

Research Article

On Metrization of the Topologies Induced by Fuzzy Metrics

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In this paper, we focus on finding the metric functions in a fuzzy metric space. After introducing the concept of a stratified function in a fuzzy metric space, we prove that the topology generated by the family of stratified functions coincides with the topology generated by the fuzzy metric. Moreover, we get the concrete form of metric function under some special conditions.

1. Introduction

Since Zadeh [1] first proposed fuzzy set theory in 1965, many researchers have defined concepts of fuzzy metric spaces and studied their properties in different ways [2–4]. Inspired by the notion of probabilistic metric spaces, Kramosil and Michalek [5] in 1975 introduced the notion of fuzzy metric, a fuzzy set in the Cartesian product $X \times X \times R$ satisfying certain conditions. Later, George and Veeramani [6] used the concept of continuous t -norms to modify this definition of fuzzy metric space and showed that every fuzzy metric space generates a Hausdorff first-countable topology. So far the GV-fuzzy metric theory has been developed by many researchers. Many important topics about the classical metric spaces were developed to fuzzy metric spaces. In this process, it was found that the theory of fuzzy metric was very different from the classical theory of metric. For example, Gregori and Romaguera [7] proved that there exists a GV-fuzzy metric space which is not completable. Gregori and Romaguera [8] characterized the class of completable strong fuzzy metric spaces. Recently, some good related results about fixed point in fuzzy metric spaces were introduced. For example, one can see the works [9–11]. Meanwhile, the fuzzy metrics have been applied to domain theory, color image processing, and analysis of algorithms (see [12–17]).

In 2000, Gregori and Romaguera [18] obtained a somewhat surprising stronger result. They proved that every GV-fuzzy metric generates a metrizable topology. This

important result connects the GV-fuzzy metric and the classical metric. However, the form of the metric function has not been explored in existing literature. It is just the main goal of the present paper.

In this paper, we first introduce the concept of a stratified function in a fuzzy metric space which is slightly different from the GV-fuzzy metric space and show that the topology induced by the family of stratified functions is compatible with the metrizable topology. Then, under some special conditions, we give the concrete metric function whose topology coincides with the metrizable topology.

The structure of the paper is as follows. In the next section, we give the preliminary notions on fuzzy metrics, with which we deal. In Section 3, we show our main results. Finally, we give our concluding remarks in Section 4.

2. Preliminaries

In this section, we first introduce some basic concepts and properties of fuzzy metric spaces.

Definition 1 (see [19]). A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a continuous t -norm if it satisfies the following conditions:

- (1) $*$ is associative and commutative
- (2) $*$ is continuous
- (3) $a * 1 = a$ for each $a \in [0, 1]$

(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0, 1]$

The following continuous t -norms are used in this paper:

$$\begin{aligned} a\Delta_1 b &= \max\{a + b - 1, 0\}, \\ a\Delta_2 b &= ab, \\ a\Delta_3 b &= \min\{a, b\}. \end{aligned} \tag{1}$$

Also, we say the t -norm Δ' is stronger than the t -norm Δ'' if $a\Delta' b \geq a\Delta'' b$ for all $a, b \in [0, 1]$. In such case, we denote it as $\Delta' \geq \Delta''$. It is easy to see that $\Delta_3 \geq \Delta_2 \geq \Delta_1$.

In the sense of Gregori and Veeramani [6], a GV-fuzzy metric is defined by the follows.

Definition 2. Let X be a nonempty set and $*$ be a continuous t -norm. A fuzzy metric M on the set X is a mapping $M: X^2 \times (0, \infty) \rightarrow (0, 1]$ satisfying the following conditions: for all $x, y, z \in X, s, t > 0$:

$$\begin{aligned} \text{(GV1)} & M(x, y, t) > 0, \\ \text{(GV2)} & M(x, y, t) = 1 \text{ if and only if } x = y, \\ \text{(GV3)} & M(x, y, t) = M(y, x, t), \\ \text{(GV4)} & M(x, y, t) * M(y, z, s) \leq M(x, z, t + s), \\ \text{(GV5)} & M(x, y, \cdot): (0, \infty) \rightarrow (0, 1] \text{ is continuous.} \end{aligned} \tag{2}$$

If M is a GV-fuzzy metric on X , then the 3-tuple $(X, M, *)$ is said to be a GV-fuzzy metric space. In that case, if confusion is not possible, we call X a GV-fuzzy metric space for short. The following is a well-known result.

