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Research Article

The Monotone Contraction Mapping Theorem

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In this paper, the fixed-point theorem for monotone contraction mappings in the setting of a uniformly convex smooth Banach space is studied. This paper provides a version of the Banach fixed-point theorem in a complete metric space.

1. Introduction

Many problems arising in different areas of mathematics, such as optimization, variational analysis, and differential equations, can be modeled by the following equation:

$$Tx = x, \tag{1}$$

where T is a nonlinear operator on any set $\mathcal{X}$ into itself. The solutions to (1) are called fixed points of T. The fixed-point theory is concerned with finding conditions on the structure that the set $\mathcal{X}$ must be endowed as well as on the properties of the operator $T\colon \mathcal{X} \longrightarrow \mathcal{X}$, in order to obtain results on the existence (and uniqueness) of fixed points, the data dependence of fixed points, and the construction of fixed points. The set or the ambient space $\mathcal{X}$ involved in fixed-point theorems covers a variety of spaces such as lattice, metric space, normed linear space, generalised metric space, uniform space, and linear topological space while the conditions imposed on the operator T are generally metrical or compactness type of conditions.

Given a complete metric space $(\mathcal{X}, d)$, the most well-studied types of self-maps are referred to as *Lipschitz mappings* (or *Lipschitz maps*, for short), which are given by the metric inequality:

$$d(Tx, Ty) \leq kd(x, y), \tag{2}$$

for all $x, y \in \mathcal{X}$, where $k > 0$ is a real number, usually referred to as the *Lipschitz constant* of T. The metric inequality (2) can be classified into three categories, *contraction mappings* for the case where $k < 1$, *nonexpansive mappings* for the case where $k = 1$, and *expansive mappings* for the case where $k > 1$. The most important property of (2) is that they are uniformly continuous. Thus, for any sequence $\{x_n\}_{n\geq 1}$ converging to x in $\mathcal{X}$, there is $d(Tx_n, Tx) = 0$ as $n \longrightarrow \infty$. The following theorem due to Banach and Steinhaus [1] is the first and simplest of the metric fixed-point theory of Lipschitz maps.

Theorem 1 (contraction mapping theorem). *Let $(\mathcal{X}, d)$ be a complete metric space and $T\colon \mathcal{X} \longrightarrow \mathcal{X}$ be a given contraction. Then, T has a unique fixed point p, and*

$$T^n(x) \longrightarrow p \;(\text{as } n \longrightarrow \infty), \quad \text{for each } x \in \mathcal{X}. \tag{3}$$

Fixed-point problems of contraction mappings always exist, and it is unique due to Theorem 1. This is a very useful result, and it has been applied in the determination of the existence and uniqueness of many results in analysis (both pure and applied) and economics (see, for instance, Border [2], Freeman [3], Picard [4], and Lindelöf [5]).

In this paper, a version of Theorem 1 in the setting of a smooth Banach spaces for monotone contraction mappings is provided. In other words, the fixed-point theorem for monotone contraction mappings in a uniformly convex smooth Banach space is proved.

Definition 1 (normalised duality mapping, see Lumer [6]). Let $\mathcal{X}$ be a Banach space with the norm $\|\cdot\|$ and let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$. Denote $\langle \cdot, \cdot \rangle$ as the duality product. The normalised duality mapping J from $\mathcal{X}$ to $\mathcal{X}^*$ is defined by

$$Jx := \left\{ f \in \mathcal{X}^* \colon \|f\|_*^2 = \|x\|^2 = \langle x, f \rangle = fx \right\}, \tag{4}$$

for all $x \in \mathcal{X}$. Hahn–Banach theorem guarantees that $Jx \neq \varnothing$ for every $x \in \mathcal{X}$. For the purposes in this work, the interest mostly lies on the case when Jx is single valued for all $x \in \mathcal{X}$, which is equivalent to the statement that $\mathcal{X}$ is a smooth Banach space. The normalised duality map J of a Banach space $\mathcal{X}$ is sequentially weakly continuous if a sequence $\{x_n\}_{n\geq 1}$ in $\mathcal{X}$ is weakly convergent to x, and then the sequence $\{Jx_n\}_{n\geq 1}$ in $\mathcal{X}^*$ is weak star convergent to Jx. That is, given that $x_n \rightharpoonup x \in \mathcal{X}$, then $\{Jx_n\}_{n\geq 1} \overset{*}{\rightharpoonup} Jx \in \mathcal{X}^*$.

