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Research Article

Asymptotic Behavior and Stationary Distribution of a Nonlinear Stochastic Epidemic Model with Relapse and Cure

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In this paper, by introducing environmental perturbation, we extend an epidemic model with graded cure, relapse, and nonlinear incidence rate from a deterministic framework to a stochastic differential one. The existence and uniqueness of positive solution for the stochastic system is verified. Using the Lyapunov function method, we estimate the distance between stochastic solutions and the corresponding deterministic system in the time mean sense. Under some acceptable conditions, the solution of the stochastic system oscillates in the vicinity of the disease-free equilibrium if the basic reproductive number $R_0 \leq 1$, while the random solution oscillates near the endemic equilibrium, and the system has a unique stationary distribution if $R_0 > 1$. Moreover, numerical simulation is conducted to support our theoretical results.

1. Introduction

Mathematical models can improve our understanding of the dynamics of infectious diseases, predict the transmission trend, and help us formulate preventive measures. The classical SIR and SIS epidemic models established by Kermack and McKendrick are one of the most important models in epidemiology [1, 2]. From then on, a large number of researchers have proposed and investigated more accurate epidemic models, taking into account different forms of incidence rate, intervention strategies, random perturbation, and other factors. In particular, the human body has certain immune mechanisms to keep itself healthy, and individuals recovered from some diseases may relapse and become reinfected [3–6]. Therefore, it is natural to consider immune effects in mathematical models. Recently, to explore infectious diseases in which infected individuals may be permanently rehabilitated or reinfected, Lahrouz et al. [7] proposed a nonlinear SIR epidemic model with relapse and graded cure as follows:

$$\begin{cases} \dot{S}(t) = \mu - \mu_1 S(t) - \beta S(t) f((I(t)) + \varepsilon I(t) + \gamma R(t), \\ \dot{I}(t) = \beta S(t) f(I(t)) - (\mu_2 + \varepsilon + \lambda) I(t) + \alpha R(t), \\ \dot{R}(t) = \lambda I(t) - (\mu_3 + \gamma + \alpha) R(t), \end{cases} \tag{1}$$

where $S(t)$, $I(t)$, and $R(t)$ denote the numbers of the population that are susceptible, infective, and recovered with temporary immunity, respectively. The parameter μ is the growth rate of S; $\mu_i\,(i = 1, 2, 3)$ denotes the death rates of S, I, and R, respectively; λ is the recovery rate of I; α, ε, and γ denote the relapse rate, the temporary immunity, and cure rate, respectively; β is the transmission rate from S to I. All the parameters are assumed to be positive, and it is biologically meaningful to suppose that $\mu_1 \leq \min\{\mu_2, \mu_3\}$. The nonlinear incidence rate $\beta S(t) f(I(t))$ in model (1) reflects the heterogeneous mixing of susceptible and infective population, and the force of infection $f(I)$ is a function of $\mathscr{C}^\infty$ on $[0, \infty)$ such that

$$f(0)=0, 0<f(I)<f'((0))I, \quad \forall I>0, (S(0), I(0), R(0)) \in \mathbb{R}_+^3. \tag{2}$$

Lahrouz et al. [7] have studied the global dynamics of system (1). The basic reproduction number is computed as

$$R_0 = \frac{\beta\mu f'(0)}{\mu_1(\mu_2+\varepsilon+\lambda-(\alpha\lambda/(\mu_3+\gamma+\alpha)))}, \tag{3}$$

and the unique disease-free equilibrium $E_0 = (\mu/\mu_1, 0, 0)$ is globally asymptotically stable if $R_0 < 1$. Under some additional conditions, system (1) has a unique endemic equilibrium E_* and it is globally asymptotically stable if $R_0 > 1$.

A large amount of research studies have found that the spread of diseases is naturally subject to random environmental perturbation, such as unpredictable human exposure and meteorological factors [8, 9]. Hence, an increasing number of stochastic epidemic models including environmental noise have been developed [10–13]. In the present paper, motivated by the approach in [14, 15], we introduce system (1) environmental noise which is directly proportional to S, I, and R and establish the following stochastic system:

$$\begin{cases} \mathrm{d}S(t) = [\mu-\mu_1 S(t)-\beta S(t) f(I(t))+\varepsilon I(t)+\gamma R(t)]\mathrm{d}t \\ \qquad +\sigma_1 S(t)\mathrm{d}B_1(t), \\ \mathrm{d}I(t) = [\beta S(t) f(I(t))-(\mu_2+\varepsilon+\lambda)I(t)+\alpha R(t)]\mathrm{d}t \\ \qquad +\sigma_2 I(t)\mathrm{d}B_2(t), \\ \mathrm{d}R(t) = [\lambda I(t)-(\mu_3+\gamma+\alpha)R(t)]\mathrm{d}t+\sigma_3 R(t)\mathrm{d}B_3(t). \end{cases} \tag{4}$$

Throughout the paper, we let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathscr{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathscr{F}_0$ contains all $\mathscr{P}$-null sets), $B_i(t)\,(i=1,2,3)$ denotes a scalar Brownian motion defined on the complete probability space Ω, and $\mathbb{R}_+^3 = \{x_i > 0, i = 1, 2, 3\}$.

The rest of the paper is organized as follows. In Section 2, we prove the existence and uniqueness of the positive solution for system (4), and some long time behavior of the solution is discussed. In Section 3, we analyze the asymptotic behavior of system (4) near the disease-free equilibrium and estimate the distance between stochastic solutions and the corresponding deterministic system in the time mean sense. In Section 4, asymptotic behavior near endemic equilibrium is analyzed, and we also obtain sufficient conditions for the existence of stationary distribution and persistence of diseases. In Section 5, numerical simulation is displayed to support our theoretical results. A brief conclusion is given in the last section.

2. Existence and Uniqueness of the Positive Solution

In this section, we present two main results. The first theorem guarantees the existence and uniqueness of the positive solution for system (4), and the second one shows some long time behavior of the solution.

Theorem 1. *For any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$, there exists a unique positive solution $(S(t), I(t), R(t))$ for system (4) on $t \geq 0$ and the solution will remain in $\mathbb{R}_+^3$ with probability one.*

Proof. Since the coefficients of system (4) are locally Lipschitz continuous, then for any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$ there is a unique local solution $(S(t), I(t), R(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. Moreover, the unique local solution to model (4) is positive by Itô's formula [16]. Therefore, it suffices to verify that the solution is global, i.e., $\tau_e = \infty$ a.s. Let $n_0 > 0$ be sufficiently large such that $S(0)$, $I(0)$, and $R(0)$ lie within the interval $[1/n_0, n_0]$. For each integer $n \geq n_0$, define the stopping times:

$$\tau_n = \inf\left\{t \in [0, \tau_e): \min\{S(t), I(t), R(t)\} \leq \frac{1}{n} \text{ or } \max\{S(t), I(t), R(t)\} \geq n\right\}. \tag{5}$$

Set $\inf\varnothing = \infty$ ($\varnothing$ represents the empty set). Note that τ_n is increasing as $n \longrightarrow \infty$. Let $\tau_\infty = \lim_{n\longrightarrow\infty}\tau_n$, then $\tau_\infty \leq \tau_e$ a.s. Now, we state that $\tau_\infty = \infty$ a.s. If this statement is violated, then there exists a constant $T > 0$ and $\varepsilon \in (0, 1)$ such that $\mathbb{P}\{\tau_\infty \leq T\} > \varepsilon$. As a consequence, there exists an integer $n_1 \geq n_0$ such that

$$\mathbb{P}\{\tau_n \leq T\} \geq \varepsilon, \quad \text{for all } n \geq n_1. \tag{6}$$

Define a nonnegative C^2-function V: $\mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ as

$$V(t) = V(S, I, R) = \left(S - C - C\ln\frac{S}{C}\right) + (I - 1 - \ln I) + (R - 1 - \ln R), \tag{7}$$

where $C > 0$ is a positive constant determined later, and the nonnegativity can be obtained from $u - 1 - \ln u \geq 0$ for any $u > 0$.

