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Research Article

$\beta(\tau_1, \tau_2)$-Continuous Multifunctions on Bitopological Spaces

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The purpose of this paper is to introduce the concepts of $\beta(\tau_1, \tau_2)$-continuous multifunctions and almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, some characterizations of $\beta(\tau_1, \tau_2)$-continuous multifunctions and almost $\beta(\tau_1, \tau_2)$-continuous multifunctions are investigated.

1. Introduction

General topology is an important mathematical branch which is applied in many fields of applied sciences. Continuity is a basic concept for the study in topological spaces. Generalization of this concept by using weaker forms of open sets such as semi-open sets [1], preopen sets [2], and β-open sets [3] is one of the main research topics of general topology. In 1983, Abd El-Monsef et al. [4] introduced the classes of β-open sets called semi-preopen sets by Andrijević in [3]; moreover, Abd El-Monsef et al. [4] introduced almost β-continuous functions in topological spaces. From 1992 to 1993, the authors [5] obtained several characterizations of β-continuity and showed that almost quasi-continuity [6] investigated by Borsik and Dobos was equivalent to β-continuity. Therefore, in 1997, Nasef and Noiri [7] investigated fundamental characterizations of almost β-continuous functions. A year later, Popa and Noiri [8] investigated further characterizations of almost β-continuous functions. In 1992, Khedr et al. [9] generalized the notions of β-open sets and investigated β-continuous functions in bitopological spaces. Furthermore, in [10, 11] from 1996 to 1999, the authors extended these functions to multifunction by introducing and characterizing the notions of β-continuous multifunctions and almost β-continuous multifunctions. In this paper, we introduce the notions of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions and investigate some characterizations of upper and lower $\beta(\tau_1, \tau_2)$-continuous multifunctions. Section 4 is devoted to introduce and study upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by $\tau_i\text{-Cl}(A)$ and $\tau_i\text{-Int}(A)$, respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-semiopen (resp., $\tau_1\tau_2$-regular open [12], $\tau_1\tau_2$-regular closed [13], $\tau_1\tau_2$-preopen [14], and $\tau_1\tau_2$-β-open [9]) if $A \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(A))$ (resp., $A = \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$, $A = \tau_1\text{-Cl}(\tau_2\text{-Int}(A))$, $A \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(A))$, and $A \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A)))$). The complement of $\tau_1\tau_2$-semiopen (resp., $\tau_1\tau_2$-preopen and $\tau_1\tau_2$-β-open) set is said to be $\tau_1\tau_2$-semiclosed (resp., $\tau_1\tau_2$-preclosed and $\tau_1\tau_2$-β-closed). The $\tau_1\tau_2$-semiclosure (resp., $\tau_1\tau_2$-preclosure [9] and $\tau_1\tau_2$-β-closure [9]) of A is defined by the intersection of $\tau_1\tau_2$-semiclosed (resp., $\tau_1\tau_2$-preclosed and $\tau_1\tau_2$-β-closed) sets containing A and is denoted by $\tau_1\tau_2\text{-sCl}(A)$ (resp., $\tau_1\tau_2\text{-pCl}(A)$ and $\tau_1\tau_2\text{-}\beta\text{Cl}(A)$). The $\tau_1\tau_2$-semiinterior (resp., $\tau_1\tau_2$-preinterior

[15] and $\tau_1\tau_2$-β-interior [16]) of A is defined by the union of $\tau_1\tau_2$-semiopen (resp. $\tau_1\tau_2$-preopen and $\tau_1\tau_2$-β-open) sets contained in A and is denoted by $\tau_1\tau_2$-sInt(A) (resp., $\tau_1\tau_2$-pInt(A) and $\tau_1\tau_2$-βInt(A)).

By a multifunction F: $X \longrightarrow Y$, we mean a point-to-set correspondence from X into Y, and we always assume that $F(x) \neq \emptyset$ for all $x \in X$. For a multifunction F: $X \longrightarrow Y$, following [17], we shall denote the upper and lower inverse of a set B of Y by $F^+(B)$ and $F^-(B)$, respectively, that is, $F^+(B) = \{x \in X \mid F(x) \subseteq B\}$ and $F^-(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\}$. In particular, $F^-(y) = \{x \in X \mid y \in F(x)\}$ for each point $y \in Y$. For each $A \subseteq X$, $F(A) = \cup_{x\in A} F(x)$. Then F is said to be surjection if $F(X) = Y$, or equivalent, if for each $y \in Y$, and there exists $x \in X$ such that $y \in F(x)$ and F is called injection if $x \neq y$ implies $F(x) \cap F(y) = \emptyset$.

A subset A of a bitopolgical space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-closed [18] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of $\tau_1\tau_2$-closed is said to be $\tau_1\tau_2$-open. The intersection of all $\tau_1\tau_2$-closed sets containing A is called $\tau_1\tau_2$-closure of A and denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets contained in A is called $\tau_1\tau_2$-interior of A and denoted by $\tau_1\tau_2$-Int(A). A subset N of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-neighbourhood (resp., $\tau_1\tau_2$-β-neighbourhood) of $x \in X$ if there exists $\tau_1\tau_2$-open (resp. $\tau_1\tau_2$-β-open) set V of (X, τ_1, τ_2) such that $x \in V \subseteq N$.

Lemma 1 (see [18]). *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A);
(2) If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$;
(3) $\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed;
(4) A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A);
(5) $\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.

Lemma 2 (see [3]). *For a subset A of a topological space (X, τ), the following properties hold:*

(1) Cl$(A) \cap G \subseteq$ Cl$(A \cap G)$ for every open set G;
(2) Int$(A \cup F) \subseteq$ Int$(A) \cup F$ for every closed set F.

Lemma 3. *For a subset A of a bitopological space (X, τ_1, τ_2), $x \in \tau_1\tau_2$-sCl(A) if and only if $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-semiopen set U containing x.*

Proof. Let $x \in \tau_1\tau_2$-sCl(A). We shall show that $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-semiopen set U containing x. Suppose that $U \cap A = \emptyset$ for some $\tau_1\tau_2$-semiopen set U containing x. Then, $A \subseteq X - U$ and $X - U$ is $\tau_1\tau_2$-semiclosed. Since $x \in \tau_1\tau_2$-sCl(A), we have $x \in \tau_1\tau_2$-sCl$(X - U) = X - U$; hence $x \notin U$, which is a contradiction that $x \in U$. Therefore, $U \cap A \neq \emptyset$.