Lemma 1 (see [20]). $M(x, y, \cdot)$ is increasing for all $x, y \in X$.

Gregori and Veeramani proved in [7] that every GV-fuzzy metric M on X generates a topology τ_M which has as a base

$$\{B_M(x, r, t): x \in X, r \in (0, 1), t > 0\}, \tag{3}$$

where

$$B_M(x, r, t) = \{y \in X: M(x, y, t) > 1 - r\}, \tag{4}$$

for all $x \in X, r \in (0, 1)$, and $t > 0$. They proved that for each $x \in X$, the family $\{B_M(x, (1/n), (1/n)): n \in \mathbf{N}\}$ is a local base at x . A sequence $\{x_n\}$ in (X, τ_M) converges to $x \in X$ if and only if $\lim_{n \rightarrow \infty} M(x_n, x, t) = 1$ for all $t > 0$. Also, by using Kelley metrization lemma [21], they also proved that τ_M is a metrizable topology.

3. Main Results

First, we introduce the concept of a stratified function in a GV-fuzzy metric space.

Definition 3. $(X, M, *)$ is a GV-fuzzy metric space. Let $r \in (0, 1)$ and $x, y \in X$; set

$$d_r(x, y) = \inf\{t > 0: M(x, y, t) > 1 - r\}. \tag{5}$$

Then, d_r is called a r -stratified function with respect to $(X, M, *)$, $\{d_r: 0 < r < 1\}$, the family of stratified functions.

To avoid the occurrence of the empty set, by a fuzzy metric in the rest of this paper, we mean a GV-fuzzy metric satisfying

$$\text{(GV6)} \lim_{t \rightarrow \infty} M(x, y, t) = 1, \quad \forall x, y \in X. \tag{6}$$

Lemma 2. Let $(X, M, *)$ be a fuzzy metric space, $r \in (0, 1)$, $t > 0$, $x, y \in X$. Then,

- (1) For any $\lambda > d_r(x, y)$, $M(x, y, \lambda) > 1 - r$
- (2) The function d_r is decreasing with respect to $r \in (0, 1)$
- (3) $B_M(x, r, t) = N_r(x, t)$, where

$$N_r(x, t) = \{y \in X: d_r(x, y) < t\}. \tag{7}$$

- (4) The function $M(x, y, \cdot)$ is strictly increasing for the fixed points $x, y \in X$, if and only if for any $r \in (0, 1)$,

$$d_r(x, y) = \inf\{t > 0: M(x, y, t) \geq 1 - r\}. \tag{8}$$

Proof

- (1) Let $\lambda > d_r(x, y)$. From Definition 3, there exists $0 < t < \lambda$ such that $M(x, y, t) > 1 - r$. So, $M(x, y, \lambda) > 1 - r$.
- (2) It follows from Lemma 1 directly.
- (3) Let $y \in B_M(x, r, t)$, that is, $M(x, y, t) > 1 - r$. Since $M(x, y, \cdot)$ is continuous and increasing, there exists $0 < t_1 < t$ such that $M(x, y, t_1) > 1 - r$. From (5), we know $d_r(x, y) \leq t_1 < t$. So, $y \in N_r(x, t)$. From the arbitrariness of y , we know that

$$B_M(x, r, t) \subseteq N_r(x, t). \tag{9}$$

Let $z \in N_r(x, t)$; then, $d_r(x, z) < t$. From (5), there exists $0 < t_2 < t$ such that $M(x, z, t_2) > 1 - r$. Therefore, $M(x, z, t) \geq M(x, z, t_2) > 1 - r$. So, $z \in B_M(x, r, t)$. From the arbitrariness of z , we know that $N_r(x, t) \subseteq B_M(x, r, t)$.

