Research Article

New Refinements and Improvements of Some Trigonometric Inequalities Based on Padé Approximant

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A multiple-point Padé approximant method is presented for approximating and bounding some trigonometric functions in this paper. We give new refinements and improvements of some trigonometric inequalities including Jordan’s inequality, Kober’s inequality, and Becker-Stark’s inequality. The analysis results show that our conclusions are better than the previous conclusions.

1. Introduction

Inequalities involving trigonometric functions are used in many areas of pure and applied mathematics. Trigonometric inequalities have attracted many researchers. Many improvements of Jordan’s inequality [1–11], Kober’s inequality [12–16], and Becker-Stark’s inequality [4, 17, 18] have been obtained. Recently, Bercu presented a Padé approximant method [19] and obtained the following inequalities:

\[-7x^2 + 60 \frac{\sin(x)}{x} < \frac{11x^4 - 360x^2 + 2520}{60x^2 + 2520}, \quad 0 < x < \frac{\pi}{2}, \quad (1)\]

\[17x^4 - 480x^2 + 1080 \frac{\cos(x)}{x} < \frac{3x^4 - 56x^2 + 120}{4x^2 + 120}, \quad 0 < x < \frac{\pi}{2}, \quad (2)\]

\[-28x^4 - 600x^2 + 7200 \frac{\tan(x)}{x} < \frac{22x^8 - 60x^6 - 4680x^4 - 237600x^2 + 2721600}{1020x^6 + 14040x^4 - 1144800x^2 + 2721600}, \quad 0 < x < 1.5701. \quad (3)\]
Zhang et al. [20] gave the improvements of inequalities (1)–(3):

\[
-5x^6 + 364x^4 - 9240x^2 + 60480 \leq \frac{\sin(x)}{x} \leq \frac{551x^4 - 22260x^2 + 166320}{15(5x^4 + 36x^2 + 11088)}, \quad 0 \leq x \leq \frac{\pi}{2}
\]

\[
-13x^6 + 660x^4 - 9720x^2 + 20160 \leq \frac{\cos(x)}{x} < \frac{313x^4 - 6900x^2 + 15120}{13x^4 + 660x^2 + 15120}, \quad 0 \leq x \leq \frac{\pi}{2}
\]

\[
\frac{21(495 - 60x^2 + x^4)}{10395 - 4725x^2 + 210x^4 - x^6} < \frac{\tan(x)}{x} < \frac{T_1(x)}{105(\pi^2 - 4x^2)} \cdot T_2(x), \quad 0 \leq x \leq \frac{\pi}{2}
\]

where

\[
T_1(x) = (\pi^4 - 840\pi^2 + 75600\pi^2 - 665280)x^6 + (210\pi^6 + 52920\pi^4 - 7620480\pi^2 + 69854400)x^4
\]

\[
+ (-17955\pi^6 + 1323000\pi^4 + 52390800\pi^2 - 628689600)x^2 + (155925(\pi^4 - 112\pi^2 + 1008))\pi^2,
\]

\[
T_2(x) = (26\pi^4 - 2664\pi^2 + 23670)x^4 + (-666\pi^4 + 73980\pi^2 - 665280)x^2
\]

\[
+ (1485\pi^4 - 166320\pi^2 + 1496880).
\]

In this paper, we present a multiple-point Padé approximant method for approximating and bounding some trigonometric functions. New refinements and improvements of related trigonometric inequalities are obtained, including Jordan’s inequality, Kober’s inequality, and Becker-Stark’s inequality. We introduce the concept of maximum error to compare our results with the previous methods. The results show that our bounds are tighter than the previous conclusions.

2. Multiple-Point Padé Approximant Method

The Padé approximant has been studied in many literature studies [19, 21–24]. In particular, Bercu et al. presented good results of several trigonometric inequalities using the Padé approximant. In this section, we present a multiple-point Padé approximant method. Given a bounded smooth function \( f(x) \), let

\[
R(x) = \frac{\sum_{i=0}^{p} a_i x^i}{1 + \sum_{j=1}^{q} b_j x^j}
\]

be a rational polynomial interpolating of \( f(x) \) at multiple points \( x_1, x_2, \ldots, x_k \) such that

\[
E^{(i)}(x_1) = 0, E^{(i)}(x_2) = 0, \ldots, E^{(i)}(x_k) = 0,
\]