Remark 1. By virtue of the Riesz representation theorem, it follows that $Jx = x$ (J is the identity map) when it is in a Hilbert space.

Throughout this paper, $\mathfrak{R}$ denotes the real part of a complex number. Also, $F(T)$ is used to denote the set of fixed points of the mapping T (that is, $F(T) = \{x \in \mathcal{C} \colon Tx = x\}$).

Definition 2 (generalised projection functional, see Alber [7]). Let $\mathcal{X}$ be a smooth Banach space and let $\mathcal{X}^*$ be the dual space of $\mathcal{X}$. The generalised projection functional $\phi(\cdot,\cdot) \colon \mathcal{X} \times \mathcal{X} \longrightarrow \mathbb{R}$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\mathfrak{R}\langle y, Jx \rangle + \|x\|^2, \tag{5}$$

for all $x, y \in \mathcal{X}$, where J is the normalised duality mapping from $\mathcal{X}$ to $\mathcal{X}^*$. It is obvious from the definition that the generalised projection functional $\phi(\cdot,\cdot)$ satisfies the following inequality:

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2, \tag{6}$$

for all $x, y \in \mathcal{X}$.

The definition for the mapping discussed in this paper is introduced in the following.

Definition 3 (monotone contraction mapping). Let $\mathcal{X}$ be a smooth Banach space and let $\mathcal{C}$ be a closed subset of $\mathcal{X}$. Then, the mapping $T \colon \mathcal{C} \longrightarrow \mathcal{C}$ is said to be a monotone contraction mapping if there exists $0 \leq c < 1$ such that for all $x, y \in \mathcal{C}$, the following two conditions are satisfied:

(1) $\mathfrak{R}\langle Tx - Ty, JTx - JTy \rangle \leq c\mathfrak{R}\langle x - y, Jx - Jy \rangle$,

(2) $\mathfrak{R}\langle T^{m+1}x - T^m y, JT^{n+1}x - JT^n y \rangle \leq 0$,

where J is the normalised duality mapping and for all $m, n \geq 0$ with $m \neq n$.

It should be noted here that, monotone contraction mappings reduce to the contraction type of mappings in (2) when in Hilbert spaces because in Hilbert spaces J is the identity mapping.

2. Preliminaries

The following proposition and lemmas are introduced that will be used in the proof of the main result. As before, all notations employed remain as defined.

Proposition 1 (see, for instance, Ezearn [8]). *Let $\mathcal{X}$ be a normed linear space. Then, for any $jx \in Jx, jy \in Jy$*

$$(\|x\| - \|y\|)^2 \leq \mathfrak{R}\langle x - y, jx - jy \rangle \leq \|x - y\| (\|x\| + \|y\|). \tag{7}$$

Thus, $\mathfrak{R}\langle x - y, jx - jy \rangle \geq 0$. Moreover, if

$$\mathfrak{R}\langle x - y, jx - jy \rangle = 0, \tag{8}$$

then $jx \in Jy$ and $jy \in Jx$; in particular, when $\mathcal{X}$ is smooth (resp., strictly convex) then equality occurs if and only if $jx = jy$ (resp., $x = y$).

Lemma 1 (see, for instance, Ezearn [8]). *Let $\mathcal{X}$ be a uniformly convex smooth Banach space. Suppose $\{u_n\}_{n\geq 1}$, $\{v_n\}_{n\geq 1} \subset \mathcal{X}$ such that*

$$\begin{gathered} \lim_{n\longrightarrow\infty} v_n = v \in \mathcal{X}, \\ \lim_{n\longrightarrow\infty} \mathfrak{R}\langle u_n - v_n, Ju_n - Jv_n \rangle = 0. \end{gathered} \tag{9}$$

Then, $\lim_{n\longrightarrow\infty} u_n = v$.

Lemma 2 (see Kamimura and Takahashi [9]). *Let $\mathcal{X}$ be a uniformly convex and smooth Banach space and let $\{x_n\}$ and $\{y_n\}$ be two sequences in $\mathcal{X}$ such that either $\{x_n\}$ or $\{y_n\}$ is bounded. If $\lim_{n\longrightarrow\infty}\phi(x_n, y_n) = 0$, then $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0$.*

3. Main Result

The proof of the main result of this paper is given in this section, which is accomplished in Theorem 2. The following lemma and proposition shall aid in arriving at the conclusion of the main result.