- (4) Suppose that (8) holds; however, $M(x, y, \cdot)$ is not strictly increasing. Then, there exist $t_1, t_2 \in \{t > 0: 0 < M(x, y, t) < 1\}$ such that $t_1 < t_2$ and $M(x, y, \cdot) \equiv 1 - r_0$ on $[t_1, t_2]$. Thus,

$$\begin{aligned} \sup\{t > 0: M(x, y, t) \leq 1 - r_0\} &\geq t_2 > t_1 \\ &\geq \inf\{t > 0: M(x, y, t) \geq 1 - r_0\}. \end{aligned} \tag{10}$$

It is easy to see that

$$\begin{aligned} \sup\{t > 0: M(x, y, t) \leq 1 - r_0\} \\ = \inf\{t > 0: M(x, y, t) > 1 - r_0\}. \end{aligned} \tag{11}$$

Therefore, $d_{r_0}(x, y) > \inf\{t > 0: M(x, y, t) \geq 1 - r_0\}$, which conflicts with (8).

Conversely, suppose $M(x, y, \cdot)$ is strictly increasing. Let

$$\begin{aligned} t_0 &= d_r(x, y) \\ &= \inf\{t > 0: M(x, y, t) > 1 - r\}. \end{aligned} \tag{12}$$

Obviously, $\inf\{t > 0: M(x, y, t) \geq 1 - r\} \leq t_0$. If $\inf\{t > 0: M(x, y, t) \geq 1 - r\} < t_0$, then there is $0 < t_2 < t_0$ such that $M(x, y, t_2) \geq 1 - r$, so $M(x, y, t_0) > 1 - r$. Since $M(x, y, \cdot)$ is left continuous at t_0 , there is $\delta > 0$ such that $M(x, y, t_0 - \delta) > 1 - r$, which conflicts with the definition of t_0 . Thus, $\inf\{t > 0: M(x, y, t) \geq 1 - r\} \geq t_0$. So,

$$\begin{aligned} d_r(x, y) &= t_0 \\ &= \inf\{t > 0: M(x, y, t) > 1 - r\}. \end{aligned} \tag{13}$$

Now, from Lemma 2 (3), it is easy to see that the topology τ_M can be induced by the family of stratified functions. That is, we obtain the following theorem.

Theorem 1. Let $D = \{d_r: 0 < r < 1\}$ be the family of stratified functions with respect to a fuzzy metric space $(X, M, *)$, $N_r(x, t)$ be defined by (8), and

$$B_x = \{N_r(x, t): r \in (0, 1), t > 0\}. \tag{14}$$

Then,

- (1) B_x is a base of neighborhoods at $x \in X$.
- (2) The topology τ_D generated by $\{B_x: x \in X\}$ coincides with the topology τ_M .

Generally, a stratified function is not a pseudometric. In fact, we have the following result.

Theorem 2. Let $(X, M, *)$ be a fuzzy metric space. A stratified function d_r ($r \in (0, 1)$) is a pseudometric on X if and only if M satisfies the following condition: for any $x, y, z \in X$, $t_1, t_2 > 0$, if $M(x, z, t_1) > 1 - r$, $M(z, y, t_2) > 1 - r$, then

$$M(x, y, t_1 + t_2) > 1 - r. \tag{15}$$

Proof. For any $r \in (0, 1)$, it is obvious that $d_r(x, y) \geq 0$, $d_r(x, y) = d_r(y, x)$, and $d_r(x, y) = 0$ when $x = y$. Thus, to complete the proof, we only have to prove that $d_r(x, y) \leq d_r(x, z) + d_r(z, y)$ if and only if M satisfies condition (15). \square

