.

$*$	f	s	n	t	e
f	f	f	f	t	t
s	s	f	s	e	t
n	n	n	f	t	t
t	t	t	t	f	f
e	e	t	e	s	f

Table 4: Table representation of hybrid structure $\widetilde{g}_{\delta}$.

Z	$\widetilde{g}$	δ
f	U	0.4
s	$\{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$	0.8
n	$\{\gamma_2, \gamma_4\}$	0.5
t	$\{\gamma_2, \gamma_4\}$	0.7
e	$\{\gamma_2, \gamma_4\}$	0.8

for all $s \in Z$.

Proof. Let $\widetilde{g}_{\delta}$ be a hybrid p-ideal of Z over U. Then,

$$\begin{aligned} &\widetilde{g}(s) \supseteq \widetilde{g}((s*r)*(t*r)) \cap \widetilde{g}(t), \\ &\widetilde{\delta}(s) \leq \vee\left\{\widetilde{\delta}((s*r)*(t*r)), \widetilde{\delta}(t)\right\}, \end{aligned} \tag{10}$$

for all $s, t, r \in Z$. Substituting s for r and 0 for t in (10), then

$$\begin{aligned} \widetilde{g}(s) &\supseteq \widetilde{g}(\widetilde{g}((s*s)*(0*s)), \widetilde{g}(0)) \\ &= \widetilde{g}(0*(0*s)) \cap \widetilde{g}(0) \\ &= \widetilde{g}(0*(0*s)), \\ \widetilde{\delta}(s) &\leq \cup\left\{\widetilde{\delta}((s*s)*(0*s)), \widetilde{\delta}(0)\right\} \\ &= \cup\left\{\widetilde{\delta}(0*(0*s)), \widetilde{\delta}(0)\right\} \\ &= \widetilde{\delta}(0*(0*s)). \end{aligned} \tag{11}$$

□

Theorem 3. *If $\widetilde{g}_{\delta}$ is a hybrid p-ideal of Z over U, then*

$$\begin{aligned} &\widetilde{g}(s) = \widetilde{g}(0*(0*s)), \\ &\widetilde{\delta}(s) = \widetilde{\delta}(0*(0*s)), \end{aligned} \tag{12}$$

for all $s \in Z$.

Proof. By (K_2), $(0*(0*s)) \leq s$. Therefore, by Proposition 1,

$$\begin{aligned} &\widetilde{g}(0*(0*s)) \supseteq \widetilde{g}(s), \\ &\widetilde{\delta}(0*(0*s)) \leq \widetilde{\delta}(s). \end{aligned} \tag{13}$$

Thus, by using (9), we have

$$\begin{aligned} &\widetilde{g}(s) = \widetilde{g}(0*(0*s)), \\ &\widetilde{\delta}(s) = \widetilde{\delta}(0*(0*s)). \end{aligned} \tag{14}$$

□

Theorem 4. *If $\widetilde{g}_{\delta}$ is a hybrid p-ideal of Z over U, then*

$$\begin{aligned} &\widetilde{g}((s*r)*(t*r)) \supseteq \widetilde{g}(s*t), \\ &\widetilde{\delta}((s*r)*(t*r)) \leq \widetilde{\delta}(s*t), \end{aligned} \tag{15}$$

for all $s, t, r \in Z$.

Proof. Let $\widetilde{g}_{\delta}$ be a hybrid p-ideal of Z over U. Note that $((s*r)*(t*r) \leq s*t)$ holds in Z. It implies that $((s*r)*(t*r))*(s*t) = 0$. Since $\widetilde{g}_{\delta}$ is a hybrid ideal of Z over U, we have

$$\begin{aligned} \widetilde{g}((s*r)*(t*r)) &\supseteq \widetilde{g}(((s*r)*(t*r))*(s*t)) \cap \widetilde{g}(s*t) \\ &= \widetilde{g}(0) \cap \widetilde{g}(s*t) \\ &= \widetilde{g}(s*t), \\ \widetilde{\delta}((s*r)*(t*r)) &\leq \cup\left\{\widetilde{\delta}(((s*r)*(t*r)) *\left(s*t)), \widetilde{\delta}(s*t)\right\}\right. \\ &= \cup\left\{\widetilde{\delta}(0), \widetilde{\delta}(s*t)\right\} \\ &= \widetilde{\delta}(s*t), \end{aligned} \tag{16}$$

for all $s, t, r \in Z$.