Let $n \geq n_0$ and $T > 0$ be arbitrary. Applying Itô's formula to V, we obtain that

$$\mathrm{d}V(S, I, R) = \mathscr{L}V\mathrm{d}t + \sigma_1 S\left(1-\frac{C}{S}\right)\mathrm{d}B_1(t) + \sigma_2 I\left(1-\frac{1}{I}\right)\mathrm{d}B_2(t) + \sigma_3 R\left(1-\frac{1}{R}\right)\mathrm{d}B_3(t), \tag{8}$$

where

$$\begin{aligned}\mathscr{L}V &= \mu + C\mu_1 + \mu_2 + \mu_3 + \varepsilon + \lambda + \gamma + \alpha + C\beta f(I) + \frac{C\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} - \mu_1 S - \frac{C\mu}{S} - \frac{CI\varepsilon}{S} - \frac{CR\gamma}{S} \\ &\quad - \mu_2 I - \frac{\beta S f(I)}{I} - \frac{\alpha R}{I} - \mu_3 R - \frac{\lambda I}{R} \\ &\le \mu + C\mu_1 + \mu_2 + \mu_3 + \varepsilon + \lambda + \gamma + \alpha + \frac{C\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} + (C\beta f'(0) - \mu_2) I \\ &\le \mu + C\mu_1 + \mu_2 + \mu_3 + \varepsilon + \lambda + \gamma + \alpha + \frac{C\sigma_1^2 + \sigma_2^2 + \sigma_3^2}{2} := K.\end{aligned} \tag{9}$$

Here, we choose C such that $C\beta f'(0) - \mu_2 \le 0$. Then, substituting the inequality into (8) yields that

$$\begin{aligned}\mathrm{d}V(S, I, R) &\le K\mathrm{d}t + \sigma_1 S\left(1 - \frac{C}{S}\right)\mathrm{d}B_1(t) + \sigma_2 I\left(1 - \frac{1}{I}\right)\mathrm{d}B_2(t) \\ &\quad + \sigma_3 R\left(1 - \frac{1}{R}\right)\mathrm{d}B_3(t).\end{aligned} \tag{10}$$

Furthermore,

$$\begin{aligned}\int_0^{\tau_n \wedge T} \mathrm{d}V(S(r), I(r), R(r)) &\le \int_0^{\tau_n \wedge T} K\mathrm{d}t + \int_0^{\tau_n \wedge T} \sigma_1 S(r)\left(1 - \frac{C}{S(r)}\right)\mathrm{d}B_1(r) \\ &\quad + \int_0^{\tau_n \wedge T} \sigma_2 I(r)\left(1 - \frac{1}{I(r)}\right)\mathrm{d}B_2(r) + \int_0^{\tau_n \wedge T} \sigma_3 R(r)\left(1 - \frac{1}{R(r)}\right)\mathrm{d}B_3(r),\end{aligned} \tag{11}$$

where $\tau_n \wedge T = \min\{\tau_n, T\}$. Taking the expectation of the above inequality yields

$$EV(S(\tau_n \wedge T), I(\tau_n \wedge T), R(\tau_n \wedge T) \le V(S(0), I(0), R(0)) + KT). \tag{12}$$

Set $\Omega_n = \{\tau_n \le T\}$ for $n \ge n_1$, then $\mathbb{P}(\Omega_k) \ge \varepsilon$ due to (6). Note that, for every $\omega \in \Omega_n$, at least one of $S(\tau_n, \omega), I(\tau_n, \omega), R(\tau_n, \omega)$ equals to either n or $1/n$. Hence,

$$V(S(\tau_n \wedge T), I(\tau_n \wedge T), R(\tau_n \wedge T)) \ge (n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 - \ln\frac{1}{n}\right). \tag{13}$$

Following from (12), we obtain

$$\begin{aligned}V(S(0), I(0), R(0)) + KT &\ge E\left[I_{\Omega_n(\omega)} V(S(\tau_n), I(\tau_n), R(\tau_n))\right] \\ &\ge \varepsilon\left((n - 1 - \ln n) \wedge \left(\frac{1}{n} - 1 - \ln\frac{1}{n}\right)\right),\end{aligned} \tag{14}$$

where $I_{\Omega_n(\omega)}$ is the indicator function of Ω_n. Taking $n \longrightarrow \infty$, we have $\infty > +\infty$, which is a contradiction. The conclusion is confirmed. □

Theorem 2. *For any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$, the solution $(S(t), I(t), R(t))$ of system (4) has the following properties:*

$$\begin{aligned}&\lim_{t \longrightarrow \infty} \frac{S(t)}{t} = 0, \\ &\lim_{t \longrightarrow \infty} \frac{I(t)}{t} = 0, \\ &\lim_{t \longrightarrow \infty} \frac{R(t)}{t} = 0, \text{ a.s.}, \\ &\lim_{t \longrightarrow \infty} \frac{\int_0^t S(r)dB_1(r)}{t} = 0, \\ &\lim_{t \longrightarrow \infty} \frac{\int_0^t I(r)dB_2(r)}{t} = 0, \\ &\lim_{t \longrightarrow \infty} \frac{\int_0^t R(r)dB_3(r)}{t} = 0, \text{ a.s.}\end{aligned} \tag{15}$$

The proof is standard, and we omit it here.

3. Asymptotic Behavior around the Disease-Free Equilibrium

For the deterministic system (1), the unique disease-free equilibrium $E_0 = (\mu/\mu_1, 0, 0)$ is globally asymptotically stable if the reproduction number $R_0 < 1$. The following theorem

shows the asymptotic behavior of the stochastic system (4) near E_0.

Theorem 3. *Let $(S(t), I(t), R(t))$ be the solution of system (4) with any initial value $(S(0), I(0), R(0)) \in \mathbb{R}_+^3$. If $R_0 \le 1, \mu_1 > 2\sigma_1^2 \vee \sigma_3^2$, and $M_1 > 0, M_2 > 0$, then*

$$\limsup_{t\longrightarrow\infty} \frac{1}{t} E \int_0^t \left[\left(\frac{3}{4}\mu_1 - \sigma_1^2\right)\left(S(r) - \frac{\mu}{\mu_1}\right)^2 + M_1 I^2(r) + M_2 R^2(r)\right] \mathrm{d}r \le \sigma_1^2 \left[\frac{\mu^2}{\mu_1^2} + \frac{(\mu_1+\mu_2+\lambda)^2 \mu}{\mu_1^2(\mu_2+\lambda)(\mu_1 - 2\sigma_1^2)}\right], \tag{16}$$

where $M_1 = ((\mu_2+\lambda)/8) - \sigma_2^2/2 - ((\mu_1+\mu_2+\lambda)^2\varepsilon^2/\mu_1(\mu_2+\lambda)(\mu_1 - 2\sigma_1^2))$, $M_2 = ((\mu_1 - \sigma_3^2)\mu_1(\mu_2+\lambda)/8\lambda) - (2(\mu_1+\mu_2+\lambda)^2(\gamma+\alpha)^2/\mu_1(\mu_2+\lambda)(\mu_1 - 2\sigma_1^2))$.