Conversely, we assume that $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-semiopen set U containing x. We shall show that $x \in \tau_1\tau_2$-sCl(A). Suppose that $x \notin \tau_1\tau_2$-sCl(A). Then, there exists a $\tau_1\tau_2$-semiclosed set F such that $A \subseteq F$ and $x \notin F$. Therefore, we obtain $X - F$ is a $\tau_1\tau_2$-semiopen set containing x such that $(X - F) \cap A = \emptyset$. This is a contradiction to $U \cap A \neq \emptyset$, and hence, $x \in \tau_1\tau_2$-sCl(A). □

Lemma 4. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties are hold:*

(1) $X - \tau_1\tau_2$-sCl$(A) = \tau_1\tau_2$-sInt$(X - A)$;
(2) $X - \tau_1\tau_2$-sInt$(A) = \tau_1\tau_2$-sCl$(X - A)$.

Proof

(1) Let $x \in X - \tau_1\tau_2$-sCl(A). Then, $x \notin \tau_1\tau_2$-sCl(A). Thus, there exists a $\tau_1\tau_2$-semiopen set V containing x such that $V \cap A = \emptyset$. Therefore, $V \subseteq X - A$, and hence $x \in \tau_1\tau_2$-sInt$(X - A)$. This shows that

$$X - \tau_1\tau_2\text{-sCl}(A) \subseteq \tau_1\tau_2\text{-sInt}(X - A). \tag{1}$$

Let $x \in \tau_1\tau_2$-sInt$(X - A)$. Consequently, there exists a $\tau_1\tau_2$-semiopen set V containing x such that $V \subseteq X - A$. Then, $V \cap A = \emptyset$. By Lemma 3, we have $x \notin \tau_1\tau_2$-sCl(A); hence, $x \in X - \tau_1\tau_2$-sCl(A). Therefore,

$$\tau_1\tau_2\text{-sInt}(X - A) \subseteq X - \tau_1\tau_2\text{-sCl}(A). \tag{2}$$

Consequently, we obtain $X - \tau_1\tau_2$-sCl$(A) = \tau_1\tau_2$-sInt$(X - A)$.

(2) This follows from (1). □

Lemma 5. *For a subset A of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $\tau_1\tau_2$-sCl$(A) = \tau_1$-Int$(\tau_2$-Cl$(A)) \cup A$;
(2) If A is τ_1-open in X, then $\tau_1\tau_2$-sCl$(A) = \tau_1$-Int$(\tau_2$-Cl$(A))$.

Proof

(1) Since $\tau_1\tau_2$-sCl(A) is $\tau_1\tau_2$-semiclosed, we have

$$\tau_1\text{-Int}\left(\tau_2\text{-Cl}\left(\tau_1\tau_2\text{-sCl}(A)\right)\right) \subseteq \tau_1\tau_2\text{-sCl}(A). \tag{3}$$

Thus, τ_1-Int$(\tau_2$-Cl$(A)) \subseteq \tau_1\tau_2$-sCl$(A)$. Hence, τ_1-Int$(\tau_2$-Cl$(A)) \cup A \subseteq \tau_1\tau_2$-sCl$(A)$. To establish the opposite inclusion, we observe that

$$\begin{aligned} &\tau_1\text{-Int}\left(\tau_2\text{-Cl}\left(\tau_1\text{-Int}\left(\tau_2\text{-Cl}(A)\right) \cup A\right)\right) \subseteq \tau_1 \\ &\text{-Int}\left(\tau_2\text{-Cl}(A) \cup A\right) = \tau_1\text{-Int}\left(\tau_2\text{-Cl}(A)\right). \end{aligned} \tag{4}$$

Therefore,

$$\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(\tau_2\text{-Cl}(A))\cup A))\subseteq\tau_1\text{-Int}(\tau_2\text{-Cl}(A))\subseteq\tau_1\text{-Int}(\tau_2\text{-Cl}(A))\cup A. \tag{5}$$

Hence, $\tau_1\text{-Int}(\tau_2\text{-Cl}(A))\cup A$ is $\tau_1\tau_2$-semiclosed. Then,

$$\tau_1\tau_2\text{-sCl}(A)\subseteq\tau_1\text{-Int}(\tau_2\text{-Cl}(A))\cup A. \tag{6}$$

Consequently, we obtain $\tau_1\tau_2\text{-sCl}(A)=\tau_1\text{-Int}(\tau_2\text{-Cl}(A))\cup A$.

(2) Let A be a τ_1-open set, then $A=\tau_1\text{-Int}(A)\subseteq\tau_1\text{-Int}(\tau_2\text{-Cl}(A))$. Therefore, by (1), we have $\tau_1\tau_2\text{-sCl}(A)=\tau_1\text{-Int}(\tau_2\text{-Cl}(A))$. □

Proposition 1. *Let (X,τ_1,τ_2) be a bitopological space and $\{A_\gamma \mid \gamma\in\Gamma\}$ a family of subsets of X. The following properties hold:*

(1) If A_γ is $\tau_1\tau_2$-β-open for each $\gamma\in\Gamma$, then $\cup_{\gamma\in\Gamma}A_\gamma$ is $\tau_1\tau_2$-β-open;

(2) If A_γ is $\tau_1\tau_2$-β-closed for each $\gamma\in\Gamma$, then $\cap_{\gamma\in\Gamma}A_\gamma$ is $\tau_1\tau_2$-β-closed.

Proof

(1) Suppose that A_γ is $\tau_1\tau_2$-β-open for each $\gamma\in\Gamma$. Then, we have $A_\gamma\subseteq\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A_\gamma)))\subseteq\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(\cup_{\gamma\in\Gamma}A_\gamma)))$, and hence, $\cup_{\gamma\in\Gamma}A_\gamma\subseteq\tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(\cup_{\gamma\in\Gamma}A_\gamma)))$. This shows that $\cup_{\gamma\in\Gamma}A_\gamma$ is $\tau_1\tau_2$-β-open.

(2) By utilizing Proposition 1 (1), the proof is obvious.

The intersection of two $\tau_1\tau_2$-β-open sets is not $\tau_1\tau_2$-β-open set as shown in the following example. □

Example 1. Let $X=\{a,b,c\}$ with topologies $\tau_1=\{\varnothing,\{a\},\{b\},\{a,b\},X\}$ and $\tau_2=\{\varnothing,\{a\},\{b\},\{a,b\},X\}$. Then, $\{a,c\}$ and $\{b,c\}$ are $\tau_1\tau_2$-β-open sets, but $\{a,c\}\cap\{b,c\}=\{c\}$ is not $\tau_1\tau_2$-β-open set.

