Hindawi
Journal of Mathematics
Volume 2020, Article ID 2417681, 9 pages
https://doi.org/10.1155/2020/2417681

Research Article

Modeling the Dependence of Barometric Pressure with Altitude Using Caputo and Caputo–Fabrizio Fractional Derivatives

Muath Awadalla,[1] **Yves Yameni Noupoue Yannick**,[2] **and Kinda Abu Asbeh**[1]

[1]*Department of Mathematics and Statistics, College of Science, King Faisal University, Hafuf, Al Ahsa 31982, Saudi Arabia*
[2]*Institute of Statistics, Biostatistics and Actuarial Sciences (ISBA), Universite Catholique de Louvain (UCLouvain), Louvain-la-Neuve 1348, Belgium*

Correspondence should be addressed to Muath Awadalla; mawadalla@kfu.edu.sa

Received 14 August 2020; Revised 31 October 2020; Accepted 16 November 2020; Published 25 November 2020

Academic Editor: Sumit Chandok

This work is dedicated to the study of the relationship between altitude and barometric atmospheric pressure. There is a consistent literature on this relationship, out of which an ordinary differential equation with initial value problems is often used for modeling. Here, we proposed a new modeling technique of the relationship using Caputo and Caputo–Fabrizio fractional differential equations. First, the proposed model is proven well-defined through existence and uniqueness of its solution. Caputo–Fabrizio fractional derivative is the main tool used throughout the proof. Then, follow experimental study using real world dataset. The experiment has revealed that the Caputo fractional derivative is the most appropriate tool for fitting the model, since it has produced the smallest error rate of 1.74% corresponding to the fractional order of derivative $\alpha = 1.005$. Caputo–Fabrizio was the second best since it yielded an error rate value of 1.97% for a fractional order of derivative $\alpha = 1.042$, and finally the classical method produced an error rate of 4.36%.

1. Introduction

Natural phenomena are commonly used to indicate something that happens randomly or without human action. Besides, there exist many phenomena that occur as result of experiment. In either case, scientists are always interested in describing phenomena. Mathematical modeling is a powerful tool used over centuries for natural phenomenon study. Following time or space axes, occurrence of a phenomenon can be trendless, or it can show known mathematical trend. Whether or not a phenomenon shows a trend over elapsed time or space, there is always at least one mathematical tool that can be used for its description and modeling. In general, two mains approaches are used in mathematical modeling. These are deterministic and nondeterministic methods. Deterministic approach is mostly concerning with differential equations [1]. The method specificity is that given an initial value problem, other values can be predicted and computed using a function which is solution to a differential equation. Derivatives and Integrals are main tools used in finding solution to differential equations. Integer values were used as order of derivative until earlier 19th century [2] when the idea of fractional derivative was introduced. Earlier works on fractional differential equations (FDE) have focused on investigation of existence and uniqueness of the solution to designed models.

In recent decades, interest has increased among researchers who investigate efficiency of FDE in solving real-life problems. The approaches used are usually similar. Given a modeling problem that can be solved using differential equations, researchers will first check if there exists classical solution. Next, they will build FDE approach and finally the obtained results will be evaluated and discussed. Some interesting applied results are found in physics [3, 4], medical sciences [5], biology [6, 7], and economics [8], which represent a very tiny part of the existing literature.

In this work, a relationship between air pressure and altitude is studied using FDE. Caputo–Fabrizio fractional derivative is used to prove the existence and uniqueness of the proposed model. Moreover, the Caputo method is used in numerical simulation. Hence, it is first proven that an FDE model is appropriate to solve the problem, and secondly, experimental studies are carried out using two different types of fractional derivative and numerical solutions since there was no analytic form of the solution.

2. Elements of Fraction Calculus

This section is dedicated to the study of some elements of fractional calculus which are required in sequel of this work.

Definition 1 (see [9]). Consider a real number $\gamma > 0$; the one parameter Mittag–Leffler function is computed as $E_\gamma(x) = \sum_{k=0}^{\infty} (x^k/\Gamma(\gamma k + 1))$, where $x \in \mathbb{R}$ and Γ represents the usual gamma function defined as $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} \mathrm{d}t$, $\forall x > 0$.

The Mittag–Leffler function is useful in fractional calculus since it is often used for representing the solution of FDE.

Definition 2 (see [9]). Consider a real number $\gamma > 0$ representing a fractional order of derivative, the Caputo fractional derivative of order γ of a function h: $[0, +\infty] \longrightarrow \mathbb{R}$ is given by the formula

$$\left({}_{\mathrm{C}}D_{0^+}^{\gamma} h\right)(t) = \begin{cases} \displaystyle\int_0^t \frac{(t-s)^{p-\gamma-1} h^{(p)}(s)}{\Gamma(n-\gamma)} \mathrm{d}s, & p-1 < \gamma < p, \\ h^{(p)}(t), & \gamma \in \mathbb{N}, \end{cases} \tag{1}$$

with p being a strictly positive integer, that is $p \in \mathbb{N}^*$.