Proof. DefineC^2 functions as follows:

$$\begin{aligned} V_1(S) &= \frac{(S - (\mu/\mu_1))^2}{2}, \\ V_2(I, R) &= \frac{\mu}{\mu_1}\left[I + \frac{\alpha}{\mu_3+\gamma+\alpha} R\right], \\ V_3(S, I) &= \frac{(S - (\mu/\mu_1) + I)^2}{2}, \\ V_4(R) &= \frac{R^2}{2}. \end{aligned} \tag{17}$$

Making use of Itô's formula, we obtain

$$\begin{aligned} \mathrm{d}V_1 &= \left(\left(S - \frac{\mu}{\mu_1}\right)(\mu - \mu_1 S - \beta S f(I) + \varepsilon I + \gamma R) + \frac{\sigma_1^2 S^2}{2}\right)\mathrm{d}t \\ &\quad + \sigma_1 S\left(S - \frac{\mu}{\mu_1}\right)\mathrm{d}B_1(t) \\ &:= \mathscr{L}V_1 \mathrm{d}t + \sigma_1 S\left(S - \frac{\mu}{\mu_1}\right)\mathrm{d}B_1(t), \end{aligned} \tag{18}$$

where

$$\begin{aligned} \mathscr{L}V_1 &= -\mu_1\left(S - \frac{\mu}{\mu_1}\right)^2 - \beta f(I)\left(S - \frac{\mu}{\mu_1}\right)^2 - \frac{\beta\mu f(I)}{\mu_1}\left(S - \frac{\mu}{\mu_1}\right) + \left(S - \frac{\mu}{\mu_1}\right)(\varepsilon I + \gamma R) + \frac{\sigma_1^2 S^2}{2} \\ &\le -\mu_1\left(S - \frac{\mu}{\mu_1}\right)^2 - \frac{\beta\mu f(I)}{\mu_1}\left(S - \frac{\mu}{\mu_1}\right) + \frac{\varepsilon^2 I^2}{\mu_1} + \frac{\mu_1}{4}\left(S - \frac{\mu}{\mu_1}\right)^2 + \frac{\gamma^2 R^2}{\mu_1} + \frac{\mu_1}{4}\left(S - \frac{\mu}{\mu_1}\right)^2 \\ &\quad + \sigma_1^2\left(S - \frac{\mu}{\mu_1}\right)^2 + \frac{\sigma_1^2\mu^2}{\mu_1^2} \\ &= -\left(\frac{\mu_1}{2} - \sigma_1^2\right)\left(S - \frac{\mu}{\mu_1}\right)^2 - \frac{\beta\mu f(I)}{\mu_1}\left(S - \frac{\mu}{\mu_1}\right) + \frac{\varepsilon^2 I^2}{\mu_1} + \frac{\gamma^2 R^2}{\mu_1} + \frac{\sigma_1^2\mu^2}{\mu_1^2}. \end{aligned} \tag{19}$$

As $R_0 \le 1$, we have

$$\begin{aligned} \mathscr{L}V_2 &= \frac{\beta\mu S f(I)}{\mu_1} - \frac{\mu(\mu_2+\varepsilon+\lambda)I}{\mu_1} + \frac{\mu\alpha\lambda I}{\mu_1(\mu_3+\gamma+\alpha)} \\ &= \frac{\beta\mu f(I)}{\mu_1}\left(S - \frac{\mu}{\mu_1}\right) + \frac{\beta\mu^2 f(I)}{\mu_1^2} - \frac{\mu(\mu_2+\varepsilon+\lambda)I}{\mu_1} + \frac{\mu\alpha\lambda I}{\mu_1(\mu_3+\gamma+\alpha)} \\ &\le \frac{\beta\mu f(I)}{\mu_1}\left(S - \frac{\mu}{\mu_1}\right) + \frac{\beta\mu^2 f\prime(0)I}{\mu_1^2} - \frac{\mu(\mu_2+\varepsilon+\lambda)I}{\mu_1} + \frac{\mu\alpha\lambda I}{\mu_1(\mu_3+\gamma+\alpha)} \\ &= \frac{\beta\mu f(I)}{\mu_1}\left(S - \frac{\mu}{\mu_1}\right) + \frac{\mu(\mu_2+\varepsilon+\lambda - (\alpha\lambda/(\mu_3+\gamma+\alpha)))}{\mu_1}(R_0 - 1)I \le \frac{\beta\mu f(I)}{\mu_1}\left(S - \frac{\mu}{\mu_1}\right). \end{aligned} \tag{20}$$

Similarly,

$$\begin{aligned}
\mathscr{L}V_3 &= -\mu_1\left(S-\frac{\mu}{\mu_1}\right)^2 + (\gamma+\alpha)\left(S-\frac{\mu}{\mu_1}\right)R - (\mu_1+\mu_2+\lambda)\left(S-\frac{\mu}{\mu_1}\right)I + (\gamma+\alpha)RI \\
&\quad - (\mu_2+\lambda)I^2 + \frac{S^2\sigma_1^2}{2} + \frac{I^2\sigma_2^2}{2} \\
&\le -\mu_1\left(S-\frac{\mu}{\mu_1}\right)^2 + \frac{\mu_1}{4}\left(S-\frac{\mu}{\mu_1}\right)^2 + \frac{(\gamma+\alpha)^2R^2}{\mu_1} + \frac{\mu_2+\lambda}{2}I^2 + \frac{(\mu_1+\mu_2+\lambda)^2}{2(\mu_2+\lambda)}\left(S-\frac{\mu}{\mu_1}\right)^2 \\
&\quad - (\mu_2+\lambda)I^2 + \frac{I^2\sigma_2^2}{2} + \frac{(\gamma+\alpha)^2R^2}{\mu_2+\lambda} + \frac{(\mu_2+\lambda)}{4}I^2 + \sigma_1^2\left(S-\frac{\mu}{\mu_1}\right)^2 + \frac{\sigma_1^2\mu^2}{\mu_1^2} \\
&= \left[-\frac{3\mu_1}{4} + \sigma_1^2 + \frac{(\mu_1+\mu_2+\lambda)^2}{2(\mu_2+\lambda)}\right]\left(S-\frac{\mu}{\mu_1}\right)^2 - \left(\frac{\mu_2+\lambda}{4} - \frac{\sigma_2^2}{2}\right)I^2 \\
&\quad + \frac{(\gamma+\alpha)^2(\mu_1+\mu_2+\lambda)}{\mu_1(\mu_2+\lambda)}R^2 + \frac{\sigma_1^2\mu^2}{\mu_1^2},
\end{aligned} \tag{21}$$

$$\begin{aligned}
\frac{\mu_1(\mu_2+\lambda)}{4\lambda}\mathscr{L}V_4 &= \frac{\mu_1(\mu_2+\lambda)IR}{4} - \frac{\mu_1(\mu_2+\lambda)(\mu_3+\gamma+\alpha)R^2}{4\lambda} + \frac{\sigma_3^2\mu_1(\mu_2+\lambda)R^2}{8\lambda} \\
&\le \frac{\mu_2+\lambda}{4}\left(\frac{I^2}{2} + \frac{\mu_1^2R^2}{2}\right) - \frac{\mu_1^2(\mu_2+\lambda)R^2}{4\lambda} + \frac{\sigma_3^2\mu_1(\mu_2+\lambda)R^2}{8\lambda} \\
&\le \frac{(\mu_2+\lambda)I^2}{8} - \frac{\mu_1^2(\mu_2+\lambda)R^2}{8\lambda} + \frac{\sigma_3^2\mu_1(\mu_2+\lambda)R^2}{8\lambda} \\
&= \frac{(\mu_2+\lambda)I^2}{8} - \frac{(\mu_1-\sigma_3^2)\mu_1(\mu_2+\lambda)R^2}{8\lambda}.
\end{aligned} \tag{22}$$

In inequalities (19)–(22), we have used the inequality $2ab \le a^2+b^2$, $(a+b)^2 \le 2a^2+2b^2$ for any $a, b \in \mathbb{R}$, and the young inequality $-(\mu_1+\mu_2+\lambda)(S-(\mu/\mu_1))I \le ((\mu_2+\lambda)/2)I^2 + ((\mu_1+\mu_2+\lambda)^2/2(\mu_2+\lambda))(S-(\mu/\mu_1))^2$.