Proposition 2. *For a subset A of a bitopological space (X,τ_1,τ_2), the following properties are hold:*

(1) $\tau_1\tau_2$-$\beta Int(A)$ is $\tau_1\tau_2$-β-open;

(2) $\tau_1\tau_2$-$\beta Cl(A)$ is $\tau_1\tau_2$-β-closed;

(3) A is $\tau_1\tau_2$-β-open if and only if $A=\tau_1\tau_2$-$\beta Int(A)$;

(4) A is $\tau_1\tau_2$-β-closed if and only if $A=\tau_1\tau_2$-$\beta Cl(A)$.

Proof. (1) and (2) follow from Proposition 1. (3) and (4) follow from (1) and (2). □

Proposition 3. *For a subset A of a bitopological space (X,τ_1,τ_2), $x\in\tau_1\tau_2$-$\beta Cl(A)$ if and only if $U\cap A\neq\varnothing$ for every $\tau_1\tau_2$-β-open set U containing x.*

Proof. This is similar to the proof of Lemma 3. □

Proposition 4. *For a subset A of a bitopological space (X,τ_1,τ_2), the following properties hold:*

(1) $X-\tau_1\tau_2$-$\beta Cl(A)=\tau_1\tau_2$-$\beta Int(X-A)$;

(2) $X-\tau_1\tau_2$-$\beta Int(A)=\tau_1\tau_2$-$\beta Cl(X-A)$.

Proof

(1) Let $x\in X-\tau_1\tau_2$-$\beta\text{Cl}(A)$. Then, $x\notin\tau_1\tau_2$-$\beta\text{Cl}(A)$; there exists a $\tau_1\tau_2$-β-open set V containing x such that $V\cap A=\varnothing$. Then, $V\subseteq X-A$, and hence, $x\in\tau_1\tau_2$-$\beta\text{Int}(X-A)$. This shows that $X-\tau_1\tau_2$-$\beta\text{Cl}(A)\subseteq\tau_1\tau_2$-$\beta\text{Int}(X-A)$. Let $x\in\tau_1\tau_2$-$\beta\text{Int}(X-A)$. Then, there exists a $\tau_1\tau_2$-β-open set V containing x such that $V\subseteq X-A$. Hence, $V\cap A=\varnothing$. By Proposition 3, we have $x\notin\tau_1\tau_2$-$\beta\text{Cl}(A)$; hence, $x\notin X-\tau_1\tau_2$-$\beta\text{Cl}(A)$. Therefore, $X-\tau_1\tau_2$-$\beta\text{Cl}(A)\supseteq\tau_1\tau_2$-$\beta\text{Int}(X-A)$. Consequently, we obtain $X-\tau_1\tau_2$-$\beta\text{Cl}(A)=\tau_1\tau_2$-$\beta\text{Int}(X-A)$.

(2) This follows from (1). □

3. Characterizations of Upper and Lower $\beta(\tau_1,\tau_2)$-Continuous Multifunctions

In this section, we introduce the notions of upper and lower $\beta(\tau_1,\tau_2)$-continuous multifunctions and investigate some characterizations of these multifunctions.

Definition 1. A multifunction $F\colon(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is said to be

(1) Upper $\beta(\tau_1,\tau_2)$-continuous at a point $x\in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, and there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U)\subseteq V$;

(2) Lower $\beta(\tau_1,\tau_2)$-continuous at a point $x\in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x)\cap V\neq\varnothing$, and there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z)\cap V\neq\varnothing$ for every $z\in U$;

(3) Upper (resp., lower) $\beta(\tau_1,\tau_2)$-continuous if F has this property at each point of X.

Example 2. Let $X=\{a,b,c\}$ with topologies $\tau_1=\{\varnothing,\{a\},\{b\},\{a,b\},X\}$ and $\tau_2=\{\varnothing,\{a\},\{a,b\},X\}$. Let $Y=\{1,2,3,4,5\}$ with topologies $\sigma_1=\{\varnothing,\{1\},\{2,3,4,5\},Y\}$ and $\sigma_2=\{\varnothing,\{2\},\{3\},\{2,3\},\{2,3,4,5\},Y\}$. A multifunction $F\colon(X,\tau_1,\tau_2)\longrightarrow(Y,\sigma_1,\sigma_2)$ is defined as follows: $F(a)=\{2,3\}$, $F(b)=\{1,2\}$, $F(c)=\{1,4,5\}$. Then, F is upper and lower $\beta(\tau_1,\tau_2)$-continuous.

Theorem 1. *A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-$\beta Int(F^+(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.*

Proof. Let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Consequently, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$. Therefore, $x \in U \subseteq F^+(V)$. Since U is $\tau_1\tau_2$-β-open, we have $x \in \tau_1\tau_2$-$\beta\mathrm{Int}(F^+(V))$.

Conversely, let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. By the hypothesis, $x \in \tau_1\tau_2$-$\beta\mathrm{Int}(F^+(V))$. There exists a $\tau_1\tau_2$-β-open set G containing x such that $G \subseteq F^+(V)$; hence, $F(G) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous at x. □

Theorem 2. *A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2$-$\beta Int(F^-(V))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \varnothing$.*

Proof. The proof is similar to that of Theorem 1. □

Theorem 3. *For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is upper $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^+(V)$ is $\tau_1\tau_2$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^-(K)$ is $\tau_1\tau_2$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-$\beta Cl(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-$Cl(B))$ for every subset B of Y;

(5) τ_1-$Int(\tau_2$-$Cl(\tau_1$-$Int(F^-(B)))) \subseteq F^-(\sigma_1\sigma_2$-$Cl(B))$ for every subset B of Y.

Proof

(1)$\Longrightarrow$(2): let V be a $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Therefore, $F(x) \subset V$, then there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$. Consequently, we obtain $x \in U \subseteq \tau_1$-$\mathrm{Cl}(\tau_2$-$\mathrm{Int}(\tau_1$-$\mathrm{Cl}(U))) \subseteq \tau_1$-$\mathrm{Cl}(\tau_2$-$\mathrm{Int}(\tau_1$-$\mathrm{Cl}(F^+(V))))$. Thus, $F^+(V) \subseteq \tau_1$-$\mathrm{Cl}(\tau_2$-$\mathrm{Int}(\tau_1$-$\mathrm{Cl}(F^+(V))))$. This shows $F^+(V)$ is $\tau_1\tau_2$-β-open in X.

(2)$\Longrightarrow$(3): this follows from the fact that $F^+(Y - B) = X - F^-(B)$ for every subset B of Y.

(3)$\Longrightarrow$(4): for each subset B of Y, $\sigma_1\sigma_2$-$\mathrm{Cl}(B)$ is $\sigma_1\sigma_2$-closed in Y. By (3), $F^-(\sigma_1\sigma_2$-$\mathrm{Cl}(B))$ is $\tau_1\tau_2$-β-closed in X; therefore,$\tau_1\tau_2$-$\beta\mathrm{Cl}(F^-(B)) \subseteq F^-(\sigma_1\sigma_2$-$\mathrm{Cl}(B))$.