Sufficiency. For any $\varepsilon > 0$, from Lemma 2 (1), we obtain

$$\begin{aligned} M\left(x, z, d_r(x, z) + \frac{\varepsilon}{2}\right) &> 1 - r, \\ M\left(z, y, d_r(z, y) + \frac{\varepsilon}{2}\right) &> 1 - r. \end{aligned} \tag{16}$$

From (15), we have

$$M(x, y, d_r(x, z) + d_r(z, y) + \varepsilon) > 1 - r. \tag{17}$$

Therefore,

$$d_r(x, y) \leq d_r(x, z) + d_r(z, y) + \varepsilon. \tag{18}$$

From the arbitrariness of $\varepsilon > 0$, we know $d_r(x, y) \leq d_r(x, z) + d_r(z, y)$.

Necessity. Suppose that $M(x, z, t_1) > 1 - r$, $M(z, y, t_2) > 1 - r$. By (GV5), there exists $\delta > 0$ such that

$$\begin{aligned} M(x, z, t_1 - \delta) &> 1 - r, \\ M(z, y, t_2 - \delta) &> 1 - r. \end{aligned} \tag{19}$$

So, $d_r(x, z) \leq t_1 - \delta$ and $d_r(z, y) \leq t_2 - \delta$. Since $d_r(x, y) \leq d_r(x, z) + d_r(z, y)$, we have $d_r(x, y) \leq t_1 + t_2 - 2\delta < t_1 + t_2$. By the definition of $d_r(x, y)$, there exists $t_0 < t_1 + t_2$ such that $M(x, y, t_0) > 1 - r$. Thus, $M(x, y, t_1 + t_2) > 1 - r$.

Remark 1. M satisfies (15) if $* = \Delta_3$.

Now, we explore the metric which induces the topology τ_M .

Definition 4. Let $R^+ = [0, \infty)$. We call function $f: R^+ \rightarrow R^+$ satisfies

- (i) The condition C_1 , if $f(0) = 0$, $f(t) \equiv 0$, and f is increasing and continuous at 0.
- (ii) The condition C_2 , if $f(0) = 0$, $f(t) > 0$ as $t > 0$, $\lim_{t \rightarrow +\infty} f(t) = +\infty$, $f(t_1 + t_2) \leq f(t_1) + f(t_2)$ for any $t_1, t_2 \in R^+$, and f is left continuous and increasing.

Theorem 3. $(X, M, *)$ is fuzzy metric space; the functions K and G satisfy the conditions C_1 and C_2 , respectively. Define a function d on $X \times X$ as

$$\begin{aligned} d(x, y) &= \inf\{s > 0: \text{if } G(t) > s, \text{ then } M(x, y, t) + K(s) \geq 1\}, \\ &\forall x, y \in X. \end{aligned} \tag{20}$$

If one of the following conditions is satisfied:

- (I) M satisfies condition (15)
- (II) $* \geq \Delta_1$, $\forall r_1, r_2 \in [0, \infty)$

$$K(r_1) + K(r_2) \leq K(r_1 + r_2), \tag{21}$$

then d is a metric on X .

Proof. First, we prove the following fact: if

$$G(t) > r > d(x, y), \tag{22}$$

then

$$M(x, y, t) \geq 1 - K(r). \tag{23}$$

In fact, if $G(t) > r > d(x, y)$, from the definition of d , we obtain there exists $0 < s < r$ such that if $G(t) > s$, then $M(x, y, t) + K(s) \geq 1$. Therefore, if $G(t) > r$, then $G(t) > s$, and hence, $M(x, y, t) + K(r) \geq M(x, y, t) + K(s) \geq 1$. That is, $M(x, y, t) \geq 1 - K(r)$.

Next, we prove d is a metric on X , that is, d satisfies the following properties: for any $x, y, z \in X$,

$$(M_1) \quad d(x, y) \geq 0, \quad d(x, y) = d(y, x)$$

$$(M_2) \quad d(x, y) = 0 \text{ if and only if } x = y$$

$$(M_3) \quad d(x, y) \leq d(x, z) + d(z, y)$$

The conclusion (M_1) is obvious.