Define a nonnegative C^2 function $V\colon \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ as follows:

$$V = V_3 + \frac{(\mu_1+\mu_2+\lambda)^2}{2(\mu_2+\lambda)((\mu_1/2)-\sigma_1^2)}(V_1+V_2) + \frac{\mu_1(\mu_2+\lambda)}{4\lambda}V_4. \tag{23}$$

Together with (19)–(22), we obtain

$$\begin{aligned}
LV &\le -\left(\frac{3\mu_1}{4} - \sigma_1^2\right)\left(S-\frac{\mu}{\mu_1}\right)^2 - \left[\frac{\mu_2+\lambda}{8} - \frac{\sigma_2^2}{2} - \frac{(\mu_1+\mu_2+\lambda)^2\varepsilon^2}{\mu_1(\mu_2+\lambda)(\mu_1-2\sigma_1^2)}\right]I^2 - \frac{\mu_1(\mu_2+\lambda)(\mu_1-\sigma_3^2)}{8\lambda}R^2 \\
&\quad + \left[\frac{(\gamma+\alpha)^2(\mu_1+\mu_2+\lambda)}{\mu_1(\mu_2+\lambda)} + \frac{(\mu_1+\mu_2+\lambda)^2(\gamma+\alpha)^2}{\mu_1(\mu_2+\lambda)(\mu_1-2\sigma_1^2)}\right]R^2 + \frac{\sigma_1^2\mu^2}{\mu_1^2} + \frac{\sigma_1^2(\mu_1+\mu_2+\lambda)^2\mu^2}{\mu_1^2(\mu_2+\lambda)(\mu_1-2\sigma_1^2)} \\
&\le -\left(\frac{3\mu_1}{4} - \sigma_1^2\right)\left(S-\frac{\mu}{\mu_1}\right)^2 - \left[\frac{\mu_2+\lambda}{8} - \frac{\sigma_2^2}{2} - \frac{(\mu_1+\mu_2+\lambda)^2\varepsilon^2}{\mu_1(\mu_2+\lambda)(\mu_1-2\sigma_1^2)}\right]I^2 - \left[\frac{\mu_1(\mu_2+\lambda)(\mu_1-\sigma_3^2)}{8\lambda}\right. \\
&\quad \left. - \frac{2(\gamma+\alpha)^2(\mu_1+\mu_2+\lambda)^2}{\mu_1(\mu_2+\lambda)(\mu_1-2\sigma_1^2)}\right]R^2 + \frac{\sigma_1^2\mu^2}{\mu_1^2} + \frac{\sigma_1^2(\mu_1+\mu_2+\lambda)^2\mu^2}{\mu_1^2(\mu_2+\lambda)(\mu_1-2\sigma_1^2)} \\
&= -\left(\frac{3\mu_1}{4} - \sigma_1^2\right)\left(S-\frac{\mu}{\mu_1}\right)^2 - M_1I^2 - M_2R^2 + \frac{\sigma_1^2\mu^2}{\mu_1^2} + \frac{\sigma_1^2(\mu_1+\mu_2+\lambda)^2\mu^2}{\mu_1^2(\mu_2+\lambda)(\mu_1-2\sigma_1^2)}.
\end{aligned} \tag{24}$$

Integrating both sides of (24) and taking expectation, we obtain

$$EV(t)-V(0)\le -E\int_0^t\left(\frac{3\mu_1}{4}-\sigma_1^2\right)\left(S(r)-\frac{\mu}{\mu_1}\right)^2\mathrm{d}r - E\int_0^t M_1I^2(r)\mathrm{d}r - E\int_0^t M_2R^2(r)\mathrm{d}r + \left[\frac{\sigma_1^2\mu^2}{\mu_1^2}+\frac{\sigma_1^2(\mu_1+\mu_2+\lambda)^2\mu^2}{\mu_1^2(\mu_2+\lambda)(\mu_1-2\sigma_1^2)}\right]t. \tag{25}$$

Hence,

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}E\int_0^t\left\{\left(\frac{3\mu_1}{4}-\sigma_1^2\right)\left(S(r)-\frac{\mu}{\mu_1}\right)^2+M_1I^2(r)+M_2R^2(r)\right\}\mathrm{d}r \le \sigma_1^2\left[\frac{\mu^2}{\mu_1^2}+\frac{(\mu_1+\mu_2+\lambda)^2\mu^2}{\mu_1^2(\mu_2+\lambda)(\mu_1-2\sigma_1^2)}\right]. \tag{26}$$

The above theorem shows that the solution of system (4) oscillates near the disease-free equilibrium E_0 in the time mean sense if $R_0\le 1$, and the magnitude of the oscillation is proportional to the intensity of noise. From the perspective of biology, the disease will be controlled in a small range if the intensity of noise is sufficiently small. □

4. Stationary Distribution and Asymptotic Behavior around the Endemic Equilibrium

In this section, we turn to the case when the reproduction number $R_0>1$ and discuss sufficient conditions for the persistence of disease. We first recall some general results. Consider ℓ-dimension stochastic equation:

$$\mathrm{d}X(t)=f(X(t))\mathrm{d}t+\sum_{r=1}^{k}g_r(X)\mathrm{d}B_r(t), \tag{27}$$

where $X(t)$ is a homogeneous Markov process in ℓ-dimension Euclidean space $\mathbb{R}^\ell$. The diffusion matrix is defined as follows:

$$A(x)=\left(a_{ij}(x)\right),\qquad a_{ij}(x)=\sum_{r=1}^{k}g_r^i(x)g_r^j(x). \tag{28}$$

Lemma 1 (see [17]). *The Markov process X(t), the solution of system (27), has a unique ergodic stationary distribution $\mu(\cdot)$, if there exists a bounded domain $D\subset\mathbb{R}^\ell$ with regular boundary and*

(i) There exists a positive number M such that $\sum_{i,j=1}^{l}a_{ij}(x)\xi_i\xi_j\ge M|\xi|^2$, $x\in D$, $\xi\in R^\ell$.

(ii) There exists a nonnegative function $V\colon D^c\longrightarrow\mathbb{R}$ such that V is twice continuously differentiable and that for some $\theta>0$, $\mathscr{L}V(x)\le-\theta$, for any $x\in D^c$. Then,

$$\mathbb{P}_x\left\{\lim_{T\longrightarrow\infty}\frac{1}{T}\int_0^T f(X(t))\mathrm{d}t=\int_{\mathbb{R}^n}f(x)\mu(\mathrm{d}x)\right\}=1, \tag{29}$$

for all $x\in\mathbb{R}^\ell$, where $f\colon\mathbb{R}^\ell\longrightarrow\mathbb{R}$ be a function integrable with respect to the measure μ.