(4)$\Longrightarrow$(5): let B be a subset of Y. By Proposition 2 (2), we obtain

$$\begin{aligned}
&\tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(F^-(B)))) \subseteq \tau_1\text{-Int}(\tau_2\text{-Cl} \\
&\quad \cdot (\tau_1\text{-Int}(\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(B))))) \subseteq \tau_1\tau_2\text{-}\beta\text{Cl}(F^-(B)).
\end{aligned} \tag{7}$$

Consequently, τ_1-$\mathrm{Int}(\tau_2$-$\mathrm{Cl}(\tau_1$-$\mathrm{Int}(F^-(B)))) \subseteq F^-(\sigma_1\sigma_2$-$\mathrm{Cl}(B))$ by (4).

(5)$\Longrightarrow$(2): let V be a $\sigma_1\sigma_2$-open set of Y so $Y - V$ is $\sigma_1\sigma_2$-closed in Y. By (5),

$$\begin{aligned}
X - F^+(V) &\supseteq X - F^+(\sigma_1\sigma_2\text{-Int}(V)) \\
&= F^-(\sigma_1\sigma_2\text{-Cl}(Y - V)) \\
&\supseteq \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(F^-(Y - V)))) \\
&= \tau_1\text{-Int}(\tau_2\text{-Cl}(\tau_1\text{-Int}(X - F^+(V)))) \\
&= X - \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(F^+(V)))).
\end{aligned} \tag{8}$$

Therefore, we obtain $F^+(V) \subseteq \tau_1$-$\mathrm{Cl}(\tau_2$-$\mathrm{Int}(\tau_1$-$\mathrm{Cl}(F^+(V))))$, and hence, $F^+(V)$ is $\tau_1\tau_2$-β-open in X.

(2)$\Longrightarrow$(1): let $x \in X$ and V be a $\sigma_1\sigma_2$-open set containing $F(x)$. By (2), $F^+(V)$ is a $\tau_1\tau_2$-β-open set containing x. Putting $U = F^+(V)$, we obtain U is a $\tau_1\tau_2$-β-open set containing x such that $F(U) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous. □

Theorem 4. *For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) F is lower $\beta(\tau_1, \tau_2)$-continuous;

(2) $F^-(V)$ is $\tau_1\tau_2$-β-open in X for every $\sigma_1\sigma_2$-open set V of Y;

(3) $F^+(K)$ is $\tau_1\tau_2$-β-closed in X for every $\sigma_1\sigma_2$-closed set K of Y;

(4) $\tau_1\tau_2$-$\beta Cl(F^+(B)) \subseteq F^+(\sigma_1\sigma_2$-$Cl(B))$ for every subset B of Y;

(5) τ_1-$Int(\tau_2$-$Cl(\tau_1$-$Int(F^+(B)))) \subseteq F^+(\sigma_1\sigma_2$-$Cl(B))$ for every subset B of Y.

Proof. It is shown similarly to the proof of Theorem 3 that the statements (1), (2), (3), (4), and (5) are equivalent. □

Definition 2 (see [18]). A collection $\mathscr{U}$ of subsets of a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-locally finite if every $x \in X$ has a $\tau_1\tau_2$-neighborhood which intersects only finitely many elements of $\mathscr{U}$.

Definition 3 (see [18]). A subset A of a bitopological space (X, τ_1, τ_2) is said to be

(1) $\tau_1\tau_2$-paracompact if every cover of A by $\tau_1\tau_2$-open sets of X is refined by a cover of A which consists of $\tau_1\tau_2$-open sets of X and is $\tau_1\tau_2$-locally finite in X;

(2) $\tau_1\tau_2$-regular if for each $x \in A$ and each $\tau_1\tau_2$-open set U of X containing x, and there exists a $\tau_1\tau_2$-open set V of X such that $x \in V \subseteq \tau_1\tau_2$-$\mathrm{Cl}(V) \subseteq U$.

Lemma 6 (see [18]). *If A is a $\tau_1\tau_2$-regular $\tau_1\tau_2$-paracompact set of a bitopological space (X, τ_1, τ_2) and U is a $\tau_1\tau_2$-open neighbourhood of A, then there exists a $\tau_1\tau_2$-open set V of X such that $A \subseteq V \subseteq \tau_1\tau_2\text{-}Cl(V) \subseteq U$.*

Definition 4. A multifunction $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is called punctually (τ_1, τ_2)-paracompact (resp., punctually (τ_1, τ_2)-regular) if for each $x \in X$, and $F(x)$ is $\tau_1\tau_2$-paracompact (resp., $\tau_1\tau_2$-regular).

For a multifunction $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, by

$$\beta\mathrm{Cl}F_{\circledast}\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2), \tag{9}$$

we denote a multifunction defined as follows: $\beta\mathrm{Cl}F_{\circledast}(x) = \sigma_1\sigma_2\text{-}\beta\mathrm{Cl}(F(x))$ for each $x \in X$.

Example 3. Let $X = \{1, 2, 3\}$ with topologies $\tau_1 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$ and $\tau_2 = \{\varnothing, \{1\}, \{1, 2\}, X\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\varnothing, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, Y\}$ and $\sigma_2 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, Y\}$. A multifunction $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(1) = \{a, b\}$, $F(2) = \{a\}$, and $F(3) = \{b\}$. Then, F is punctually (τ_1, τ_2)-paracompact.

Example 4. Let $X = \{1, 2, 3\}$ with topologies $\tau_1 = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, X\}$ and $\tau_2 = \{\varnothing, \{1\}, \{1, 2\}, X\}$. Let $Y = \{a, b, c\}$ with topologies $\sigma_1 = \{\varnothing, \{b, c\}, Y\}$ and $\sigma_2 = \{\varnothing, \{a, c\}, Y\}$. A multifunction $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(1) = F(2) = F(3) = \{a, b\}$. Then, F is punctually (τ_1, τ_2)-regular.