Lemma 3. *Let $\mathcal{C}$ be a closed subset of a uniformly convex smooth Banach space $\mathcal{X}$ and let $T \colon \mathcal{C} \longrightarrow \mathcal{C}$ be a monotone contraction mapping. Then, T is (point-wise) continuous on C.*

Proof. Suppose $x_n \longrightarrow x \in \mathcal{C}$ as $n \longrightarrow \infty$. Then, by Definition 3, the following is obtained:

$$\mathfrak{R}\langle Tx_n - Tx, JTx_n - JTx \rangle \leq c\mathfrak{R}\langle x_n - x, Jx_n - Jx \rangle. \tag{10}$$

Since $x_n \longrightarrow x$ as $n \longrightarrow \infty$, (10) reduces to

$$\lim_{n\longrightarrow\infty} \mathfrak{R}\langle Tx_n - Tx, JTx_n - JTx \rangle \leq 0, \tag{11}$$

which by Proposition 1, implies that

$$\lim_{n\longrightarrow\infty} \mathfrak{R}\langle Tx_n - Tx, JTx_n - JTx \rangle = 0. \tag{12}$$

From (12), put $v_n = Tx$. Then, $\lim_{n\longrightarrow\infty} v_n = Tx$. By Lemma 1, $u_n = Tx_n \longrightarrow Tx$ as $n \longrightarrow \infty$ which completes the proof. □

Proposition 2. *Let $\mathcal{C}$ be a closed subset of a uniformly convex smooth Banach space $\mathcal{X}$ and let $T \colon \mathcal{C} \longrightarrow \mathcal{C}$ be a* *monotone contraction mapping. Then, T has at most one fixed point.*

Proof. Suppose that $u, v \in F(T)$ with $u \neq v$. Then by Definition 3,

$$\begin{aligned} \Re\langle Tu - Tv, JTu - JTv\rangle &\leq c\Re\langle u - v, Ju - Jv\rangle, \\ \Re\langle u - v, Ju - Jv\rangle &\leq c\Re\langle u - v, Ju - Jv\rangle, \end{aligned} \tag{13}$$

which reduces to

$$(1 - c)\Re\langle u - v, Ju - Jv\rangle \leq 0. \tag{14}$$

Since $0 \leq c < 1$, then $(1 - c) > 0$, and it follows from (14) that

$$\Re\langle u - v, Ju - Jv\rangle \leq 0, \tag{15}$$

which by Proposition 1, implies that

$$\Re\langle u - v, Ju - Jv\rangle = 0. \tag{16}$$

Since $\mathcal{X}$ is a strictly convex space, by Proposition 1, it is obtained that $u = v$ which completes the proof. □

The main result of this paper is stated and proved in the following.

Theorem 2 (monotone contraction mapping theorem). *Let $\mathcal{C}$ be a closed subset of a uniformly convex smooth Banach space $\mathcal{X}$ and let $T: \mathcal{C} \longrightarrow \mathcal{C}$ be a monotone contraction mapping. Then, T has a unique fixed point, that is, $F(T) = \{p\}$, and that the Picard iteration associated to T, that is, the sequence defined by $x_n = T(x_{n-1}) = T^n(x_0)$ for all $n \geq 1$ converges to p for any initial guess $x_0 \in \mathcal{X}$.*

Proof. To prove existence of a fixed point, it is shown that for any given $x_0 \in \mathcal{X}$, the Picard iteration $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. For $n > m$, the following evaluation is obtained:

$$\begin{aligned} \Re\langle x_n - x_m, Jx_n - Jx_m\rangle &= \Re\langle x_n - x_m, (Jx_n - Jx_{n-1}) \\ &\quad + (Jx_{n-1} - Jx_{n-2}) + Jx_{n-2} - Jx_{m+1} + (Jx_{m+1} - Jx_m)\rangle \\ &= \sum_{k=m}^{n-1} \Re\langle x_n - x_m, Jx_{k+1} - Jx_k\rangle \\ &= \sum_{k=m}^{n-1} \Re\langle (x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + x_{n-2} - x_{m+1} \\ &\quad + (x_{m+1} - x_m), Jx_{k+1} - Jx_k\rangle \\ &= \sum_{k=m}^{n-1}\sum_{\substack{l=m \\ k\neq l}}^{n-1} \Re\langle x_{l+1} - x_l, Jx_{k+1} - Jx_k\rangle \\ &= \sum_{k=m}^{n-1} \Re\langle x_{k+1} - x_k, Jx_{k+1} - Jx_k\rangle \\ &\quad + \sum_{k=m}^{n-1}\sum_{\substack{l=m \\ k\neq l}}^{n-1} \Re\langle x_{l+1} - x_l, Jx_{k+1} - Jx_k\rangle. \end{aligned} \tag{17}$$