For the deterministic system (1), there exists at least one positive equilibrium $E_*=(S_*,I_*,R_*)$ if $R_0>1$. Moreover, assume the condition

$$(I-I_*)\left(\frac{f(I)}{I}-\frac{f(I_*)}{I_*}\right)\le 0,\quad \text{for all } I\ge 0, \tag{30}$$

holds; then, the equilibrium E_* is unique and globally stable according to Theorem 5.1 in [7].

Theorem 4. *Let $R_0>1$, and assume conditions (2) and (30) hold. If $\min\{m_1S_*^2,m_2I_*^2,m_3R_*^2\}>\delta>0$, $(m_i>0, i=1,2,3)$, then for any initial value $(S(0),I(0),R(0))\in\mathbb{R}_+^3$, system (4) has a unique stationary distribution and the ergodicity hold. Especially, we have*

$$\lim_{t\longrightarrow\infty}\frac{1}{t}E\int_0^t\left[m_1(S(r)-S_*)^2+m_2(I(r)-I_*)^2+m_3(R(r)-R_*)^2\right]\mathrm{d}r\le\delta, \tag{31}$$

Here, $m_1=(\mu_1/2)-\sigma_1^2, m_2=((\mu_2+\lambda)/4)-\sigma_2^2, m_3=((\mu_2+\lambda)(\mu_3+\gamma+\alpha))/\lambda^2((\mu_3+\gamma+\alpha)/2-\sigma_3^2)-((\alpha+\gamma)^2/2\mu_1)-((\alpha+\gamma+(\alpha\mu_1+\mu_2+\lambda)/\beta f I_*))))^2)/\mu_2+\lambda)$, and $\delta=\sigma_1^2S_*^2+\sigma_2^2I_*^2+(\sigma_2^2I_*^2(\mu_1+\mu_2+\lambda)/2\beta f(I_*))+((\mu_2+\lambda)(\mu_3+\gamma+\alpha)/\lambda^2)\sigma_3^2R_*^2$.

Proof. In order to prove the existence of a unique ergodic stationary distribution, it suffices to verify conditions (i) and (ii) in Lemma 1. To begin with, it is easy to see that $E_*=(S_*,I_*,R_*)$ satisfies

$$\begin{aligned}\mu&=\mu_1S_*+\beta S_*f(I_*)-\varepsilon I_*-\gamma R_*,\\ \beta S_*f(I_*)&=(\mu_2+\varepsilon+\lambda)I_*-\alpha R_*,\\ \lambda I_*&=(\mu_3+\gamma+\alpha)R_*.\end{aligned} \tag{32}$$

Define nonnegative C^2 functions as follows:

$$V_1(S,I)=\frac{(S-S_*+I-I_*)^2}{2},\qquad V_2(I)=I-I_*-I_*\log\frac{I}{I_*},\qquad V_3(R)=\frac{(R-R_*)^2}{2}. \tag{33}$$

By a standard calculation, we obtain

$$\begin{aligned}\mathscr{L}V_1 &= -\mu_1(S-S_*)^2-(\mu_1+\mu_2+\lambda)(S-S_*)(I-I_*)+(\alpha+\gamma)(S-S_*)(R-R_*)-(\mu_2+\lambda)(I-I_*)^2\\&\quad+(\alpha+\gamma)(I-I_*)(R-R_*)+\frac{S^2\sigma_1^2+I^2\sigma_2^2}{2}\\&\le -\frac{\mu_1}{2}(S-S_*)^2-(\mu_2+\lambda)(I-I_*)^2-(\mu_1+\mu_2+\lambda)(S-S_*)(I-I_*)+(\alpha+\gamma)(I-I_*)(R-R_*)\\&\quad+\frac{(\alpha+\gamma)^2}{2\mu_1}(R-R_*)^2+\frac{S^2\sigma_1^2+I^2\sigma_2^2}{2},\end{aligned}\tag{34}$$

$$\begin{aligned}\mathscr{L}V_2 &= (I-I_*)\left[\beta S\left(\frac{f(I)}{I}-\frac{f(I_*)}{I_*}\right)+\frac{\beta f(I_*)}{I_*}(S-S_*)-\frac{\alpha R}{II_*}(I-I_*)+\frac{\alpha}{I_*}(R-R_*)\right]+\frac{\sigma_2^2I_*}{2}\\&=\beta S(I-I_*)\left(\frac{f(I)}{I}-\frac{f(I_*)}{I_*}\right)+\frac{\beta f(I_*)}{I_*}(S-S_*)(I-I_*)-\frac{\alpha R}{I_*I}(I-I_*)^2\\&\quad+\frac{\alpha}{I_*}(R-R_*)(I-I_*)+\frac{\sigma_2^2I_*}{2}\\&\le\frac{\beta f(I_*)}{I_*}(S-S_*)(I-I_*)+\frac{\alpha}{I_*}(R-R_*)(I-I_*)+\frac{\sigma_2^2I_*}{2},\end{aligned}\tag{35}$$

$$\begin{aligned}\mathscr{L}V_3 &= (R-R_*)[\lambda I-(\mu_3+\gamma+\alpha)R-\lambda I_*+(\mu_3+\gamma+\alpha)R_*]+\frac{\sigma_3^2R^2}{2}\\&=\lambda(I-I_*)(R-R_*)-(\mu_3+\gamma+\alpha)(R-R_*)^2+\frac{\sigma_3^2R^2}{2}\\&\le\frac{\lambda^2}{2(\mu_3+\gamma+\alpha)}(I-I_*)^2+\frac{\mu_3+\gamma+\alpha}{2}(R-R_*)^2-(\mu_3+\gamma+\alpha)(R-R_*)^2\\&\quad+\sigma_3^2(R-R_*)^2+\sigma_3^2R_*^2\\&=\frac{\lambda^2}{2(\mu_3+\gamma+\alpha)}(I-I_*)^2-\left(\frac{\mu_3+\gamma+\alpha}{2}-\sigma_3^2\right)(R-R_*)^2+\sigma_3^2R_*^2.\end{aligned}\tag{36}$$

Then, define a nonnegative C^2 function $V\colon \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$:

$$V(t)=V(S,I,R)=V_1(S,I)+\frac{I_*(\mu_1+\mu_2+\lambda)}{\beta f(I_*)}V_2(I)+\frac{(\mu_2+\lambda)(\mu_3+\gamma+\alpha)}{\lambda^2}V_3(R).\tag{37}$$