Lemma 7. *Let (X, τ_1, τ_2) be a bitopological space. Then, $\tau_1\tau_2\text{-}\beta Cl(A) \subseteq \tau_1\tau_2\text{-}Cl(A)$ for every subset A of X.*

Proof. Let $x \in X - \tau_1\tau_2\text{-}\mathrm{Cl}(A)$. By Lemma 1 (5), $x \in \tau_1\tau_2\text{-}\mathrm{Int}(X - A)$, and there exists a $\tau_1\tau_2$-open set V such that $x \in V \subseteq X - A$. Since every $\tau_1\tau_2$-open set is $\tau_1\tau_2$-β-open, we have $x \in \tau_1\tau_2\text{-}\beta\mathrm{Int}(X - A)$. By Proposition 4 (1), $x \in X - \tau_1\tau_2\text{-}\beta\mathrm{Cl}(A)$, so $X - \tau_1\tau_2\text{-}\mathrm{Cl}(A) \subseteq X - \tau_1\tau_2\text{-}\beta\mathrm{Cl}(A)$. Consequently, we obtain $\tau_1\tau_2\text{-}\beta\mathrm{Cl}(A) \subseteq \tau_1\tau_2\text{-}\mathrm{Cl}(A)$. □

Lemma 8. *If $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is punctually (τ_1, τ_2)-paracompact and punctually (τ_1, τ_2)-regular, then $\beta ClF^{+}_{\circledast}(V) = F^{+}(V)$ for every $\sigma_1\sigma_2$-open V of Y.*

Proof. Let V be a $\sigma_1\sigma_2$-open set V of Y and $x \in \beta\mathrm{Cl}F^{+}_{\circledast}(V)$. Then, we have $\sigma_1\sigma_2\text{-}\beta\mathrm{Cl}(F(x))$ and $F(x) \subseteq V$. Therefore, we have $x \in F^{+}(V)$, and hence $\beta\mathrm{Cl}F^{+}_{\circledast}(V) \subseteq F^{+}(V)$. On the contrary, let $x \in F^{+}(V)$. Then, $F(x) \subseteq V$, and by Lemma 6, there exists a $\sigma_1\sigma_2$-open set U of Y such that $F(x) \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(U) \subseteq U \subseteq V$. By Lemma 7, we have $\sigma_1\sigma_2\text{-}\beta\mathrm{Cl}(F(x)) \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(U) \subseteq V$. This shows that $x \in \beta\mathrm{Cl}F^{+}_{\circledast}(V)$, and hence, $F^{+}(V) \subseteq \beta\mathrm{Cl}F^{+}_{\circledast}(V)$. Consequently, we obtain $\beta\mathrm{Cl}F^{+}_{\circledast}(V) = F^{+}(V)$. □

Theorem 5. *Let $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be punctually (τ_1, τ_2)-paracompact and punctually (τ_1, τ_2)-regular. Then F is upper $\beta(\tau_1, \tau_2)$-continuous if and only if $\beta ClF_{\circledast}\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous.*

Proof. Suppose that F is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $\beta\mathrm{Cl}F_{\circledast}(x) \subseteq V$. By Lemma 8, we have $x \in \beta\mathrm{Cl}F^{+}_{\circledast}(V) = F^{+}(V)$. Since F is upper $\beta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq V$. Since $F(z)$ is $\sigma_1\sigma_2$-paracompact and $\sigma_1\sigma_2$-regular for each $z \in U$, by Lemma 6, there exists a $\sigma_1\sigma_2$-open set H such that $F(z) \subseteq H \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(H) \subseteq V$. By Lemma 7, we have $\sigma_1\sigma_2\text{-}\beta\mathrm{Cl}(F(z)) \subseteq \sigma_1\sigma_2\text{-}\mathrm{Cl}(H) \subseteq V$ for each $z \in U$, and hence, $\beta\,\mathrm{Cl}F_{\circledast}(U) \subseteq V$. This shows that $\beta\mathrm{Cl}F_{\circledast}$ is upper $\beta(\tau_1, \tau_2)$-continuous.

Conversely, suppose that $\beta\mathrm{Cl}F_{\circledast}\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $F(x) \subseteq V$. By Lemma 8, we have $x \in F^{+}(V) = \beta\mathrm{Cl}F^{+}_{\circledast}(V)$, and hence, $\beta\mathrm{Cl}F_{\circledast}(x) \subseteq V$. Since $\beta\mathrm{Cl}F_{\circledast}$ is upper $\beta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set U of containing x such that $\beta\mathrm{Cl}F_{\circledast}(U) \subseteq V$; hence, $F(U) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous. □

Lemma 9. *For a multifunction $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, it follows that for each $\sigma_1\sigma_2$-β-open set V of Y $\beta ClF^{-}_{\circledast}(V) = F^{-}(V)$.*

Proof. Suppose that V is a $\sigma_1\sigma_2$-β-open set Y. Let $x \in \beta\mathrm{Cl}F^{-}_{\circledast}(V)$. Then, $\sigma_1\sigma_2\text{-}\beta\mathrm{Cl}(F(x)) \cap V \neq \varnothing$. Hence, $F(x) \cap V \neq \varnothing$. Therefore, we obtain $x \in F^{-}(V)$. This shows that $\beta\mathrm{Cl}F^{-}_{\circledast}(V) \subseteq F^{-}(V)$. On the contrary, let $x \in F^{-}(V)$. Then, we have $\varnothing \neq F(x) \cap V \subseteq \sigma_1\sigma_2\text{-}\beta\mathrm{Cl}(F(x)) \cap V$. Thus, $x \in \beta\mathrm{Cl}F^{-}_{\circledast}(V)$. This shows that $F^{-}(V) \subseteq \beta\mathrm{Cl}F^{-}_{\circledast}(V)$. Consequently, we obtain $\beta\mathrm{Cl}F^{-}_{\circledast}(V) = F^{-}(V)$. □

Theorem 6. *A multifunction $F\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous if and only if $\beta ClF_{\circledast}\colon (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous.*

Proof. By utilizing Lemma 9, this can be proved similarly to that of Theorem 5.

For a multifunction $F\colon X \longrightarrow Y$, the graph multifunction $G_F\colon X \longrightarrow X \times Y$ is defined as follows: $G_F(x) = \{x\} \times F(x)$ for every $x \in X$. □

Lemma 10 (see [14]). *The following hold for a multifunction $F\colon X \longrightarrow Y$:*

(i) $G^{+}_{F}(A \times B) = A \cap F^{+}(B)$;

(ii) $G^{-}_{F}(A \times B) = A \cap F^{-}(B)$

for any subsets $A \subseteq X$ and $B \subseteq Y$.

Lemma 11. *Let (X, τ_1, τ_2) be a bitopological space. If A is $\tau_1\tau_2$-β-open and B is $\tau_1\tau_2$-open in X, then $A \cap B$ is $\tau_1\tau_2$-β-open.*

Proof. Suppose that A is $\tau_1\tau_2$-β-open and B is $\tau_1\tau_2$-open in X. Then, we have $A \subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A)))$ and $B = \tau_1\text{-Int}(B) = \tau_2\text{-Int}(B)$. By Lemma 2 (1),

$$\begin{aligned} A \cap B &\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A))) \cap B \\ &\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A)) \cap B) \\ &= \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A) \cap B)) \\ &\subseteq \tau_1\text{-Cl}(\tau_2\text{-Int}(\tau_1\text{-Cl}(A \cap B))). \end{aligned} \tag{10}$$

Consequently, we obtain $A \cap B$ is $\tau_1\tau_2$-β-open. □

Definition 5 (see [18]). A bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-compact if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.