Definition 3 (see [9]). Consider a function $f \in H^1(c, d), c > d$ and $\gamma \in [0, 1]$; it follows that a new version of the Caputo fractional derivative is defined as

$$\left({}_{\mathrm{CF}}D_0^{\gamma} f\right)(x) = \frac{M(\gamma)}{(1-\gamma)} \int_c^x f'(u) \exp\left[-\frac{\gamma}{1-\gamma}(x-u)\right] \mathrm{d}u. \tag{2}$$

Moreover, if the function f is such that $f \notin H^1[c, d]$, then another new fractional derivative known as Caputo–Fabrizio fractional derivative is obtained as follows:

$$\left({}_{\mathrm{CF}}D_0^{q} f\right)(x) = \frac{M(\gamma)}{(1-\gamma)} \int_0^x f'(u) \exp\left[-\frac{\gamma}{1-\gamma}(x-u)\right] \mathrm{d}u. \tag{3}$$

Definition 4 (see [9]). Consider a real number $\gamma > 0$ representing a fractional order of integral; the Caputo–Fabrizio fractional integral of a continuous function f: $[0, +\infty) \longrightarrow \mathbb{R}$ is defined as follows (see [10]):

$$\left({}_{\mathrm{CF}}I_{0^+}^{\gamma} f\right)(x) = \frac{2(1-\gamma)}{2M(\gamma) - \gamma M(\gamma)} f(x) + \frac{2\gamma}{2M(\gamma) - \gamma M(\gamma)} \int_0^x f(u) \mathrm{d}u, \quad x \geq 0, \tag{4}$$

where $M(\gamma)$ is the normalization function having the property $M(0) = M(1) = 1$.

Definition 5 (see [9]). The Riemann–Liouville fractional integral of order $\gamma > 0$ for a function h: $[0, +\infty) \longrightarrow \mathbb{R}$ is defined as

$$\left({}_{\mathrm{RL}}I_{0^+}^{\gamma} h\right)(t) = \frac{1}{\Gamma(\gamma)} \int_0^x (x-u)^{\gamma-1} h(u) \mathrm{d}u, \tag{5}$$

provided that the right-hand side of the integral is pointwise defined on the open interval $(0, +\infty)$.

Definition 6 (see [9]). The Riemann–Liouville fractional derivative of order $\gamma > 0$ of a function h: $[0, +\infty) \longrightarrow \mathbb{R}$ is given by

$$\left({}_{\mathrm{RL}}D_{0^+}^{\gamma} h\right)(x) = \frac{1}{\Gamma(p-\gamma)} \left(\frac{\mathrm{d}^p}{\mathrm{d}x^p}\right) \int_0^x (x-u)^{p-\gamma-1} h(u) \mathrm{d}u, \tag{6}$$

where $p - 1 \leq \gamma < p$, $p \in \mathbb{N}$.

3. Literature Review on the Classical Model

In this section, a survey of existing works carried out on the study of variation of air pressure as function of altitude is provided. Air resistance is highly considered in aerodynamic and mechanics. In August 1960, a US army captain known as Joseph Kittinger [11] has experimentally proven the effect of air resistance during a free fall he did from 102,800 feet above the earth surface. His main interest was to study the air resistance during his jump. From various studies on the air resistance, it was proven that given a free-falling object, the air resistance is not a perfect function of the object speed, but, in many cases, the moving speed is often enough as a variable to describe the resistance. In general, if the falling object has a speed v, then air resistance is well approximated by

$$\mathrm{Ar}(v) = \lambda_1 v - \lambda_2 v|v|, \tag{7}$$

where λ_1 and λ_2 are constants. It has been proven also that if the object speed is above 24 m/s but below the sound speed, the linear term of equation (7) can be neglected and the air resistance is approximated by

$$\mathrm{Ar}(v) = -\lambda v|v|. \tag{8}$$

In the full report of the experiment by Joseph Kittinger [11], air pressure and resistance depend on altitude. In their work, Grigorie et al. [12] studied the measurement of an aircraft altitude using the air pressure. More important fact is that the authors described the atmospheric layers and how each layers' temperature and pressure are characterized. Standard Atmosphere is a reference manual adopted by International Civil Aviation Organization [13] and the US National oceanic and atmospheric administration [14]. It provides standardized data for variation of air pressure as a function of altitude. West, in his work [15], stated two main reasons why the barometric air pressure declines with an increasing of altitude. The physical principles behind these reasons are given by the following hydrostatic equation:

$$\mathrm{d}P = -\rho g\, dh, \tag{9}$$

and the ideal gas law:

$$P = \frac{\rho RT}{M}, \tag{10}$$

where h represents altitude; ρ is the density and it depends on altitude; g is the gravity acceleration at height h; R is the universal gas constant; M is the molecular weight of the air; P is the pressure; and T is the absolute temperature.

Note: it is common to denote the pressure and the density by $P(h)$ and $\rho(h)$, respectively, as they are functions of altitude.

A solution to differential equation (9) is obtained prior to establishment of the relation between $P(h)$ and $\rho(h)$. The gas state equation is given by

$$PV = n_M RT, \tag{11}$$

where n_M is the number of moles.

On the contrary, the relationships between the air mass m, volume V, mole mass M, and density ρ are defined as

$$m = n_M M \Leftrightarrow n_M = \frac{m}{M}, \tag{12}$$

$$\rho = \frac{m}{V} \Leftrightarrow m = \rho V. \tag{13}$$

Equations (12) and (13) inserted into equation (11) leads to the following relation:

$$\rho = \frac{M}{\mathrm{RT}} P. \tag{14}$$

A substitution of equation (14) into equation (9) leads to

$$\begin{cases} \dfrac{\mathrm{d}P}{\mathrm{d}h} = -\dfrac{gM}{\mathrm{RT}} P, \\ P(0) = P_0. \end{cases} \tag{15}$$

It is easy to obtain the solution of equation (15) through simple integration as follows:

$$\begin{aligned} \frac{\mathrm{d}P}{\mathrm{d}h} = -\frac{gM}{\mathrm{RT}} P &\Longrightarrow \frac{\mathrm{d}P}{P} = -\frac{gM}{RT}\mathrm{d}h \\ &\Longrightarrow \int \frac{\mathrm{d}P}{P} = -\frac{gM}{\mathrm{RT}} \int \mathrm{d}h \\ &\Longrightarrow \ln(P) = -\frac{gM}{\mathrm{RT}} h + \mathrm{cte} \\ &\Longrightarrow P(h) = e^{\mathrm{cte}} e^{-(gM/\mathrm{RT})h}. \end{aligned} \tag{16}$$