The following theorem gives a condition for a hybrid ideal to be a hybrid p-ideal. □

Theorem 5. *Let $\widetilde{g}_{\delta}$ be a hybrid ideal of Z over U satisfying the following condition:*

$$\begin{aligned} &\widetilde{g}(s*t) \supseteq \widetilde{g}((s*r)*(t*r)), \\ &\widetilde{\delta}(s*t) \leq \widetilde{\delta}((s*r)*(t*r)), \end{aligned} \tag{17}$$

for all $s, t, r \in Z$. Then, $\widetilde{g}_{\delta}$ is a hybrid p-ideal of Z over U.

Proof. Suppose that $\widetilde{g}_{\delta}$ is a hybrid ideal of Z satisfying (17). Then,

$$\begin{aligned} \widetilde{g}(s) &\supseteq \widetilde{g}(s*t) \cap \widetilde{g}(t) \\ &\supseteq \widetilde{g}((s*r)*(t*r)) \cap \widetilde{g}(t), \\ \widetilde{\delta}(s) &\leq \cup\left\{\widetilde{\delta}(s*t), \widetilde{\delta}(t)\right\} \\ &\leq \cup\left\{\widetilde{\delta}((s*r)*(t*r)), \widetilde{\delta}(t)\right\}, \end{aligned} \tag{18}$$

for all $s, t, r \in Z$, as required. □

Definition 5. A HS $\tilde{g}_\delta$ in Z over U is said to be a hybrid h-ideal of Z over U if it satisfies (HI_1) and (HI_4)

$$(\forall s,t,r \in Z)\quad \begin{pmatrix} \tilde{g}(s*r)\supseteq\tilde{g}(s*(t*r))\cap\tilde{g}(t), \\ \tilde{\delta}(s*r)\leq \cup\left\{\tilde{\delta}(s*(t*r)),\tilde{\delta}(t)\right\} \end{pmatrix}.$$

Example 3. Let $U=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5\}$ be the initial universe set. As a set of parameters, we consider $Z=\{f,s,n,t,e\}$. The operation $*$ is defined on Z by Table 5.

Then, $(Z,*,f)$ is a (BCI)-algebra. Define a HS $\tilde{g}_\delta$ in Z over U by the following Table 6.

It can be easily checked that $\tilde{g}_\delta$ is a hybrid h-ideal of Z over U.

Table 5: Cayley table of the binary operation*.

$*$	f	s	n	t	e
f	f	f	f	t	t
s	s	f	s	e	t
n	n	n	f	t	t
t	t	t	t	f	f
e	e	t	e	s	f

Table 6: Table representation of hybrid structure $\tilde{g}_\delta$.

Z	$\tilde{g}$	$\tilde{\delta}$
f	U	0.1
s	$\{\gamma_2,\gamma_4\}$	0.7
n	$\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$	0.3
t	$\{\gamma_2,\gamma_3,\gamma_4\}$	0.5
e	$\{\gamma_2,\gamma_4\}$	0.7

Theorem 6. *For any (BCI)-algebra, every hybrid h-ideal is a hybrid ideal.*

Proof. Proof. Let $\tilde{g}_\delta$ be a hybrid h-ideal of Z over U. Since $s*0=s$ for all $s\in Z$, we have

$$\begin{aligned} \tilde{g}(s*0)&\supseteq\tilde{g}(s*(t*0))\cap\tilde{g}(t)\\ \tilde{g}(s)&\supseteq\tilde{g}(s*t)\cap\tilde{g}(t)\\ &\text{and}\\ \tilde{\delta}(s*0)&\leq\cup\left\{\tilde{\delta}(s*(t*0)),\tilde{\delta}(t)\right\}\\ \tilde{\delta}(s)&\leq\cup\left\{\tilde{\delta}(s*t),\tilde{\delta}(t)\right\}, \end{aligned} \tag{19}$$

for all $s,t\in Z$. Hence, $\tilde{g}_\delta$ is a hybrid ideal of Z over U.

In general, the converse of Theorem 6 is not valid, as the following example shows. □

Example 4. Let $U=\{\gamma_1,\gamma_2,\gamma_3,\gamma_4,\gamma_5\}$ be the initial universe set. For a set of parameters $Z=\{p,s,t,w,q\}$, we define a binary operation by the Cayley table in Table 7.