By ρ_i, we denote the product topology $\tau_i \times \sigma_i$ for $i = 1, 2$.

Theorem 7. *Let F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ be a multifunction such that $F(x)$ is $\sigma_1\sigma_2$-compact for each $x \in X$. Then F is upper $\beta(\tau_1, \tau_2)$-continuous if and only if G_F: $(X, \tau_1, \tau_2) \longrightarrow (X \times Y, \rho_1, \rho_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous.*

Proof. Suppose that F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and W be a $\rho_1\rho_2$-open set of $X \times Y$ containing $G_F(x)$. For each $y \in F(x)$, there exist $\tau_1\tau_2$-open set $U(y)$ of X and $\sigma_1\sigma_2$-open set $V(y)$ of Y such that $(x, y) \in U(y) \times V(y) \subseteq W$. The family $\{V(y) \mid y \in F(x)\}$ is $\sigma_1\sigma_2$-open cover of $F(x)$, and there exists a finite number of points, say, $y_1, y_2, \ldots, y_n$ in $F(x)$ such that $F(x) \subseteq \cup\{V(y_i) \mid 1 \leq i \leq n\}$. Put

$$\begin{aligned} U &= \cap\{U(y_i) \mid 1 \leq i \leq n\}, \\ V &= \cup\{V(y_i) \mid 1 \leq i \leq n\}. \end{aligned} \tag{11}$$

Then, we have U is $\tau_1\tau_2$-open in X and V is $\sigma_1\sigma_2$-open in Y such that $\{x\} \times F(x) \subseteq U \times V \subseteq W$. Since F is upper $\beta(\tau_1, \tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set G containing x such that $F(G) \subseteq V$. By Lemma 10, we have $U \cap G \subseteq U \cap F^+(V) = G_F^+(U \times V) \subseteq G_F^+(W)$. By Lemma 11, $U \cap G$ is $\tau_1\tau_2$-β-open in X and $G_F(U \cap G) \subseteq W$. This shows that G_F is upper $\beta(\tau_1, \tau_2)$-continuous.

Conversely, suppose that G_F: $(X, \tau_1, \tau_2) \longrightarrow (X \times Y, \rho_1, \rho_2)$ is upper $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open containing $F(x)$. Since $X \times V$ is $\rho_1\rho_2$-open in $X \times Y$ and $G_F(x) \subseteq X \times V$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $G_F(U) \subseteq X \times V$. Therefore, by Lemma 10, $U \subseteq G_F^+(X \times V) = F^+(V)$ and so $F(U) \subseteq V$. This shows that F is upper $\beta(\tau_1, \tau_2)$-continuous. □

Theorem 8. *A multifunction F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous if and only if G_F: $(X, \tau_1, \tau_2) \longrightarrow (X \times Y, \rho_1, \rho_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous.*

Proof. Suppose that F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and W be a $\rho_1\rho_2$-open set of $X \times Y$ such that $G_F(x) \cap W \neq \varnothing$. There exists $y \in F(x)$ such that $(x, y) \in W$, and hence, $(x, y) \in U \times V \subseteq W$ for some $\tau_1\tau_2$-open set U of X and $\sigma_1\sigma_2$-open set V of Y. Since $F(x) \cap V \neq \varnothing$, there exists a $\tau_1\tau_2$-β-open set G containing x such that $F(z) \cap V \neq \varnothing$ for each $z \in G$; hence $G \subseteq F^-(V)$. By Lemmas 10 and 11, we have $U \cap G \subseteq U \cap F^-(V) = G_F^-(U \times V) \subseteq G_F^-(W)$. Moreover, $U \cap G$ is a $\tau_1\tau_2$-β-open set containing x, and hence, G_F is lower $\beta(\tau_1, \tau_2)$-continuous.

Conversely, suppose that F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is lower $\beta(\tau_1, \tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $F(x) \cap V \neq \varnothing$. Since $X \times V$ is $\rho_1\rho_2$-open in $X \times Y$ and

$$G_F(x) \cap (X \times V) = (\{x\} \times F(x)) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \varnothing. \tag{12}$$

Then, there exists a $\tau_1\tau_2$-β-open set U containing x such that $G_F(z) \cap (X \times V) \neq \varnothing$ for each $z \in U$. By Lemma 10, we obtain $U \subseteq G_F^-(X \times V) = F^-(V)$. This shows that F is lower $\beta(\tau_1, \tau_2)$-continuous. □

4. Characterizations of Upper and Lower Almost $\beta(\tau_1, \tau_2)$-Continuous Multifunctions

In this section, we introduce the concepts of upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, several interesting characterizations of these multifunctions are discussed.

Definition 6. A multifunction F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be

(1) Upper almost $\beta(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, and there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))$;

(2) Lower almost $\beta(\tau_1, \tau_2)$-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \varnothing$, and there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)) \neq \varnothing$ for every $z \in U$;

(3) upper almost (resp., lower almost) $\beta(\tau_1, \tau_2)$-continuous if F has this property at each point of X.

Remark 1. For a multifunction F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following implication holds:

$$\text{upper } \beta(\tau_1, \tau_2) - \text{continuity} \Longrightarrow \text{upper almost } \beta(\tau_1, \tau_2) - \text{continuity}. \tag{13}$$

The converse of the implication is not true in general. We present an example for the implication as follows.

Example 5. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\varnothing, \{a\}, \{a, b\}, X\}$. Let $Y = \{1, 2, 3, 4, 5\}$ with topologies $\sigma_1 = \{\varnothing, \{1\}, \{2, 3, 4, 5\}, Y\}$ and $\sigma_2 = \{\varnothing, \{2\}, \{3\}, \{2, 3\}, \{2, 3, 4, 5\}, Y\}$. A multifunction F: $(X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(a) = \{1\}$, $F(b) = \{2, 3\}$, and $F(c) = \{4, 5\}$. Then, F is upper almost $\beta(\tau_1, \tau_2)$-continuous, but F is not upper $\beta(\tau_1, \tau_2)$-continuous.

Remark 2. For a multifunction $F\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$, the following implication holds:

$$\text{lower } \beta(\tau_1,\tau_2) - \text{continuity} \Longrightarrow \text{lower almost } \beta(\tau_1,\tau_2) - \text{continuity}. \tag{14}$$

The converse of the implication is not true in general. We present an example for the implication as follows.