Using the initial condition stated as at $h = 0$ $P(0) = P_0$, it follows that the analytic solution to equation (15) is

$$P(h) = P_0 e^{-(gM/\mathrm{RT})h}. \tag{17}$$

4. Modeling the Pressure with Fractional Derivative

This section describes a model of the relationship between air pressure and altitude using fractional derivative is proposed. Assume that the derivative term in the differential equation (15) is taken in the fractional sense. Without loss of generality, let it be the Caputo fractional derivative; then, the following model is obtained:

$$\begin{cases} {}_C D_{0^+}^{\gamma} P(h) = -\dfrac{gM}{\mathrm{RT}} P(h), \\ P(0) = P_0 < \infty, \end{cases} \tag{18}$$

where γ is the fractional order of derivative.

Existence and uniqueness to the problem equation (18) is proven as follows prior to numerical simulation.

Applying the Caputo–Fabrizio fractional integral to equation (18) leads to

$$\begin{aligned} P(h) - P(0) = {} & \frac{2(1-\gamma)}{2M(\gamma) - \gamma M(\gamma)}\left(-\frac{gM}{\mathrm{RT}} P\right) \\ & + \frac{2\gamma}{2M(\gamma) - \gamma M(\gamma)} \int_0^h -\frac{gM}{\mathrm{RT}} P(s)\mathrm{d}s. \end{aligned} \tag{19}$$

For conformity and simplicity in notation, let the kernel function be chosen as

$$K(h, P(h)) = -\frac{gM}{\mathrm{RT}} P(h). \tag{20}$$

Denote by $H = C([0, T], \mathbb{R})$ the Banach space of all functions f, such that f are continuous from the closed interval $[0, T]$ to $\mathbb{R}$; moreover, let the space be endowed with the norm defined as $\|P\| = \sup_{0\le h\le T}|P(h)|$. Let an operator $E: H \longrightarrow H$ be defined as follows:

$$\begin{aligned} (EP)(h) = P(0) & + \frac{2(1-\gamma)}{2M(\gamma) - \gamma M(\gamma)} K(h, P(h)) \\ & + \frac{2\gamma}{2M(\gamma) - \gamma M(\gamma)} \int_0^h K(u, P(u))\mathrm{d}u. \end{aligned} \tag{21}$$

Equations (20) and (21) are tools used in sequel process of proving existence and uniqueness of the solution. Beside these equations are also some lemmas which are also set as preliminaries tools in the process of proving the existence and uniqueness of the solution to the proposed fractional model equation (18).

Lemma 1 (Nonlinear alternative of Leray–Shauder type, see [6]).

Given the open subset V of a Banach space S, $0 \in V$, and $G: \overline{V} \longrightarrow S$ be a contraction such that $G(\overline{V})$ is bounded then

(i) G has a fixed point in $\overline{V}$

(ii) $\exists \mu \in (0,1)$ and $v \in \partial V$ such that $v = \mu G(v)$ holds.

Lemma 2 (Arzela–Ascoli Theorem, see [6]). $G \subset C(S, \mathbb{R})$ is compact if and only if it is closed, bounded, and equicontinuous.

Lemma 3 (Krasnoselskii's Theorem, see [6]). Given the Banach space $(E, \|\cdot\|)$, closed convex $B \subseteq E$, A is open, where $A \subset B$, and $P \in A$, assume that $G\colon \overline{A} \longrightarrow B$ can be written as $G = G_1 + G_2$. In addition, $G(\overline{A})$ if it is a bounded set in B satisfying

(i) $G_1\colon \overline{A} \longrightarrow B$ is continuous and completely continuous

(ii) $G_2\colon \overline{A} \longrightarrow B$ is a contraction, a continuous nondecreasing function $\omega\colon [0,\infty] \longrightarrow [0,\infty]$, with $\omega(a_1) > a_1, a_1 > 0$, such that $|G_2(a_1) - G_2(a_2)| \leq \phi(\|a_1 - a_2\|)$, for any $a_1, a_2 \in \overline{A}$

Then, it follows that

(i) $G(a_0) = a_0, a_0 \in \overline{A}$

(ii) $\exists\, a \in \partial A$ and $\lambda \in (0,1)$ with $a = \lambda G(a) + (1-\lambda)P$

Lemma 4 (Banach's contraction mapping principle, see [6]).

Let $(S_., d_.)$ be a complete metric space; if $G\colon S_. \longrightarrow S$ is a contraction, then

(i) G has a unique fixed point $s_. \in S$, that is, $G(s_.) = s$

(ii) $\forall s_0 \in S$, we have $\lim_{n\to\infty} G^n(s_0) = s$, with $d(G^n(s_0), s) \leq (l_G^n/1 - l_G)d(s_0, G^n(s_0))$

4.1. Existence of Solutions

Theorem 1. *Let $K\colon [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function defined in a way that the following two assumptions hold:*

(i) $\exists P_K > 0$ such that $|K(h, P_1) - K(h, P_2)| \leq P_K |P_1 - P_2|, \forall h \in [0,T], \forall P_1, P_2 \in \mathbb{R}$

(ii) $|K(h, P)| \leq y(h)$, $\forall (h, P) \in [0,T] \times \mathbb{R}$, where, $y \in C([0,T], \mathbb{R}^+)$ with $\sup_{0\leq h\leq T}|y(h)| = \|y\|$

In addition, let it be assumed that the following relation holds on the given quantity $P_K(2\gamma/(2M(\gamma) - \gamma M(\gamma)))T < 1$. Having the above setting implies the existence of at least one solution to the initial value problem given by equation (18).