Then, $(Z,*,p)$ is a (BCI)-algebra. Define a HS $\tilde{g}_\delta$ in Z over U by Table 8.

It is routine to verify that $\tilde{g}_\delta$ is a hybrid ideal but not hybrid h-ideal since

Table 7: Cayley table of the binary operation*.

$*$	p	s	t	w	q
p	p	p	q	w	t
s	s	p	q	w	t
t	t	t	p	q	w
w	w	w	t	p	q
q	q	q	w	t	p

Table 8: Table representation of hybrid structure $\tilde{g}_\delta$.

Z	$\tilde{g}$	$\tilde{\delta}$
p	U	0.2
s	$\{\gamma_1,\gamma_2,\gamma_3,\gamma_4\}$	0.3
t	$\{\gamma_2,\gamma_3\}$	0.4
w	$\{\gamma_2,\gamma_3\}$	0.4
q	$\{\gamma_2,\gamma_3\}$	0.4

$$\begin{aligned} &\tilde{g}(q*t)=\tilde{g}(w)=\{\gamma_2,\gamma_3\}\not\supseteq U=\tilde{g}(q*(p*t))\cap\tilde{g}(p)=\tilde{g}(q*q)\cap\tilde{g}(p)=\tilde{g}(p)\cap\tilde{g}(p),\\ &\tilde{\delta}(q*t)=\tilde{\delta}(w)=0.4\not\leq\cup\left\{\tilde{\delta}(q*(p*t)),\tilde{\delta}(p)\right\}=\cup\left\{\tilde{\delta}(q*q),\tilde{\delta}(p)\right\}=\cup\left\{\tilde{\delta}(p),\tilde{\delta}(p)\right\}=0.2. \end{aligned} \tag{20}$$

Theorem 7. *Let $\tilde{g}_\delta$ be a hybrid h-ideal of Z over U. Then,*

$$\begin{aligned} \tilde{g}(0*s)&\supseteq\tilde{g}(s),\\ \tilde{\delta}(0*s)&\leq\tilde{\delta}(s), \end{aligned} \tag{21}$$

for all $s\in Z$.

Proof. Let $\tilde{g}_\delta$ be a hybrid h-ideal of Z over U. Then,

$$\begin{aligned} \tilde{g}(s*r)&\supseteq\tilde{g}(s*(t*r))\cap\tilde{g}(t),\\ \tilde{\delta}(s*r)&\leq\cup\left\{\tilde{\delta}(s*(t*r)),\tilde{\delta}(t)\right\}. \end{aligned} \tag{22}$$

Substituting s by 0, r by s, and t by s, we have

$$\begin{aligned} \tilde{g}(0*s)&\supseteq\tilde{g}(0*(s*s))\cap\tilde{g}(s)=\tilde{g}(0)\cap\tilde{g}(s)\\ &=\tilde{g}(s),\\ \tilde{\delta}(0*s)&\leq\cup\left\{\tilde{\delta}(0*(s*s)),\tilde{\delta}(s)\right\}\\ &=\cup\left\{\tilde{\delta}(0),\tilde{\delta}(s)\right\}=\tilde{\delta}(s). \end{aligned} \tag{23}$$

□

Definition 6. A HS $\tilde{g}_\delta$ in Z over U is said to be a hybrid a-ideal of Z over U if it satisfies (HI_1) and (HI_5)

$$(\forall s,t,r \in Z)\quad \begin{pmatrix} \tilde{g}(t*s)\supseteq\tilde{g}((s*r)*(0*t))\cap\tilde{g}(r), \\ \tilde{\delta}(t*s)\leq \cup\left\{\tilde{\delta}((s*r)*(0*t)),\tilde{\delta}(r)\right\}. \end{pmatrix}.$$

Example 5. Consider $\tilde{g}_\delta$ represented in Example 1. It is routine to verify that $\tilde{g}_\delta$ is a hybrid a-ideal of Z over U.