\[
i_1 = 0, 1, \ldots, i_1, i_2 = 0, 1, \ldots, i_2, \ldots, i_k = 0, 1, \ldots, l_k,
\]

where \( E(x) = (1 + \sum_{i=1}^{q} b_i x^i) \cdot f(x) - (\sum_{i=0}^{p} a_i x^i) \) and \( p \geq 0 \) and \( q \geq 1 \) are two given integers. There are \( p + q + 1 \) unknowns in equation (9), \( a_i \) and \( b_j, \quad i = 0, 1, 2, \ldots, p, j = 1, 2, \ldots, q \). By selecting suitable values of \( l_1, l_2, \ldots, l_k \), we can obtain the polynomial \( R(x) \) by solving equation (9).

The general Padé approximant method is a special case of the multiple-point Padé approximant. Here, we just need to consider one point. If \( f \) can be written as a formal power series \( f(x) = c_0 + c_1 x + c_2 x^2 + \cdots \), where the coefficients \( c_j, j = 0, 1, 2, \ldots \), are constant. Taylor’s expansion is one of the most common ways to get a power series of a function. The Padé approximant \( R(f)(x) \) of degree \( (p,q) \) of the function \( f \) is determined by

\[
\begin{align*}
a_0 &= c_0, \\
a_1 &= c_0 b_1 + c_1, \\
a_2 &= c_0 b_2 + c_1 b_1 + c_2, \\
& \vdots \\
a_p &= c_0 b_p + \cdots + c_{p-1} b_1 + c_p, \\
0 &= c_{p+1} + c_p b_1 + \cdots + c_{p+q} b_q, \\
& \vdots \\
0 &= c_{p+q} + c_{p+q-1} b_1 + \cdots + c_p b_q.
\end{align*}
\]

The Padé approximant is considered the “best” approximation of a function by a rational function of a given degree. The rational approximation is also good for series with alternation terms and poor polynomial convergence.
This is our motivation of using the Padé approximant to approximate trigonometric functions and improve these trigonometric inequalities. Different values of \( p \) and \( q \) will affect the approximate performance. By selecting suitable values of \( p \) and \( q \), we can obtain the “best” approximant. Let \((p, q) = (k, k)\), and we can obtain a simple result. The result is a special case of (10).

It is well known that

\[
\tan(x) = \sum_{n=1}^{\infty} T(n)x^{2n-1},
\]

where

\[
T(n) = \frac{2^{2n}(2^{2n}-1)}{(2n)!}[B_{2n}],
\]

for \( x \in (0, \pi/2) \), \( n \in \mathbb{N}_0 \), and \( B_i \) are Bernoulli’s numbers.

Using the Padé approximant and equation (11), we obtain a better approximation of tangent function. Here, we need to pay attention to the value of \( c_j \) in formula (10). Let \( c_{2j} = 0 \), \( c_{2j-1} = T(j) \), and \( T(j) \) is given in (12); we can obtain the Padé approximant of \( \tan(x) \). In the same way, we can also obtain the Padé approximants of other trigonometric functions.

Table 1 gives the comparison between the Padé approximant and the Taylor series expansion of tangent function. It is easy to see that the maximum approximation error of the Padé approximant is less than the error of the corresponding Taylor polynomial. The advantage of the Padé approximant is more obvious with the increase of the polynomial degree. The bottom row of Table 1 shows the maximum approximation error of the Taylor polynomial is \( 6.0401 \times 10^{-3} \); however, the maximum approximation error of the Padé approximant is \( 2.2531 \times 10^{-9} \). At the same time, we can find that the form of the Padé approximant is simpler because of its lower degree.

3. New Improvements of Jordan’s, Kober’s, and Becker-Stark’s Inequalities

In this section, we give new improvements of Jordan’s, Kober’s, and Becker-Stark’s inequalities based on the Padé approximant.