Proof of Theorem 1. Consider the close set $B_r = \{P \in H, \|P\| \leq r\}$ with the radius defined such that $r \geq P_0 + ((|2(1-\gamma)/(2M(\gamma) - \gamma M(\gamma))|) + (|2\gamma/(2M(\gamma) - \gamma M(\gamma))|T))\|y\|$.

Moreover, let us introduce the following two operators, E_1 and E_2, defined within the domain of B_r and given as follows:

$$(E_1 P)(h) = P(0) + \frac{2(1-\gamma)}{2M(\gamma) - \gamma M(\gamma)} K(h, P(h)),$$
$$(E_2 P)(h) = \frac{2\gamma}{2M(\gamma) - \gamma M(\gamma)} \int_0^h K(s, P(s))\mathrm{d}s. \tag{22}$$

Given two elements $P_1, P_2 \in B_r$, then it follows $\|E_1 P_1 + E_2 P_2\| \leq ((|2(1-\gamma)/(2M(\gamma) - \gamma M(\gamma))|) + (|2\gamma/(2M(\gamma) - \gamma M(\gamma))|T))\|y\| \leq r$; hence, $E_1 P_1 + E_2 P_2 \in B_r$.

The next step at this point of the proof is to show that operator E_2 is a contraction. One can observe first of all that $\forall h \in [0,T], \forall P_1, P_2 \in B_r$, and the following relation holds:

$$\left|E_2 P_1(h) - E_2 P_2(h)\right| = \left|\frac{2\gamma}{2M(\gamma) - \gamma M(\gamma)} \int_0^h (K(s, P_1(s)) - K(s, P_2(s)))\mathrm{d}s\right| \leq \frac{2\gamma}{2M(\gamma) - \gamma M(\gamma)} h\left|K(s, P_1(s)) - K(s, P_2(s))\right|. \tag{23}$$

Then,

$$\left\|E_2 P_1 + E_2 P_2\right\| \leq \frac{2\gamma}{2M(\gamma) - \gamma M(\gamma)} T p_K \left\|P_1 - P_2\right\| \leq \left\|P_1 - P_2\right\|. \tag{24}$$

Equation (24) clearly shows that operator E_2 is a contraction.

On the contrary, operator E_1 is continuous as a result of the continuity of P. Operator E_1 is uniformly bounded as $\|E_1 P\| \leq P_0 + (2(1-\gamma)/(2M(\gamma) - \gamma M(\gamma)))\|y\|$.

The compactness of operator E_1 is also proven as follows. For $h_1, h_2 \in [0,T]$, such that $(h_1 < h_2)$, it follows that

$$\left\|(E_1 P)(h_2) - (E_1 P)(h_1)\right\| \leq \frac{2(1-\gamma)}{2M(\gamma) - \gamma M(\gamma)} \left|K(h_2, P(h_2)) - K(h_1, P(h_1))\right|. \tag{25}$$

The right-hand side of the inequality defined by equation (25) approaches zero as $h_1 \longrightarrow h_2$. Moreover, on can also observe that the quantity $\|(E_1P)(h_2) - (E_1P)(h_1)\|$ does not depend on P, which implies the relative compactness of the operator E_1. Recall Lemma 2 is enough to conclude that E_1 is a compact operator on the closed set B_r. Moreover, recalling Lemma 3, the existence of a solution to equation (18) is thus proven. Q.E.D.

4.2. Uniqueness of the Solution

Theorem 2. *Let K: $[0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous satisfying (i) and assume moreover that $P_K((2(1-\gamma)/(2M(\gamma)-\gamma M(\gamma))) + (\gamma/(2M(\gamma)-\gamma M(\gamma)))T) < 1$; then, the solution to the initial value problem equation (18) is unique.*

Proof of Theorem 2. Consider the close set $B_r = \{p \in H, \|p\| \leq r\}$ with

$$r \geq \frac{P_0 + N((2(1-\gamma)/(2M(\gamma)-\gamma M(\gamma))) + (2\gamma/(2M(\gamma)-\gamma M(\gamma)))T)}{1 - P_K((2(1-\gamma)/(2M(\gamma)-\gamma M(\gamma))) + (2\gamma/(2M(\gamma)-\gamma M(\gamma)))T)}, \quad \text{where } N = \sup_{0\leq h\leq T} |K(h,0)|. \tag{26}$$

The first step of the proof is to show that $EB_r \subset B_r$. To do so, observe that $\forall P \in B_r, \forall h \in [0,T]$, and it follows that

$$|(EP)(h)| \leq P_0 + \frac{2(1-\gamma)}{2M(\gamma)-\gamma M(\gamma)}|K(h,P(h))| + \frac{2\gamma}{2M(\gamma)-\gamma M(\gamma)}\int_0^h |K(s,P(s))|\mathrm{d}s. \tag{27}$$

On the contrary, the following inequality is derived from the quantity or norm of $|K(h,P(h))|$:

$$\begin{aligned} |K(h,P(h))| &= |K(h,P(h)) - K(h,0) + K(h,0)| \\ &\leq |K(h,P(h)) - K(h,0)| + |K(h,0)| \\ &\leq P_K\|P\| + N \\ &\leq P_K r + N, \end{aligned} \tag{28}$$

Using equations (27) and (28) leads to the following relation:

$$\|EP\| \leq P_0 + \left(\frac{2(1-\gamma)}{2M(\gamma)-\gamma M(\gamma)} + \frac{2\gamma}{2M(\gamma)-\gamma M(\gamma)}T\right)(P_K r + N) \leq r. \tag{29}$$

Equation (29) implies that $EP \in B_r, \forall P \in B_r$ which means that, in general, $EB_r \subset B_r$.