Theorem 8. *Every hybrid a-ideal is both a hybrid subalgebra and a hybrid ideal of Z over U.*

Proof. Let $\tilde{g}_\delta$ be a hybrid a-ideal of Z over U. By substituting $t = r = 0$ in (HI_5) and using (K_3) and (P_1), we obtain

$$\begin{aligned}\tilde{g}(0 * s) &\supseteq \tilde{g}((s * 0) * (0 * 0)) \cap \tilde{g}(0) = \tilde{g}(s * 0) \cap \tilde{g}(0) = \tilde{g}(s),\\ \tilde{\delta}(0 * s) &\leq \cup\left\{\tilde{\delta}((s * 0) * (0 * 0)), \tilde{\delta}(0)\right\} = \tilde{\delta}(s),\end{aligned} \tag{24}$$

for all $s, t, r \in Z$.

Again setting $s = r = 0$ in (HI_5), using (P_1), (K_3), (HI_1), and above equations, we obtain

$$\begin{aligned}\tilde{g}(t * 0) &\supseteq \tilde{g}((0 * 0) * (0 * t)) \cap \tilde{g}(0)\\ &= \tilde{g}(0 * (0 * t)) \cap \tilde{g}(0) = \tilde{g}(0 * (0 * t)) \supseteq \tilde{g}(0 * t),\end{aligned} \tag{25}$$

which implies $\tilde{g}(t) \supseteq \tilde{g}(0 * t)$

$$\begin{aligned}\tilde{\delta}(t * 0) &\leq \cup\left\{\tilde{\delta}(0 * (0 * t)), \tilde{\delta}(0)\right\}\\ &= \tilde{\delta}(0 * (0 * t)) \leq \tilde{\delta}(0 * t),\end{aligned} \tag{26}$$

which implies $\tilde{\delta}(t) \leq \tilde{\delta}(0 * t)$ for all $t \in Z$. Then, from (HI_5) and (P_1), it implies that

$$\begin{aligned}\tilde{g}(t) &\supseteq \tilde{g}(0 * t) \supseteq \tilde{g}((t * r) * (0 * 0)) \cap \tilde{g}(r)\\ &= \tilde{g}(t * r) \cap \tilde{g}(r),\\ \tilde{\delta}(t) &\leq \tilde{\delta}(0 * t) \leq \cup\left\{\tilde{\delta}((t * r) * (0 * 0)), \tilde{\delta}(r)\right\}\\ &\leq \cup\left\{\tilde{\delta}(t * r), \tilde{\delta}(r)\right\},\end{aligned} \tag{27}$$

for all $t, r \in Z$. Hence, $\tilde{g}_\delta$ is a hybrid ideal of Z over U.

Now, for any $t, r \in Z$ and using above equations, we have

$$\begin{aligned}\tilde{g}(t * r) &\supseteq \tilde{g}((t * r) * t) \cap \tilde{g}(t)\\ &= \tilde{g}(0 * r) \cap \tilde{g}(t) \supseteq \tilde{g}(t) \cap \tilde{g}(r),\\ \tilde{\delta}(t * r) &\leq \cup\left\{\tilde{\delta}((t * r) * t), \tilde{\delta}(t)\right\}\\ &= \cup\left\{\tilde{\delta}(0 * r), \tilde{\delta}(t)\right\} \leq \cup\left\{\tilde{\delta}(t), \tilde{\delta}(r)\right\}.\end{aligned} \tag{28}$$

Therefore, $\tilde{g}_\delta$ is a hybrid subalgebra of Z over U. The converse of Theorem 8 is not generally valid as seen in the following example. □

Table 9: Cayley table of the binary operation*.

$*$	p	s	t
p	p	p	p
s	s	p	s
t	t	t	p

Table 10: Table representation of hybrid structure $\tilde{g}_\delta$.

Z	$\tilde{g}$	$\tilde{\delta}$
p	U	0.6
s	$\{\gamma_1, \gamma_2, \gamma_3\}$	0.7
t	$\{\gamma_1, \gamma_2, \gamma_3\}$	0.7

Example 6. Let $U = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ be the initial universe set. For a set of parameters $Z = \{p, s, t\}$, we define a binary operation by the Cayley table in Table 9.

Then, $(Z, *, p)$ is a (BCI)-algebra. Let $\tilde{g}_\delta$ be a HS in Z over U which is given by Table 10.