**Theorem 1.** For \( \forall x \in \Gamma = [0, \pi/2] \), we have that

\[
\frac{3113510400 - 498960000x^2 + 22619520x^4 - 451440x^6 + 4620x^8 - 23x^{10}}{19958400 (156 + x^2)} \leq \frac{\sin(x)}{x} \leq \frac{19958400 - 3144960x^2 + 136080x^4 - 2448x^6 + 19x^8}{181440 (110 + x^2)} \]

(13)

**Proof.** Inequality (13) is equivalent to

\[
\begin{cases}
19958400 (156 + x^2) \sin(x) - (3113510400 - 498960000x^2 + 22619520x^4 - 451440x^6 + 4620x^8 - 23x^{10})x \geq 0, \\
181440 (110 + x^2) \sin(x) - (19958400 - 3144960x^2 + 136080x^4 - 2448x^6 + 19x^8)x \leq 0.
\end{cases}
\]

(14)

It is well known that

\[
\alpha(x) - \frac{x^{15}}{130767436800} \leq \sin(x) \leq x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} + \frac{x^9}{362880} - \frac{x^{11}}{39916800} + \frac{x^{13}}{6227020800} = \alpha(x).
\]

(15)

By inequality (15), we have that
Theorem 2. For \( \forall x \in \Gamma \), we have that
\[
\frac{131040 - 62160x^2 + 3814x^4 - 59x^6}{2(65520 + 1680x^2 + 17x^4)} \leq \cos(x) \\
\leq \frac{2(65520 + 1680x^2 + 17x^4)\cos(x) - (131040 - 62160x^2 + 3814x^4 - 59x^6) \geq 0}{20160(90 + x^2)\cos(x) - (1814400 - 887040x^2 + 65520x^4 - 1680x^6 + 17x^8) \leq 0}.
\]

It is well known that
\[
\beta(x) = \frac{x^{14}}{87178291200} \leq \cos(x) \\
\leq 1 - \frac{x^2}{2} + \frac{x^4}{24} - \frac{x^6}{720} + \frac{x^8}{40320} - \frac{x^{10}}{3628800} + \frac{x^{12}}{479001600} = \beta(x).
\]

By inequality (19), we have that
\[
2(65520 + 1680x^2 + 17x^4)\cos(x) - (131040 - 62160x^2 + 3814x^4 - 59x^6) \\
\geq 2\left(65520 + 1680x^2 + 17x^4\right)\left(\beta(x) - \frac{x^{14}}{87178291200}\right) - (131040 - 62160x^2 + 3814x^4 - 59x^6) \\
= \frac{x^{12}}{43589145600}\left(8321040 - 168168x^2 + 1414x^4 - 17x^6\right) \\
\geq \frac{x^{12}}{43589145600}\left(8321040 - 168168 \times 2^2 - 17 \times 2^6\right) \geq 0, \quad \forall x \in \Gamma,
\]
\[
20160(90 + x^2)\cos(x) - (1814400 - 887040x^2 + 65520x^4 - 1680x^6 + 17x^8) \\
\leq 20160(90 + x^2)\beta(x) - (1814400 - 887040x^2 + 65520x^4 - 1680x^6 + 17x^8) \\
= \frac{x^{12}}{23760}(-42 + x^2) \leq 0, \quad \forall x \in \Gamma.
\]
Theorem 3. For $\forall x \in \Gamma$, we have that

$$\frac{T_3(x)}{T_4(x)} < \frac{\tan(x)}{x} < \frac{T_1(x)}{105 (\pi^2 - 4x^2) \cdot T_2(x)}, \quad 0 \leq x \leq \frac{\pi}{2}$$

(22)

Then, the proof of inequality (17) is completed. \( \square \) where $T_1(x)$ and $T_2(x)$ are defined in inequality (6).

$$T_3(x) = \left[(-1679616\sqrt{3} - 2654208\sqrt{2} - 710208)\pi + 16376256\sqrt{3} + 21233664\sqrt{2} - 35230464\right]x^8$$
$$+ \pi[(3079296\sqrt{3} + 4644864\sqrt{2} + 1270080)\pi - 27713664\sqrt{3} - 39813120\sqrt{2} + 62923392]x^7$$
$$+ \pi^2[(-2297808\sqrt{3} - 3280896\sqrt{2} - 934596)\pi + 18869436\sqrt{3} + 30228480\sqrt{2} - 45416304]x^6$$
$$+ \pi^3[(894240\sqrt{3} + 1198080\sqrt{2} + 366210)\pi - 6632442\sqrt{3} - 11870208\sqrt{2} + 16935408]x^5$$
$$+ \pi^4[(-191484\sqrt{3} - 238592\sqrt{2} - 81833)\pi + 1272105\sqrt{3} + 2535424\sqrt{2} - 3429916]x^4$$
$$+ \pi^5\left(21384\sqrt{3} + 24576\sqrt{2} + \frac{20295}{2}\right)\pi - 252963\sqrt{3} - 278528\sqrt{2} + 355544 \right]x^3$$
$$+ \pi^6\left(-972\sqrt{3} - 1024\sqrt{2} - \frac{1153}{2}\right)\pi + 5103\sqrt{3} + 12288\sqrt{2} - 14572 \right]x^2 + \pi^8$$