According to (34)–(36), it implies

$$\begin{aligned}\mathscr{L}V &\le -\frac{\mu_1}{2}(S-S_*)^2-\frac{\mu_2+\lambda}{2}(I-I_*)^2-\left[\frac{(\mu_2+\lambda)(\mu_3+\gamma+\alpha)}{\lambda^2}\left(\frac{\mu_3+\gamma+\alpha}{2}-\sigma_3^2\right)-\frac{(\alpha+\gamma)^2}{2\mu_1}\right](R-R_*)^2+\frac{S^2\sigma_1^2}{2}\\&\quad+\frac{I^2\sigma_2^2}{2}+\left(\alpha+\gamma+\frac{\alpha(\mu_1+\mu_2+\lambda)}{\beta f(I_*)}\right)(I-I_*)(R-R_*)+\frac{\sigma_2^2I_*^2(\mu_1+\mu_2+\lambda)}{2\beta f(I_*)}+\frac{(\mu_2+\lambda)(\mu_3+\gamma+\alpha)\sigma_3^2R_*^2}{\lambda^2}\\&\le -\left(\frac{\mu_1}{2}-\sigma_1^2\right)(S-S_*)^2-\left(\frac{\mu_2+\lambda}{4}-\sigma_2^2\right)(I-I_*)^2-\left[\frac{(\mu_2+\lambda)(\mu_3+\gamma+\alpha)}{\lambda^2}\left(\frac{\mu_3+\gamma+\alpha}{2}-\sigma_3^2\right)-\frac{(\alpha+\gamma)^2}{2\mu_1}\right.\\&\quad\left.-\frac{(\alpha+\gamma+(\alpha(\mu_1+\mu_2+\lambda))/\beta f(I_*))^2}{\mu_2+\lambda}\right](R-R_*)^2+\frac{\sigma_2^2I_*^2(\mu_1+\mu_2+\lambda)}{2\beta f(I_*)}+\frac{(\mu_2+\lambda)(\mu_3+\gamma+\alpha)}{\lambda^2}\sigma_3^2R_*^2+\sigma_1^2S_*^2+\sigma_2^2I_*^2\\&=-m_1(S-S_*)^2-m_2(I-I_*)^2-m_3(R-R_*)^2+\delta\coloneqq F(t).\end{aligned}\tag{38}$$

Denote an ellipsoid $\Sigma = \{(S, I, R): m_1(S - S_*)^2 + m_2(I - I_*)^2 + m_3(R - R_*)^2 \le \delta\}$, then the ellipsoid Σ lies entirely in $\mathbb{R}_+^3$ if $0 < \delta < \min\{m_1 S_*^2, m_2 I_*^2, m_3 R_*^2\}$. Take D to be any neighborhood of Σ with $\overline{D} \subseteq \mathbb{R}_+^3$, then there exists some $C > 0$ such that $\mathscr{L}V \le -C$ for any $x \in \mathbb{R}_+^3 \setminus D$. That is, condition (ii) holds.

The diffusion matrix of system (4) is given by

$$A = \begin{bmatrix} \sigma_1^2 S^2 & 0 & 0 \\ 0 & \sigma_2^2 I^2 & 0 \\ 0 & 0 & \sigma_3^2 R^2 \end{bmatrix}. \tag{39}$$

Choose $M = \min_{(S,I,R)\in\overline{D}\subset\mathbb{R}_+^3}\{\sigma_1^2 S^2, \sigma_2^2 I^2, \sigma_3^2 R^2\}$, we have

$$\sum_{i,j=1}^{3}\left(\sum_{k=1}^{3} a_{ik}(S,I,R)a_{jk}(S,I,R)\right)\xi_i\xi_j\xi_k = \sigma_1^2 S^2\xi_1^2 + \sigma_2^2 I^2\xi_2^2 + \sigma_3^2 R^2\xi_3^2 \ge M|\xi|^2, \quad (\xi_1, \xi_2, \xi_3) \in \mathbb{R}_+^3. \tag{40}$$

Then, condition (i) in Lemma 1 is satisfied. Therefore, according to Lemma 1, the stochastic system (4) has a unique stationary distribution and it is ergodic. Moreover, the dynamical behavior around the endemic equilibrium E_* satisfies

$$\lim_{t\longrightarrow\infty}\frac{1}{t}E\int_0^t\left[m_1(S(r) - S_*)^2 + m_2(I(r) - I_*)^2 + m_3(R(r) - R_*)^2\right]\mathrm{d}r \le \delta. \tag{41}$$

The proof is completed.

Theorem 4 shows that, under certain conditions, the solution of system (4) will oscillate around the endemic equilibrium E_* for the long time. Furthermore, the following theorem indicates that the disease will be almost surely persistent in the time mean sense. □

Theorem 5. *Suppose the conditions in Theorem 4 hold; then, the solution of system (4) has the property that*

$$\begin{aligned} &\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t S(r)\mathrm{d}r \ge \frac{m_1 S_*^2 - \delta}{2m_1 S_*}, \\ &\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t I(r)\mathrm{d}r \ge \frac{m_2 I_*^2 - \delta}{2m_2 I_*}, \\ &\liminf_{t\longrightarrow\infty}\frac{1}{t}\int_0^t R(r)\mathrm{d}r \ge \frac{m_3 R_*^2 - \delta}{2m_3 R_*}, \text{ a.s.} \end{aligned} \tag{42}$$

Proof. According to the proof process of Theorem 4, we have

$$\begin{aligned} \mathrm{d}V(t) \le{}& F(t)\mathrm{d}t + (S - S_* + I - I_*)(\sigma_1 S\mathrm{d}B_1(t) + \sigma_2 I\mathrm{d}B_2(t)) \\ &+ \frac{\sigma_2 I_*(\mu_1 + \mu_2 + \lambda)(I - I_*)}{\beta f(I_*)}\mathrm{d}B_2(t) \\ &+ \frac{\sigma_3(\mu_2 + \lambda)(\mu_3 + \gamma + \alpha)(R - R_*)R}{\lambda^2}\mathrm{d}B_3(t). \end{aligned} \tag{43}$$

Integrating the above inequality from 0 to t, we obtain

$$V(t) - V(0) \le \int_0^t F(r)\mathrm{d}r + M(t), \tag{44}$$

where

$$\begin{aligned} M(t) ={}& \int_0^t (S(r) - S_* + I(r) - I_*)(\sigma_1 S(r)\mathrm{d}B_1(r) + \sigma_2 I(r)\mathrm{d}B_2(r)) + \frac{\sigma_2 I_*(\mu_1 + \mu_2 + \lambda)}{\beta f(I_*)}\int_0^t (I(r) - I_*)\mathrm{d}B_2(r) \\ &+ \frac{\sigma_3(\mu_2 + \lambda)(\mu_3 + \gamma + \alpha)}{\lambda^2}\int_0^t (R(r) - R_*)R(r)\mathrm{d}B_3(r). \end{aligned} \tag{45}$$

From the strong law of large numbers for local martingales, we have $\lim_{t\longrightarrow\infty}(M(t)/t) = 0$ a.s. Thus, $\liminf_{t\longrightarrow\infty}(\int_0^t F(r)\mathrm{d}r/t) \ge 0$ a.s., which implies that

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\int_0^t\left[m_1(S(r) - S_*)^2 + m_2(I(r) - I_*)^2 + m_3(R(r) - R_*)^2\right]\mathrm{d}r \le \delta, \text{ a.s.} \tag{46}$$