Then, it directs to show that $\tilde{g}_\delta$ is a hybrid subalgebra and hybrid ideal but not hybrid a-ideal as

$$\begin{aligned}\tilde{g}(s * p) &= \tilde{g}(s) = \{\gamma_1, \gamma_2, \gamma_3\} \supseteq U\\ &= \tilde{g}((p * p) * (p * s)) \cap \tilde{g}(p)\\ &= \tilde{g}(p * p) \cap \tilde{g}(p) = \tilde{g}(p) \cap \tilde{g}(p),\\ \tilde{\delta}(s * p) &= \tilde{\delta}(s)\\ &= 0.7 \leq \cup\left\{\tilde{\delta}((p * p) * (p * s)), \tilde{\delta}(p)\right\}\\ &= \cup\left\{\tilde{\delta}(p * p), \tilde{\delta}(p)\right\} = \cup\left\{\tilde{\delta}(p), \tilde{\delta}(p)\right\}\\ &= 0.6.\end{aligned} \tag{29}$$

Theorem 9. *If $\tilde{g}_\delta$ is a hybrid a-ideal of Z over U, then $\tilde{g}((s * r) * (0 * t)) \supseteq \tilde{g}(s * (r * t))$ and $\tilde{\delta}((s * r) * (0 * t)) \leq \tilde{\delta}(s * (r * t))$ for all $s, t, r \in Z$.*

Proof. Suppose that $\tilde{g}_\delta$ is a hybrid a-ideal of Z over U. Since $(s * r) * (0 * t) = (s * r) * ((r * t) * r) \leq s * (r * t)$, $(s * r) * (0 * t) * (s * (r * t)) = 0$.

By Theorem 8, $\tilde{g}_\delta$ is a hybrid ideal of Z over U. Thus,

$$\begin{aligned}\widetilde{g}((s*r)*(0*t)) &\supseteq \widetilde{g}(((s*r)*(0*t))*(s*(r*t)))\cap\widetilde{g}(s*(r*t))\\ &=\widetilde{g}(0)\cap\widetilde{g}(s*(r*t))\\ &\supseteq\widetilde{g}(s*(r*t)),\\ \widetilde{\delta}((s*r)*(0*t)) &\leq \cup\left\{\widetilde{\delta}(((s*r)*(0*t))*(s*(r*t))),\widetilde{\delta}(s*(r*t))\right\}\\ &=\cup\left\{\widetilde{\delta}(0),\widetilde{\delta}(s*(r*t))\right\}\\ &\leq\widetilde{\delta}(s*(r*t)),\end{aligned} \tag{30}$$

for every $s,t,r\in Z$. □

Theorem 10. *Let $\widetilde{g}_\delta$ be a hybrid ideal of Z over U satisfying the following inequality:*

$$\begin{aligned}\widetilde{g}(t*s)&\supseteq\widetilde{g}(s*(0*t)),\\ \widetilde{\delta}(t*s)&\leq\widetilde{\delta}(s*(0*t)),\end{aligned} \tag{31}$$

for all $s,t\in Z$, then $\widetilde{g}_\delta$ is a hybrid a-ideal of Z over U.

Proof. Since $\widetilde{g}_\delta$ is a hybrid ideal of Z over U,

$$\begin{aligned}\widetilde{g}(t*s)&\supseteq\widetilde{g}(s*(0*t))\\ &\supseteq\widetilde{g}((s*(0*t))*r)\cap\widetilde{g}(r)\\ &\supseteq\widetilde{g}((s*r)*(0*t))\cap\widetilde{g}(r),\\ \widetilde{\delta}(t*s)&\leq\widetilde{\delta}(s*(0*t))\\ &\leq\cup\left\{\widetilde{\delta}((s*(0*t))*r),\widetilde{\delta}(r)\right\}\\ &\leq\cup\left\{\widetilde{\delta}((s*r)*(0*t)),\widetilde{\delta}(r)\right\},\end{aligned} \tag{32}$$

for all $s,t,r\in Z$. Hence, $\widetilde{g}_\delta$ is a hybrid a-ideal of Z over U. □

4. Conclusion

In this paper, we apply the notion of hybrid structure to the ideal theory in (BCI)-algebras. In fact, we have discussed the notions of hybrid p-ideal, hybrid h-ideal, and a-ideal in (BCI)-algebras and investigated several related properties. Furthermore, it has been shown that every hybrid p-ideal (or h-ideal or a-ideal) is a hybrid ideal in a (BCI)-algebra but converse need not be true in general, and in support, examples for each case are provided. Moreover, characterizations of hybrid p-ideal, hybrid h-ideal, and hybrid a-ideal in (BCI)-algebras are discussed.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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