The second step of this proof is to show that the operator E is a contraction. For that, observe that $\forall P_1, P_2 \in H$, and the following relation holds:

$$|EP_1(h) - EP_2(h)| \leq \frac{2(1-\gamma)}{2M(\gamma)-\gamma M(\gamma)}|K(h,P_1(h)) - K(h,P_2(h))| + \frac{2\gamma}{2M(\gamma)-\gamma M(\gamma)}\int_0^h |(K(s,P_1(s)) - K(s,P_2(s)))|\mathrm{d}s. \tag{30}$$

The relation given by equation (30) leads to

$$\|EP_1 - EP_2\| \leq \left(\frac{2(1-\gamma)}{2M(\gamma)-\gamma M(\gamma)} + \frac{2\gamma}{2M(\gamma)-\gamma M(\gamma)}T\right)P_K\|P_1 - P_2\| \leq \|P_1 - P_2\|. \tag{31}$$

From equation (31), it is clearly observed that operator E is a contraction. Hence, using Lemma 4, the solution to the initial value problem defined by equation (18) is unique over the closed interval $[0,T]$. Q.E.D.

Having proven the existence and the uniqueness of a solution to the problem defined by equation (18), it is important to investigate the shape of the said solution.

Lemma 5 (see page 135 in [9]). Given $\gamma > 0$, denote by n the integer part of γ and $\beta \in \mathbb{R}$. Consider the initial value problem defined by

$$D_{0^+}^{\gamma} f(t) = \beta f(t) + g(t), \tag{32}$$

with $f^{(i)}(0) = f_0^{(i)}$, $i = 0, 1, \ldots, n-1$, and $g \in C[0,T]$ is a given function. Equation (32) has a solution expressible in the following form:

$$f(t) = \sum_{i=0}^{n-1} f_0^{(i)} w_i(t) + \tilde{f}(t), \tag{33}$$

with

$$\tilde{f}(t) = \begin{cases} J_0^\gamma g(t), & \text{if } \beta = 0, \\ \dfrac{1}{\beta}\displaystyle\int_0^t g(t-s) w_0'(s) \mathrm{d}s & \text{if } \beta \neq 0, \end{cases} \tag{34}$$

where $w_i(t) = J_0^i e_\gamma(t)$, $i = 0, 1, \ldots, n-1$, *and* $e_\gamma(t) = E_\gamma(\beta t^\gamma)$.

Note: if γ in Lemma 5 is such that $0 < \gamma < 1$, then solution to the problem equation (32) is expressed using equations (33) and (34) as

$$f(t) = f_0^{(0)} E_\gamma(\beta t^\gamma) + \gamma \int_0^t g(t-s) s^{\gamma-1} E_\gamma'(\beta t^\gamma) \mathrm{d}s. \tag{35}$$

Using Lemma 5, the analytic form of the exact solution to equation (18), with the derivative taken in the Caputo sense is given by

$$_C P(h) = P_0 E_\alpha\left(-\frac{gM}{\mathrm{RT}} h^\alpha\right). \tag{36}$$

Lemma 6 (see [16]). Let $\gamma > 0$ and n be a natural number such that $\gamma \in (n-1, 1)$; if $f \in AC^n[a, b]$, then the unique solution of the following initial value problem

$$\left\{ {}_{\mathrm{CF}}D_{0^+}^\gamma f(t) = \sigma(t), \quad t \geq 0, f(0) = f_0, \right. \tag{37}$$

is given by

$$f(t) = f(0) + a_\gamma(\sigma(t) - \sigma(0)) + b_\gamma I^1 \sigma(t), \quad t \geq 0, \tag{38}$$

where $I^1\sigma(t)$ *is the primitive of* σ *and* $a_\gamma = (2(1-\gamma)/((2-\gamma)M(\gamma)))$, $b_\gamma = (2\gamma/((2-\gamma)M(\gamma)))$.

Considering the Caputo–Fabrizio fractional derivative, the initial value problem equation (18) becomes

$$\left\{ {}_{\mathrm{CF}}D_{0^+}^\alpha P(h) = -\frac{gM}{\mathrm{RT}} P(h), P(0) = P_0. \right. \tag{39}$$

Recalling Lemma 6, the analytic form of the exact solution to equation (39) is written as

$$_{\mathrm{CF}}P(h) = P_0 e^{-(2\alpha/(2(\mathrm{RT}/gM)+2(1-\alpha)))h}. \tag{40}$$

5. Simulation Studies

The experimental dataset used in this section was retrieved from Sable Systems International website [16], where real data from an experiment that investigates the relationship between altitude and atmospheric pressure is given. The measurement unit of the pressure is KPa. Altitude is given in meters. The experimental dataset contains negative as well as positive values of altitude. However, we conducted the study using only data from the positive altitude. The constant values which are in equations (15), (36), and (39) are the following $P_0 = 101.33$; $g = 9.82$; $M = 29$; $R = 8.31$; $T = 273000$; $T_0 = 293000$; and $\beta = (200 \times T_0)^{-1}$. The Caputo model equation (36) and the Caputo–Fabrizio model equation (40) were used to fit the true dataset. Prior to fitting the data, optimization routines were used to determine the fractional order of derivative for which each model would best fit the dataset. The routine used here consists of using all the possible value of the fractional order of derivative in the interval [0.9, 1.2] with a moving step of 10-2, while recording alongside the error percentage for each value.