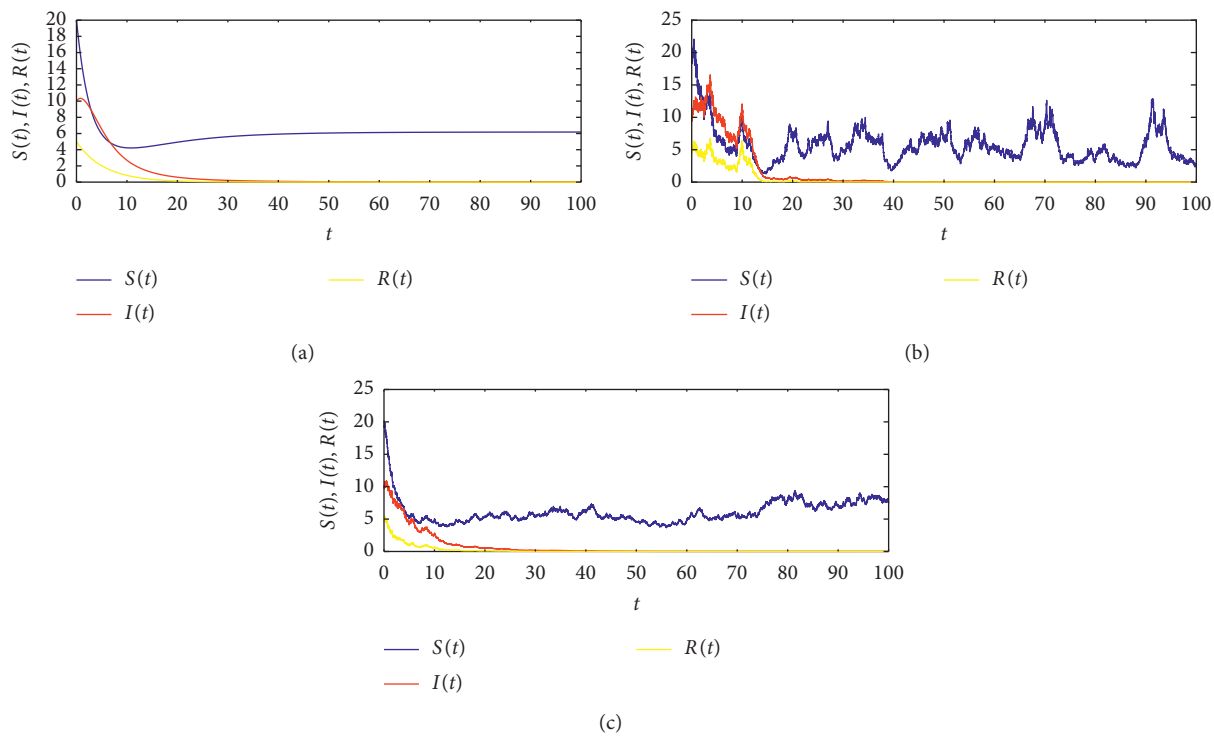


Figure 1: (a) The solution of deterministic system (1). (b) The solutions of stochastic system (4), where $\sigma_1 = 0.28, \sigma_2 = 0.256$, and $\sigma_3 = 0.396$. (c) The solution of stochastic system (4), where $\sigma_1 = 0.1, \sigma_2 = 0.15$, and $\sigma_3 = 0.25$.

Furthermore, we have

$$\begin{aligned}
&\limsup_{t\longrightarrow\infty} \frac{1}{t}\int_0^t m_1\left(S(r)-S_*\right)^2 \mathrm{d}r \le \delta, \text{ a.s.},\\
&\limsup_{t\longrightarrow\infty} \frac{1}{t}\int_0^t m_2\left(I(r)-I_*\right)^2 \mathrm{d}r \le \delta, \text{ a.s.},\\
&\limsup_{t\longrightarrow\infty} \frac{1}{t}\int_0^t m_3\left(R(r)-R_*\right)^2 \mathrm{d}r \le \delta, \text{ a.s.}
\end{aligned} \tag{47}$$

Due to $2S_*^2 - 2S_*S = 2S_*(S_* - S) \le S_*^2 + (S_* - S)^2$, it deduces

$$S \ge \frac{S_*}{2} - \frac{(S_* - S)^2}{2S_*}. \tag{48}$$

Substituting (48) into (47), we obtain

$$\begin{aligned}
&\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t S(r)\mathrm{d}r \ge \frac{S_*}{2} - \frac{\delta}{2m_1S_*} = \frac{m_1S_*^2-\delta}{2m_1S_*}, \text{ a.s.},\\
&\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t I(r)\mathrm{d}r \ge \frac{m_2I_*^2-\delta}{2m_2I_*}, \text{ a.s.},\\
&\liminf_{t\longrightarrow\infty} \frac{1}{t}\int_0^t R(r)\mathrm{d}r \ge \frac{m_3R_*^2-\delta}{2m_3R_*}, \text{ a.s.}
\end{aligned} \tag{49}$$

Therefore, the disease is persistent in the sense of time mean. The proof is completed. □

5. Numerical Simulations

In this section, we choose the saturation incidence function $f(I) = 1/(1+\theta I)$ as in [7]. Using Milstein's higher order method [18], we obtain the following discrete equations with respect to system (4):

$$\begin{cases}
S_{k+1} = S_k + \left(\mu - \mu_1 S_k - \dfrac{\beta S_k I_k}{1+\theta I_k} + \varepsilon I_k + \gamma R_k\right)\triangle t \\
\qquad + \sigma_1 S_k \xi_{1,k}\sqrt{\triangle t} + \dfrac{\sigma_1^2}{2} S_k\left(\xi_{1,k}^2\triangle t - \triangle t\right), \\
I_{k+1} = I_k + \left(\dfrac{\beta S_k I_k}{1+\theta I_k} - (\mu_2+\varepsilon+\lambda)I_k + \alpha R_k\right)\triangle t \\
\qquad + \sigma_2 I_k \xi_{2,k}\sqrt{\triangle t} + \dfrac{\sigma_2^2}{2} I_k\left(\xi_{2,k}^2\triangle t - \triangle t\right), \\
R_{k+1} = R_k + (\lambda I_k - (\mu_3+\gamma+\alpha)R_k)\triangle t + \sigma_3 R_k \xi_{3,k}\sqrt{\triangle t} \\
\qquad + \dfrac{\sigma_3^2}{2} R_k\left(\xi_{3,k}^2\triangle t - \triangle t\right).
\end{cases} \tag{50}$$

Here, the time increment $\triangle t > 0$, and $\xi_{1,k}$, $\xi_{2,k}$, and $\xi_{3,k}, k = 1, 2, \ldots, n$, are independent Gaussian random variables $N(0,1)$, and σ_k, $k = 1, 2, 3$, are intensities of white noise. Choose the initial value $(S(0), I(0), R(0)) = (20, 10, 5)$,