By Definition 3 (second part), it is obvious that $\sum_{k=m}^{n-1}\sum_{\substack{l=m\\k\neq l}}^{n-1} \Re\langle x_{l+1} - x_l, Jx_{k+1} - Jx_k\rangle \leq 0$ and as a result,

$$\Re\langle x_n - x_m, Jx_n - Jx_m\rangle \leq \sum_{k=m}^{n-1} \Re\langle x_{k+1} - x_k, Jx_{k+1} - Jx_k\rangle. \tag{18}$$

Applying Definition 3 (first part) several times, (18) reduces to

$$\begin{aligned} \Re\langle x_n - x_m, Jx_n - Jx_m\rangle &\leq \left(c^m + c^{m+1} + \cdots + c^{n-1}\right) \\ &\quad \cdot \Re\langle x_1 - x_0, Jx_1 - Jx_0\rangle, \\ &= c^m\left(1 + c + c^2 + \cdots + c^{n-m-1}\right) \\ &\quad \cdot \Re\langle x_1 - x_0, Jx_1 - Jx_0\rangle, \\ &= \left(\frac{c^m - c^n}{1 - c}\right)\Re\langle x_1 - x_0, Jx_1 - Jx_0\rangle. \end{aligned} \tag{19}$$

Since $0 \leq c < 1$, as $m, n \longrightarrow \infty$, then

$$\lim_{n,m\longrightarrow\infty} \Re\langle x_n - x_m, Jx_n - Jx_m\rangle = 0. \tag{20}$$

Since $2\Re\langle x_n - x_m, Jx_n - Jx_m\rangle = \phi(x_n, x_m) + \phi(x_m, x_n)$, by (20), then $\lim_{n,m\longrightarrow\infty}\phi(x_n, x_m) = \lim_{n,m\longrightarrow\infty}\phi(x_m, x_n) = 0$.

By Definition 2, it is obvious that

$$\left(\|x_n\| - \|x_m\|\right)^2 \leq \phi(x_n, x_m). \tag{21}$$

Since $\phi(x_n, x_m) \longrightarrow 0$ as $n, m \longrightarrow \infty$, then for any $\epsilon > 0$, there exists a natural number $N(\epsilon)$ such that for all $m, n \geq N(\epsilon)$,

$$\left|\|x_n\| - \|x_m\|\right| < \epsilon, \tag{22}$$

which implies that the sequence $\{x_n\}_{n\geq 0}$ is bounded.

Now, since either $\{x_n\}_{n\geq 0}$ or $\{x_m\}_{m\geq 0}$ is bounded and the fact that $\lim_{n,m\longrightarrow\infty}\phi(x_n, x_m) = \lim_{n,m\longrightarrow\infty}\phi(x_m, x_n) = 0$, then by Lemma 2,

$$\lim_{n,m\longrightarrow\infty} \|x_n - x_m\| = 0. \tag{23}$$

This implies that $\{x_n\}_{n\geq 0}$ is a Cauchy sequence. Since $\mathcal{X}$ is complete, there exists $p \in \mathcal{X}$ such that $x_n \longrightarrow p$ as $n \longrightarrow \infty$. By Lemma 3, T is a continuous self-map, and the following is obtained:

$$p = \lim_{n\longrightarrow\infty} x_{n+1} = \lim_{n\longrightarrow\infty} T(x_n) = T\left(\lim_{n\longrightarrow\infty} x_n\right) = Tp. \tag{24}$$

Hence, p is a fixed point of T. By Proposition 2, T has at most one fixed point, and it is deduced that for every choice of $x_0 \in \mathcal{X}$, the Picard iteration converges to the same value p, that is, the unique fixed point of T which completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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