Figure 1 shows that the Caputo method has a singularity for $\alpha = 0.999$, in fact the error rate produced by $\alpha = 0.999$ suddenly jumped to 705.7%. Figure 2, on the contrary, shows the Caputo–Fabrizio error rate variation over the closed interval [0.9, 1.2]. Unlike the Caputo method, the curve depicting the error rate is smoothed over the interval. This means there is no singularity for this method. Interestingly, the minimum error, 1.97% appears at $\alpha = 1.042$. Recall that when $\alpha = 1$, this is the classical case.

Figures 3 and 4 depict the graphs of atmospheric pressure variation with altitude. Figure 3 shows the predicted data using the classical method alongside the true data, whereas Figure 4 shows the predicted data using the Caputo–Fabrizio method (α= 1.042) alongside the true dataset. It is clearly observed from both figures that modeling with the Caputo–Fabrizio method is better than with the classical method.

Recall from Figure 1 that the Caputo method reaches the minimum error at $\alpha = 1.005$ and a singularity at $\alpha = 0.99$. Figure 5 shows the predicted data using the Caputo method (α= 1.005) alongside the true data. One can clearly see that the error is very small. In order to observe prediction behavior at a singular point, Figure 6 was plotted using predicted data by the Caputo method (α= 0.99) alongside the true dataset. The graph shows a weird behavior due to singularity.

6. Physical Meaning and Motivation of the Study

This section provides physical meaning of the relationship between altitude and atmospheric pressure. It also motivates the study. Recall that accuracy is the main metric of phenomena modeling being deterministic or stochastic. The relationship between altitude and atmospheric pressure and their consequences are well known. The said relationship finds its application in some of our daily activities. For instance, an airplane flies at high altitude and its atmospheric pressure decreases with altitude prompting pilot to use special pressurizing instruments to maintain the pressure of the plane at a convenient level. Failure to regulate a plane atmospheric pressure is a serious threat to the passenger's lives. A sad history of aviation happened in 2005, precisely on Sunday 14th August 2005 when the crew of Helios flight 522 en route to Athens from Cyprus omitted to activate the pressurizing instrument on board. The plane was then deprived from oxygen and crashed [17, 18]. A morality of the Helios flight 522 case is that aviation needs to provide airplane with sophisticate barometers able produce the atmospheric pressure as a function of plane altitude with high accuracy.

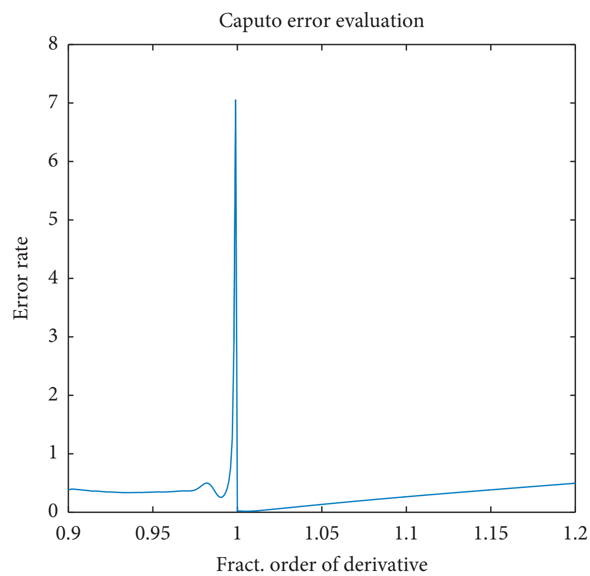

Figure 1: Error rate variation: at $\alpha = 1.005$, Caputo minimizes the error rate to 1.74%.

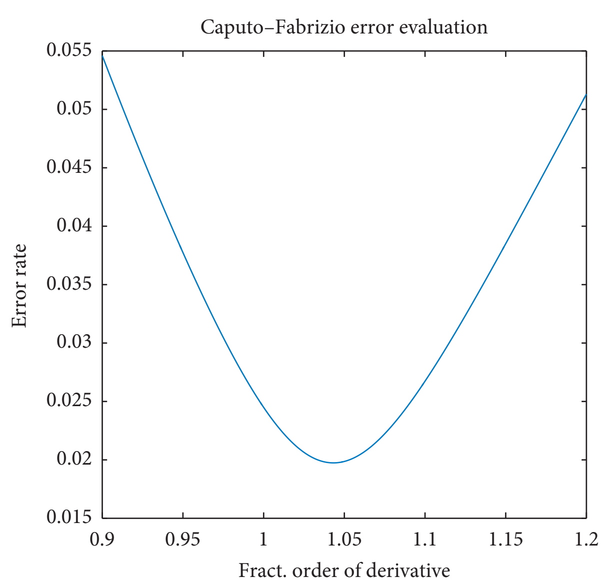

Figure 2: Error rate variation: at $\alpha = 1.042$, Caputo–Fabrizio minimizes the error rate to 1.97%.

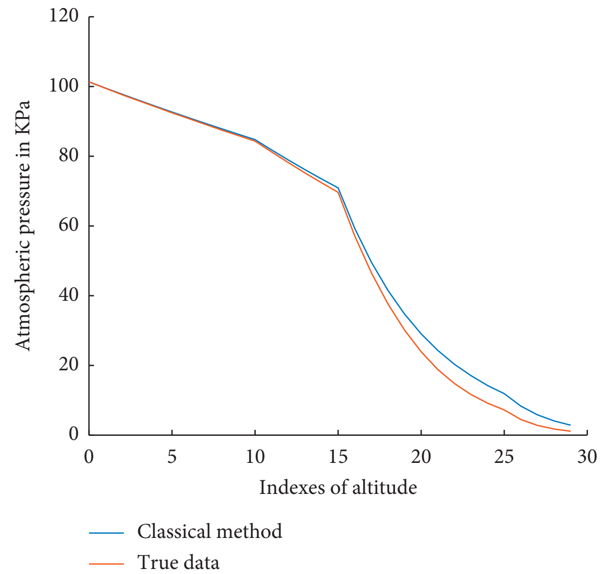

Figure 3: Classical method: error rate is 4.36%.