(23)

$$T_4(x) = \pi[(-279936\sqrt{3} - 663552\sqrt{2} - 150336)\pi + 5038848\sqrt{3} + 2654208\sqrt{2} - 7537536]x^7$$
$$+ \pi^2[(-606528\sqrt{3} + 1492992\sqrt{2} + 347328)\pi - 11617344\sqrt{3} + 5308416\sqrt{2} + 16604352]x^6$$
$$+ \pi^3[(-546264\sqrt{3} - 1400832\sqrt{2} - 337140)\pi + 11209104\sqrt{3} + 4276224\sqrt{2} - 15206904]x^5$$
$$+ \pi^4[265032\sqrt{3} + 709632\sqrt{2} + 178146)\pi - 5860188\sqrt{3} - 1769472\sqrt{2} + 7497936]x^4$$
$$+ \pi^5[(-74844\sqrt{3} - 209408\sqrt{2} - 55319)\pi + 1790424\sqrt{3} + 395264\sqrt{2} - 2148668]x^3$$
$$+ \pi^6\left[12312\sqrt{3} + 35968\sqrt{2} + \frac{20179}{2}\right]\pi - 319059\sqrt{3} - 45056\sqrt{2} + 357852 \right]x^2$$
$$+ \pi^7\left(-2187\sqrt{3} - 3328\sqrt{2} - 1000\right)\pi + 30618\sqrt{3} + 2048\sqrt{2} - \frac{64111}{2} \right]x^1$$
$$+ \pi^8\left[81\sqrt{3} + 128\sqrt{2} + \frac{83}{2}\right]\pi - 1215\sqrt{3} + 1185 \right]x^0 + \pi^9$$

(24)

Proof. The left side $(T_3(x)/T_4(x)) < (\tan(x)/x)$ of inequality (22) is equivalent to

$$e_3(x) = \sin(x) - T_3(x) \geq 0,$$
$$e_4(x) = \cos(x) - T_4(x) \leq 0,$$

where $\sin(x) = (\sin(x)/x)$.

It is obvious that

$$e_3(x) = \sin(x) - T_3(x) \geq 0,$$
$$e_4(x) = \cos(x) - T_4(x) \leq 0,$$

where $\sin(x) = (\sin(x)/x)$.
Table 2: Comparison of the maximum errors between sinc($x$) and its bounds for different methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\text{MaxError}_l$</th>
<th>$\text{MaxError}_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bercu [19] (inequality (1))</td>
<td>$2.5981 \times 10^{-3}$</td>
<td>$6.2382 \times 10^{-3}$</td>
</tr>
<tr>
<td>Zhang et al. [20] (inequality (4))</td>
<td>$1.0615 \times 10^{-6}$</td>
<td>$1.7998 \times 10^{-6}$</td>
</tr>
<tr>
<td>Results of this paper (inequality (13))</td>
<td>$1.3042 \times 10^{-10}$</td>
<td>$1.3411 \times 10^{-8}$</td>
</tr>
</tbody>
</table>

Table 3: Comparison of the maximum errors between $\cos(x)$ and its bounds for different methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\text{MaxError}_l$</th>
<th>$\text{MaxError}_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bercu [19] (inequality (2))</td>
<td>$6.8710 \times 10^{-4}$</td>
<td>$6.8832 \times 10^{-4}$</td>
</tr>
<tr>
<td>Zhang et al. [20] (inequality (5))</td>
<td>$1.3987 \times 10^{-5}$</td>
<td>$2.9435 \times 10^{-5}$</td>
</tr>
<tr>
<td>Results of this paper (inequality (17))</td>
<td>$2.9194 \times 10^{-7}$</td>
<td>$2.0648 \times 10^{-7}$</td>
</tr>
</tbody>
</table>