Example 6. Let $X = \{a,b\}$ with topologies $\tau_1 = \{\varnothing, \{b\}, X\}$ and $\tau_2 = \{\varnothing, X\}$. Let $Y = \{1,2,3,4\}$ with topologies $\sigma_1 = \{\varnothing, \{1,2\}, Y\}$ and $\sigma_2 = \{\varnothing, \{1\}, \{1,2\}, Y\}$. A multifunction $F\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is defined as follows: $F(a) = \{1,2\}$ and $F(b) = \{3\}$. Then, F is lower almost $\beta(\tau_1,\tau_2)$-continuous, but F is not lower $\beta(\tau_1,\tau_2)$-continuous.

Theorem 9. *A multifunction $F\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is upper almost $\beta(\tau_1,\tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2\text{-}\beta Int(F^+(\sigma_1\sigma_2\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y containing $F(x)$.*

Proof. Let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Then, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-sCl}(V)$. Then, $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-sCl}(V))$. Therefore, $x \in \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$.

Conversely, let V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Moreover, we have $x \in \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$. There exists a $\tau_1\tau_2$-β-open set G containing x such that $G \subseteq F^+(\sigma_1\sigma_2\text{-sCl}(V))$, and hence $F(G) \subseteq \sigma_1\sigma_2\text{-sCl}(V) = \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V))$. This shows that F is upper almost $\beta(\tau_1,\tau_2)$-continuous at x. □

Theorem 10. *A multifunction $F\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is lower almost $\beta(\tau_1,\tau_2)$-continuous at $x \in X$ if and only if $x \in \tau_1\tau_2\text{-}\beta Int(F^-(\sigma_1\sigma_2\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \varnothing$.*

Proof. The proof is similar to that of Theorem 9. □

Theorem 11. *For a multifunction $F\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is upper almost $\beta(\tau_1,\tau_2)$-continuous;

(2) For each $x \in X$ and each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U of X containing x such that $F(U) \subseteq \sigma_1\sigma_2\text{-}sCl(V)$;

(3) $F^+(V) \subseteq \tau_1\tau_2\text{-}\beta Int(F^+(\sigma_1\sigma_2\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V of Y;

(4) $\tau_1\tau_2\text{-}\beta Cl(F^-(\sigma_1\sigma_2\text{-}sInt(K))) \subseteq F^-(K)$ for every $\sigma_1\sigma_2$-closed set K of Y.

Proof

(1)$\Longrightarrow$(2): the proof follows from Definition 6 (1).

(2)$\Longrightarrow$(3): let V be a $\sigma_1\sigma_2$-open set of Y and $x \in F^+(V)$. Then, $F(x) \subseteq V$, and there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(U) \subseteq \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)) = \sigma_1\sigma_2\text{-sCl}(V)$. Therefore, we have $x \in U \subseteq F^+(\sigma_1\sigma_2\text{-sCl}(V))$. Thus, $x \in \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$. Consequently, we obtain $F^+(V) \subseteq \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$.

(3)$\Longrightarrow$(4): let K be a $\sigma_1\sigma_2$-closed set of Y. Then, since $Y - K$ is $\sigma_1\sigma_2$-open, we obtain

$$\begin{aligned} X - F^-(K) &= F^+(Y - K) \\ &\subseteq \tau_1\tau_2\text{-}\beta\text{Int}\left(F^+(\sigma_1\sigma_2\text{-sCl}(Y - K))\right) \\ &= \tau_1\tau_2\text{-}\beta\text{Int}\left(F^+(Y - \sigma_1\sigma_2\text{-sInt}(K))\right) \\ &= \tau_1\tau_2\text{-}\beta\text{Int}\left(X - F^-(\sigma_1\sigma_2\text{-sInt}(K))\right) \\ &= X - \tau_1\tau_2\text{-}\beta\text{Cl}\left(F^-(\sigma_1\sigma_2\text{-sInt}(K))\right). \end{aligned} \tag{15}$$

Therefore, we obtain $\tau_1\tau_2\text{-}\beta\text{Cl}(F^-(\sigma_1\sigma_2\text{-sInt}(K))) \subseteq F^-(K)$.

(4)$\Longrightarrow$(3): the proof is obvious.

(3)$\Longrightarrow$(1): let $x \in X$ and V be a $\sigma_1\sigma_2$-open set containing $F(x)$. Then, we have $x \in \tau_1\tau_2\text{-}\beta\text{Int}(F^+(\sigma_1\sigma_2\text{-sCl}(V)))$. Therefore, by Theorem 9, F is upper almost $\beta(\tau_1,\tau_2)$-continuous at x. □

Theorem 12. *For a multifunction $F\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$, the following properties are equivalent:*

(1) F is lower almost $\beta(\tau_1,\tau_2)$-continuous;

(2) For each $x \in X$ and each $\sigma_1\sigma_2$-open set V containing $F(x)$, there exists a $\tau_1\tau_2$-β-open set U containing x such that $F(z) \cap \sigma_1\sigma_2\text{-}sCl(V)$ for each $z \in U$;

(3) $F^-(V) \subseteq \tau_1\tau_2\text{-}\beta Int(F^-(\sigma_1\sigma_2\text{-}sCl(V)))$ for every $\sigma_1\sigma_2$-open set V;

(4) $\tau_1\tau_2\text{-}\beta Cl(F^+(\sigma_1\sigma_2\text{-}sInt(K))) \subseteq F^+(K)$ for every $\sigma_1\sigma_2$-closed set K.

Proof. By utilizing Theorem 10, this can be similar to Theorem 11. □

Theorem 13. *If a multifunction $F\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is lower almost $\beta(\tau_1,\tau_2)$-continuous, then $\beta ClF_{\circledast}\colon (X,\tau_1,\tau_2) \longrightarrow (Y,\sigma_1,\sigma_2)$ is lower almost $\beta(\tau_1,\tau_2)$-continuous.*

Proof. Suppose that F is lower almost $\beta(\tau_1,\tau_2)$-continuous. Let $x \in X$ and V be a $\sigma_1\sigma_2$-open set of Y such that $\beta\text{Cl}F_{\circledast}(x) \cap V \neq \varnothing$. By Lemma 9, we have $x \in \beta\text{Cl}F_{\circledast}^-(V) = F^-(V)$. Since F is lower almost $\beta(\tau_1,\tau_2)$-continuous, there exists a $\tau_1\tau_2$-β-open set U containing x such that

$$F(z) \cap \sigma_1\text{-Int}\left(\sigma_2\text{-Cl}(V)\right) \neq \varnothing, \quad \text{for each } z \in U. \tag{16}$$