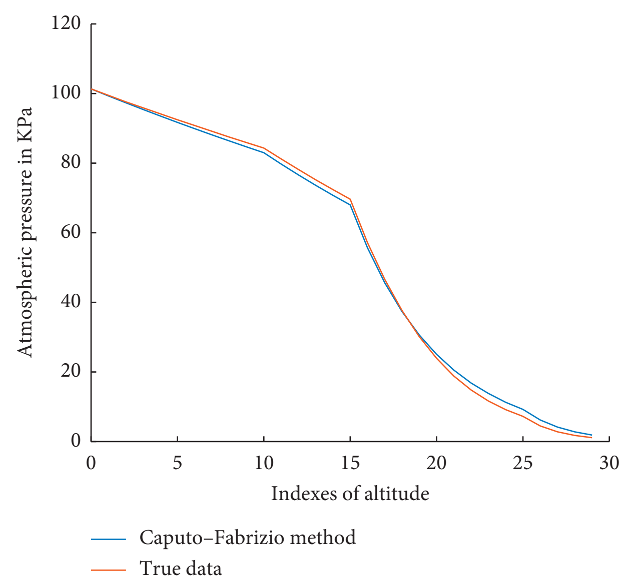

Figure 4: Caputo–Fabrizio with $\alpha = 1.042$: the error rate is 1.97%.

Another field of application where controlling atmospheric pressure is required is foods processing and its supply chain. Indeed, we live in globalization era, thus human and goods are subject to long range traveling over the world. It is common, for instance, for one living in western part of the world to consume dates (fruit) that was produce in the Eastern part. That is just a basic example to highly the fact in foods cans supply chain, atmospheric pressure is a determine element to consider. In case there is high atmospheric pressure different between the manufacturing hub and the end market, special care must be taken for the goods to remain safe.

In this work, the statement "classical approach" is used to refer to the relationship between altitude and atmospheric

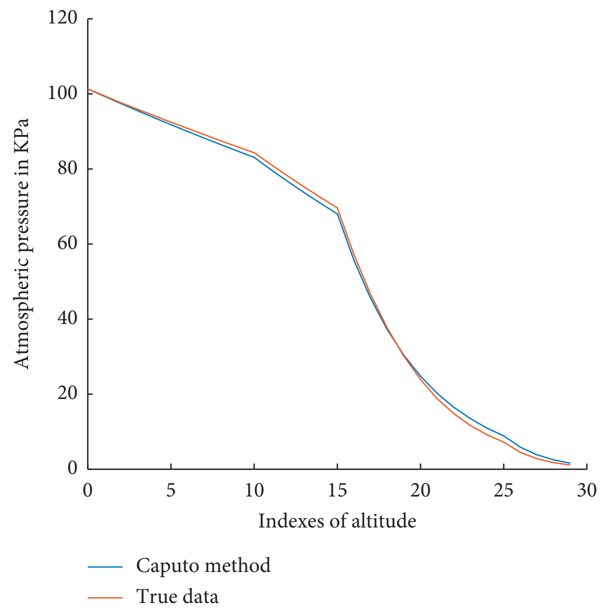

Figure 5: Caputo with $\alpha = 1.005$: the error rate is 1.74%.

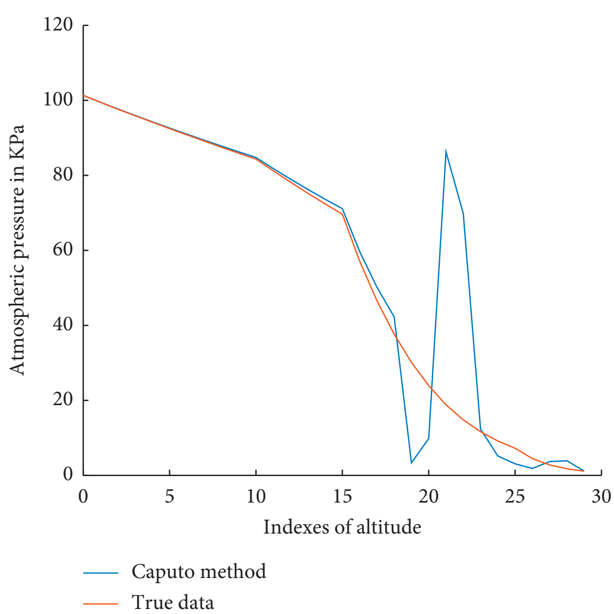

Figure 6: Caputo with $\alpha = 0.99$: the error rate is 25.48%.

pressure model as introduced in [11–15], whereas "fractional approach" refers to the proposed model built upon fractional derivative. The main result of this work is that "fractional approach" can best fit the model than "classical approach." A real-life application of the results obtained in this work is that the proposed model can be used to produce pressure related devices such barometers.

7. Conclusion and Discussion

The aims of this work were all reached. In fact, fractional differential equation was used to build and to solve altitude-pressure relationship which is well known in the classical literature. Without loss of generality, Caputo–Fabrizio fractional derivative was used to prove the existence and uniqueness of a proposed fractional model of atmospheric pressure variation with respect to altitude.