Table 4: Comparison of the maximum errors between $\tan(x)$ and its bounds for different methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>$\text{MaxError}_l$</th>
<th>$\text{MaxError}_u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zhang et al. [20] (inequality (6))</td>
<td>$2.0690 \times 10^{-1}$</td>
<td>$2.7941 \times 10^{-6}$</td>
</tr>
<tr>
<td>Results of this paper (inequality (22))</td>
<td>$3.0498 \times 10^{-5}$</td>
<td>$2.7941 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

\[
sinc^{(9)}(x) = \frac{g(x)}{x^{10}},
\]

\[
\cos^{(9)}(x) = -\sin(x) \leq 0,
\]

where \(g(x) = (x^9 - 72x^7 + 3024x^5 - 60480x^3 + 362880x)\) \(\cos(x) + (-9x^8 + 504x^6 - 1520x^4 + 181440x^2 - 362880)\) \(\sin(x)\), and we have \(g'(x) = -x^8 \sin(x) \leq 0\); then, \(g(x)\) is a monotonically decreasing function in \((0, \pi/2)\), \(g(x) \geq g(0) = 0\). Therefore, \(sinc^{(9)}(x) \leq 0\) for \(x \in (0, \pi/2)\).

By the definition of \(e_3(x)\) and \(e_4(x)\), we have
\[
e_3(0) = e_3\left(\frac{\pi}{6}\right) = e_3\left(\frac{\pi}{4}\right) = e_3\left(\frac{\pi}{2}\right) = e_3'(0) = e_3'\left(\frac{\pi}{6}\right) = e_3'\left(\frac{\pi}{4}\right) = 0,
\]

\[
e_4(0) = e_4\left(\frac{\pi}{6}\right) = e_4\left(\frac{\pi}{4}\right) = e_4\left(\frac{\pi}{2}\right) = e_4'(0) = e_4'\left(\frac{\pi}{6}\right) = e_4'\left(\frac{\pi}{4}\right) = 0,
\]

which mean inequality (25) is valid [25]. Then, we have \((T(x)/T(x)) < (\tan(x)/x)\). The proof of \((\tan(x)/x) < (T(x)/105(\pi^2 - 4x^2))\cdot T(x)\) is obtained from [20].

The proof of Theorem 3 is completed.

4. Conclusions and Analysis

In this paper, a multiple-point Padé approximant method is presented for approximating and bounding some trigonometric functions. We find that the Padé approximant is a better approximation of trigonometric functions. The conclusion is verified in Table 1. We give new refinements and improvements of Jordan’s, Kober’s, and Becker-Stark’s inequalities based on the Padé approximant. In order to compare our results with the previous methods, we introduce the concept of the maximum error. The maximum error is the most important index to measure the upper and lower bounds of an inequality. \(\text{MaxError}_l\) denotes the maximum error between a function and its lower bound. \(\text{MaxError}_u\) denotes the maximum error between a function and its upper bound.

Table 2 gives the comparison of the maximum errors between \(\text{sinc}(x)\) and its bounds for different methods. It is obvious that the results of this paper are superior to the previous conclusions. The upper and lower bounds of inequality (13) are tighter than inequalities (1) and (4). The results of \(\cos(x)\) are presented in Table 3. \(\text{MaxError}_l\) and \(\text{MaxError}_u\) of inequality (17) is the smallest of three methods in Table 3.

Table 4 gives the comparison of the maximum errors between \(\tan(x)\) and its bounds for this paper and Zhang et al.’s paper [20]. Because inequality (3) holds in \((0, 1.5701)\), not in \([0, \pi/2]\), we no longer consider the
results of literature [19] in the comparison of $\tan(x)$. Table 4 shows that the upper bound of inequality (6) has a maximum error of $2.7941 \times 10^{-6}$; however, the maximum error of the lower bound of inequality (6) is $2.0690 \times 10^{-1}$. This paper greatly improves the lower bound of $\tan(x)$. The maximum error of the lower bound of inequality (22) in this paper reaches $3.0498 \times 10^{-5}$. The maximum error of the lower bound of inequality (22) is far less than that of inequality (6).

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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References