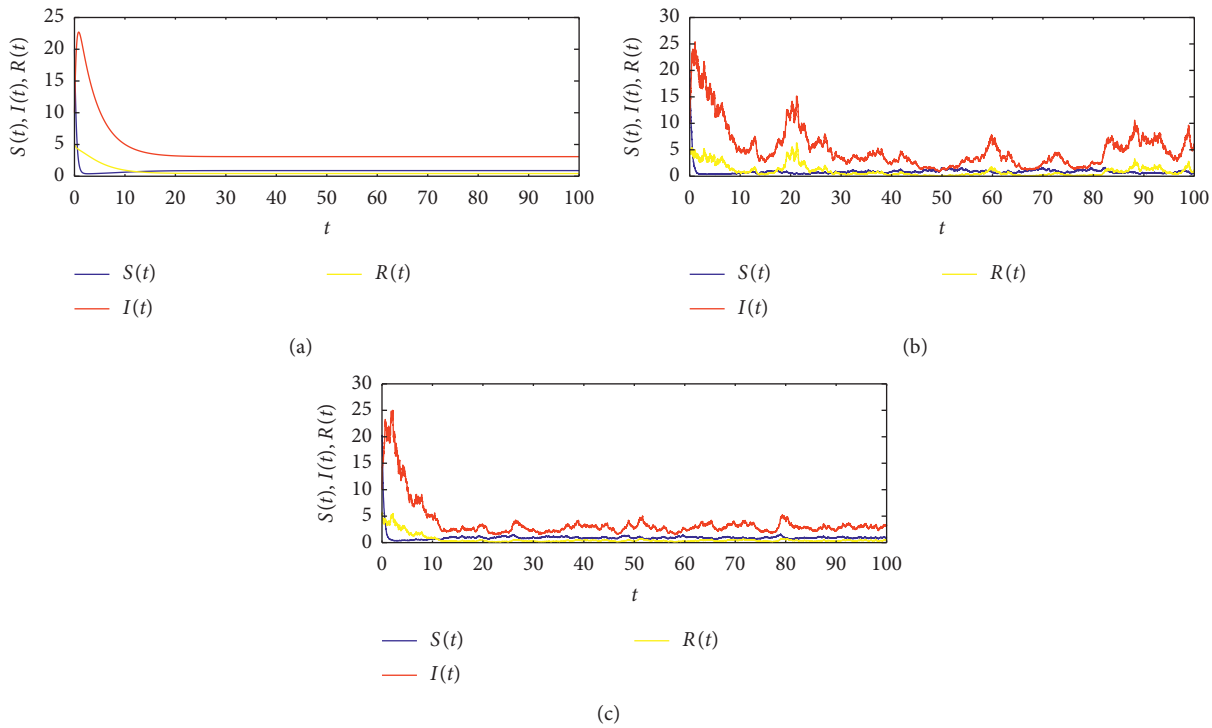


Figure 2: (a) The solution of deterministic system (1). (b) The solutions of stochastic system (4), where $\sigma_1 = 0.29, \sigma_2 = 0.25$, and $\sigma_3 = 0.5$. (c) The solution of stochastic system (4), where $\sigma_1 = 0.2, \sigma_2 = 0.2$, and $\sigma_3 = 0.3$.

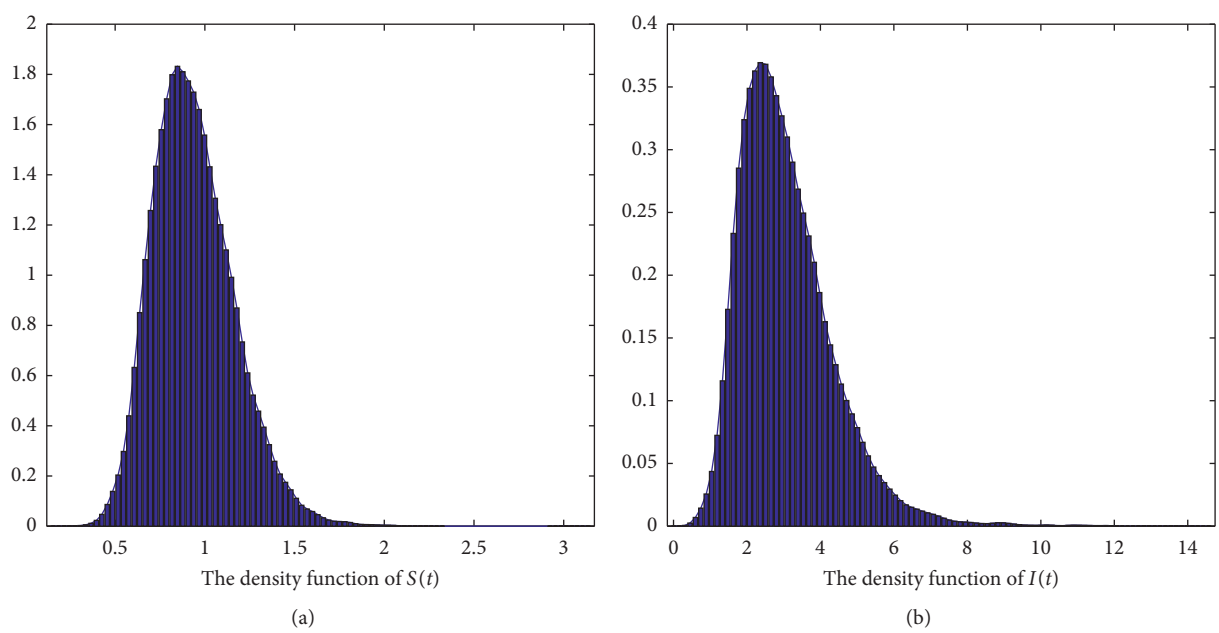
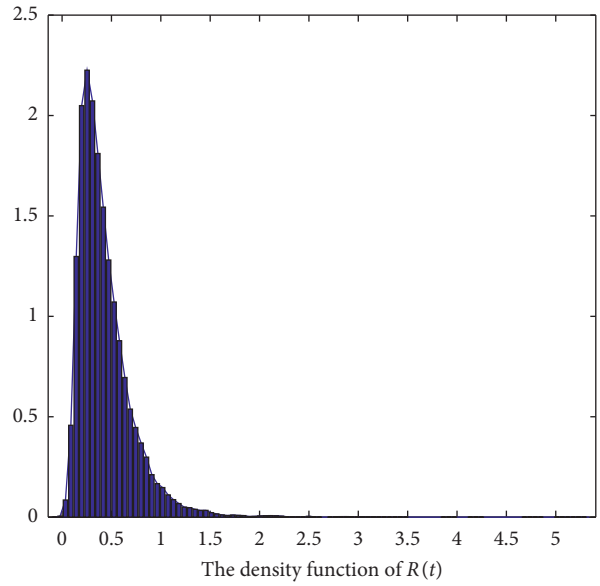


Figure 3: Continued.



(c)

Figure 3: The histograms of the solution $(S(t), I(t), R(t))$ of stochastic system (4) with the environmental white noise intensity $\sigma_1 = 0.2, \sigma_2 = 0.2$, and $\sigma_3 = 0.3$. (a) The density function of $S(t)$. (b) The density function of $I(t)$. (c) The density function of $R(t)$.

and the value of parameters as $\mu = 0.99, \mu_1 = 0.16, \mu_2 = 0.3, \mu_3 = 0.2, \beta = 0.04, \theta = 0.1,$ $\varepsilon = 0.002, \gamma = 0.0007, \lambda = 0.007$, and $\alpha = 0.0001$. The selection of these parameters satisfies the condition of Theorem 3, and numerical simulation is displayed in Figure 1. Moreover, we can see that when the noise intensity decreases, and the vibration of the solution for system (4) also decreases.

Similarly, take the value of parameters as $\mu = 0.99, \mu_1 = 0.18, \mu_2 = 0.1999, \mu_3 = 0.5, \beta = 0.4, \theta = 0.1, \varepsilon = 0.002, \gamma = 0.007, \lambda = 0.07$, and $\alpha = 0.0001$. It is easy to compute that $R_0 > 1$. The numerical simulation is shown in Figure 2, from which one can see that the solution of stochastic system vibrates around the endemic equilibrium E_*. Moreover, Figure 3 shows the histograms of $S(t), I(t)$, and $R(t)$, and it indicates the existence of a unique stationary distribution for system (4), and the disease will be almost surely persistent in the time mean sense.

6. Conclusion

Since most systems in the real world are disturbed by random and unpredictable perturbation, we introduce environmental noise of white noise type into the transmission of disease and study a stochastic version of a nonlinear SIRS epidemic model with relapse and cure. The reproduction number R_0 is a threshold parameter. If the conditions of Theorems 3 and 4 hold, according to $R_0 \leq 1$ or $R_0 > 1$, we prove that the solution of the stochastic model oscillates in the vicinity of the disease-free equilibrium and the endemic equilibrium, respectively, and the fluctuation intensity is proportional to the white noise intensity. Throughout the paper, we use numerical simulations to illustrate our theoretical results. In particular, we also prove that the stochastic SIR model has a unique ergodic stationary distribution, and the disease will be prevalent in the sense of time mean under some conditions. However, based on our theoretical analysis and numerical simulations, it is not clear that the disease can be eradicated.

Data Availability

All data generated or analyzed during this work are included in this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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