After proving that the fractional differential equation model of the relationship altitude-pressure was well defined as well as that it has a unique solution, it was worth to find that solution. Caputo and Caputo–Fabrizio methods were both selected to express the solution of the defined problem.

In the numerical simulation, the performance of each of the approach used was evaluated. The Caputo method showed singularity points for some values of the fractional order of derivative which were close to 1. However, it has best fit the experimental data since it has yield an error rate which was as smaller as 1.74% for a fractional order of derivative $\alpha = 1.005$. This performance was followed by the Caputo–Fabrizio method which for $\alpha = 1.042$ has produced a minimum error rate of 1.97%. Finally, the classical approach has produced an error rate of 4.36%. As a matter of fact, the Caputo fractional derivative or the Caputo–Fabrizio fractional derivative performed better than their classical counterpart in solving the problem defined in this work. This result can be added to the nonexhaustive list of problems for which fraction calculus approach occurred to be more efficient than existing classical approach in finding their solutions.

Data Availability

The dataset used in this study was retrieved from the following website: https://www.sablesys.com/support/technical-library/barometric-pressure-vs-altitude-table/.

Disclosure

The authors planned to present preliminaries results of this work at Southeastern-Atlantic Regional Conference on Differential Equations on October 26-27, 2019, at Embry-Riddle Aeronautical University, Daytona Beach Campus [19]. However, due to reasons beyond control, the authors could not make it to the conference venue; hence, the project was aborted.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors, M. A, Y. Y. Y, and K. A., contributed to each part of this work equally and read and approved the final version of the manuscript.

Acknowledgments

The authors extend their appreciation to the Deputyship for Research & Innovation Ministry of Education in Saudi Arabia for funding this research work through the project no. IFT20135.

References

[1] H. J. Sussmann, "On the gap between deterministic and stochastic ordinary differential equations," *The Annals of Probability*, vol. 6, no. 1, pp. 19–41, 1978.

[2] L. Debnath, "A brief historical introduction to fractional calculus," *International Journal of Mathematical Education in Science and Technology*, vol. 35, no. 4, pp. 487–501, 2004.

[3] Y. Z. Povstenko, "Fractional heat conduction equation and associated thermal stress," *Journal of Thermal Stresses*, vol. 28, no. 1, pp. 83–102, 2004.

[4] M. Awadalla and Y. Yameni, "Modeling exponential growth and exponential decay real phenomena by Ψ-caputo fractional derivative," *Journal of Advances in Mathematics and Computer Science*, vol. 28, no. 2, pp. 1–13, 2018.

[5] M. Awadalla, Y. Y. Yameni, and K. Abuassba, "A new fractional model for the cancer treatment by radiotherapy using the hadamard fractional derivative," 2019.

[6] Y. Y. Y. Noupoue, Y. Tandoğdu, and M. Awadalla, "On numerical techniques for solving the fractional logistic differential equation," *Advances in Difference Equations*, vol. 2019, no. 1, Article ID 108, 2019.

[7] C. Ionescu, A. Lopes, D. Copot, J. A. T. Machado, and J. H. T. Bates, "The role of fractional calculus in modeling biological phenomena: a review," *Communications in Nonlinear Science and Numerical Simulation*, vol. 51, pp. 141–159, 2017.

[8] J. T. Machado and M. E. Mata, "Pseudo phase plane and fractional calculus modeling of western global economic downturn," *Communications in Nonlinear Science and Numerical Simulation*, vol. 22, no. 1-3, pp. 396–406, 2015.

[9] K. Diethelm, *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type*, Springer Science & Business Media, Berlin, Germany, 2010.

[10] J. Losada and J. J. Nieto, "Properties of a new fractional derivative without singular Kernel," *Progress in Fractional Differentiation and Applications*, vol. 1, no. 2, pp. 87–92, 2015.

[11] P. Mohazzabi and J. H. Shea, "High-altitude free fall," *American Journal of Physics*, vol. 64, no. 10, pp. 1242–1246, 1996.

[12] T. L. Grigorie, L. Dinca, J. I. Corcau, and O. Grigorie, "Aircrafts' altitude measurement using pressure information: barometric altitude and density altitude," *WSEAS Transactions on Circuits and Systems*, vol. 9, no. 7, pp. 1109–2734, 2010.

[13] J. Schenkman, "International civil aviation organization," 1955.

[14] U. S. Atmosphere, *National Oceanic and Atmospheric Administration*, National Aeronautics and Space Administration, United States Air Force, Washington, DC, USA, 1976.

[15] J. B. West, "Prediction of barometric pressures at high altitudes with the use of model atmospheres," *Journal of Applied Physiology*, vol. 81, no. 4, pp. 1850–1854, 1996.

[16] J. Richardson, *Barometric Pressure vs. Altitude Table*, Sable Systems International, North Las Vegas, NV, USA, 2019, https://www.sablesys.com/support/technical-library/barometric-pressure-vs-altitude-table/.

[17] S. W. Dekker, "Past the edge of chaos," *Human Factors and Aerospace Safety*, vol. 5, no. 3, p. 267, 2005.

[18] C. K. Allison, K. M. Revell, R. Sears, and N. A. Stanton, "Systems theoretic accident model and process (STAMP) safety modelling applied to an aircraft rapid decompression event," *Safety Science*, vol. 98, pp. 159–166, 2017.

[19] M. Awadalla, Y. Yannick, and K. Abuassba, "Modeling the dependence of atmospheric pressure with altitude using Caputo and Caputo-Fabrizio fractional derivatives," 2020, https://commons.erau.edu/searcde/2019-39th/presentation/9/.