Therefore, $\sigma_1\sigma_2\text{-}\beta\text{Cl}(F(z)) \cap \sigma_1\text{-Int}(\sigma_2\text{-Cl}(V)) \neq \varnothing$ for each $z \in U$, and hence,

$$\beta\text{Cl}F_{\circledast}(z) \cap \sigma_1\text{-Int}\left(\sigma_2\text{-Cl}(V)\right) \neq \varnothing, \quad \text{for each } z \in U. \tag{17}$$

This shows that $\beta\mathrm{Cl}F_{\circledast}$ is lower almost $\beta(\tau_1, \tau_2)$-continuous. □

Definition 7. Let (X, τ_1, τ_2) be a bitopological space. The β-frontier of a subset A of X, denoted by $\tau_1\tau_2$-$\beta\mathrm{Fr}(A)$, is defined by

$$\begin{aligned}\tau_1\tau_2\text{-}\beta\mathrm{Fr}(A) &= \tau_1\tau_2\text{-}\beta\mathrm{Cl}(A) \cap \tau_1\tau_2\text{-}\beta\mathrm{Cl}(X - A)\\ &= \tau_1\tau_2\text{-}\beta\mathrm{Cl}(A) - \tau_1\tau_2\text{-}\beta\mathrm{Int}(A).\end{aligned} \tag{18}$$

Theorem 14. *The set of all points x of X at which a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is not upper L $\beta(\tau_1, \tau_2)$-continuous is identical with the union of the $\tau_1\tau_2$-β-frontier of the upper inverse images of $\sigma_1\sigma_2$-open sets containing $F(x)$.*

Proof. Let $x \in X$ at which F is not upper $\beta(\tau_1, \tau_2)$-continuous. There exists a $\sigma_1\sigma_2$-open set V containing $F(x)$ such that $U \cap (X - F^+(V)) \neq \varnothing$ for every $\tau_1\tau_2$-β-open set U containing x. Then, we have

$$x \in \tau_1\tau_2\text{-}\beta\mathrm{Cl}\left(X - F^+(V)\right) = X - \tau_1\tau_2\text{-}\beta\mathrm{Int}\left(F^+(V)\right), \tag{19}$$

and $x \in F^+(V)$. Hence, we obtain $x \in \tau_1\tau_2$-$\beta\mathrm{Fr}(F^+(V))$.

Conversely, suppose that V is a $\sigma_1\sigma_2$-open set containing $F(x)$ such that $x \in \tau_1\tau_2$-$\beta\mathrm{Fr}(F^+(V))$. If F is upper $\beta(\tau_1, \tau_2)$-continuous at x, there exists a $\tau_1\tau_2$-β-open set U containing x such that $U \subseteq F^+(V)$. This implies that $x \in \tau_1\tau_2$-$\beta\mathrm{Int}(F^+(V))$. This is a contradiction; hence, F is not upper $\beta(\tau_1, \tau_2)$-continuous. □

Theorem 15. *The set of all points x of X at which a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is not lower $\beta(\tau_1, \tau_2)$-continuous is identical with the union of the $\tau_1\tau_2$-β-frontier of the lower inverse images of $\sigma_1\sigma_2$-open sets meeting $F(x)$.*

Proof. The proof is similar to that of Theorem 14.

To discuss the relationships between upper and lower almost $\beta(\tau_1, \tau_2)$-continuous multifunctions and another type of continuity, we define upper and lower $\beta(\tau_1, \tau_2)$-continuous for multifunctions. □

Definition 8. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is said to be

(1) Upper (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $F(x)$, and there exists a $\tau_1\tau_2$-open set U containing x such that $F(U) \subseteq V$;

(2) Lower (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y such that $F(x) \cap V \neq \varnothing$, and there exists a $\tau_1\tau_2$-open set U containing x such that $F(z) \cap V \neq \varnothing$ for every $z \in U$;

(3) Upper (resp., lower) (τ_1, τ_2)-continuous if F has this property at each point of X.

Remark 3. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following implication holds:

$$\text{upper } (\tau_1, \tau_2) - \text{continuity} \Longrightarrow \text{upper almost } \beta(\tau_1, \tau_2) - \text{continuity}. \tag{20}$$

The converse of the implication is not true in general. We present an example for the implication as follows.

Example 7. Let $X = \{a, b, c\}$ with topologies $\tau_1 = \{\varnothing, \{a\}, \{b\}, \{a, b\}, X\}$ and $\tau_2 = \{\varnothing, \{a\}, \{a, b\}, X\}$. Let $Y = \{1, 2, 3, 4, 5\}$ with topologies $\sigma_1 = \{\varnothing, \{1\}, \{2, 3, 4, 5\}, Y\}$ and $\sigma_2 = \{\varnothing, \{2\}, \{3\}, \{2, 3\}, \{2, 3, 4, 5\}, Y\}$. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(a) = \{1\}$, $F(b) = \{2, 3\}$, and $F(c) = \{4, 5\}$. Then, F is upper almost $\beta(\tau_1, \tau_2)$-continuous, but F is not upper (τ_1, τ_2)-continuous.

Remark 4. For a multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$, the following implication holds:

$$\text{lower } (\tau_1, \tau_2) - \text{continuity} \Longrightarrow \text{lower almost } \beta(\tau_1, \tau_2) - \text{continuity}. \tag{21}$$

The converse of the implication is not true in general. We present an example for the implication as follows.

Example 8. Let $X = \{a, b\}$ with topologies $\tau_1 = \{\varnothing, \{b\}, X\}$ and $\tau_2 = \{\varnothing, X\}$. Let $Y = \{1, 2, 3, 4\}$ with topologies $\sigma_1 = \{\varnothing, \{1, 2\}, Y\}$ and $\sigma_2 = \{\varnothing, \{1\}, \{1, 2\}, Y\}$. A multifunction $F: (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)$ is defined as follows: $F(a) = \{1, 2\}$ and $F(b) = \{3\}$. Then, F is lower almost $\beta(\tau_1, \tau_2)$-continuous, but F is not lower (τ_1, τ_2)-continuous.

5. Conclusion

The notion of continuity for multifunctions is an important concept in general topology as well as other branches of mathematics; furthermore, the application of continuity for multifunctions has appeared in many fields of sciences. This article deals with the concepts of $\beta(\tau_1, \tau_2)$-continuous multifunctions and almost $\beta(\tau_1, \tau_2)$-continuous multifunctions. Moreover, some characterizations of $\beta(\tau_1, \tau_2)$-continuous multifunctions and almost $\beta(\tau_1, \tau_2)$-continuous multifunctions are obtained. For multifunctions, the relationships between upper and lower almost $\beta(\tau_1, \tau_2)$-continuous and the other types of continuity are discussed in Section 4.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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