For the conclusion (M_2) , it is easy to see that $d(x, y) = 0$ if $x = y$. Now, we suppose $d(x, y) = 0$; however, $x \neq y$. Then, there exists $t_0 > 0$ such that $M(x, y, t_0) \neq 1$, that is, $M(x, y, t_0) < 1$. Since K is continuous at 0, there exists $0 < r_0 < G(t_0)$ such that $K(r_0) < 1 - M(x, y, t_0)$. This is in direct contradiction to (23). Thus, $d(x, y) = 0$ implies that $x = y$.

To prove (M_3) , we take $r_1 > d(x, z)$ and $r_2 > d(z, y)$ arbitrarily. From (23), we know

(i) If $G(t) > r_1$, then

$$M(x, z, t) \geq 1 - K(r_1) \geq 1 - K(r_1 + r_2). \quad (24)$$

(ii) If $G(t) > r_2$, then

$$M(z, y, t) \geq 1 - K(r_2) \geq 1 - K(r_1 + r_2). \quad (25)$$

Now, suppose that $G(t) > r_1 + r_2$. Let

$$t_0 = \inf\{s: 0 < s \leq t, G(s) > r_1\}. \quad (26)$$

Obviously, $t_0 \leq t$. It is easy to prove that $G(t_0) \leq r_1$. In fact, if $G(t_0) > r_1$, from the left continuity of G , there exists $\eta > 0$ such that $G(t_0 - \eta) > r_1$. By the definition of t_0 , we conclude that $t_0 \leq t_0 - \eta$, a contradiction. The fact $G(t_0) \leq r_1$ implies that $G(t - t_0) \geq G(t) - G(t_0) > r_2$. By the left continuity of G again, we know there exists $\varepsilon > 0$ such that $G(t - (t_0 + \varepsilon)) = G(t - t_0 - \varepsilon) > r_2$. By the definition of t_0 , there exists $0 < s_1 \leq t$ such that $G(s_1) > r_1$ and $s_1 \leq t_0 + \varepsilon$. Noting that G is increasing, we obtain that $G(t_0 + \varepsilon) \geq G(s_1) > r_1$. From (i) and (ii), we have

$$\begin{aligned} M(x, z, t_0 + \varepsilon) &\geq 1 - K(r_1) \geq 1 - K(r_1 + r_2) \\ M(z, y, t - (t_0 + \varepsilon)) &\geq 1 - K(r_2) \\ &\geq 1 - K(r_1 + r_2). \end{aligned} \quad (27)$$

Combining conditions (I) and (II), we get $M(x, y, t) \geq 1 - K(r_1 + r_2)$.

By the definition of d , we know $d(x, y) \leq r_1 + r_2$. Since $r_1 > d(x, z)$ and $r_2 > d(z, y)$, we obtain

$$d(x, y) \leq d(x, z) + d(z, y), \quad (28)$$

directly.

Theorem 4. $(X, M, *)$ is a fuzzy metric space. M satisfies condition (15) or $* \geq \Delta_1$. Let

$$\rho(x, y) = \inf\{a > 0: M(x, y, a) + a \geq 1\}, \quad \forall x, y \in X. \quad (29)$$

Then, ρ is a metric on X and the topology τ_ρ induced by ρ coincides with the topology τ_M .

Proof. Let $G(t) = t, K(t) = t, \forall t \geq 0$. Then, K and G satisfy the conditions C_1 and C_2 , respectively. Besides, K satisfies condition (21). From Theorem 3, we know that

$$d(x, y) = \inf\{s > 0: \text{if } t > s, \text{ then } M(x, y, t) + s \geq 1\}, \quad (30)$$

$$\forall x, y \in X,$$

is a metric on X . Thus, to show that ρ is a metric, we only need to show $d(x, y) = \rho(x, y), \forall x, y \in X$. In fact, if $a > 0, M(x, y, a) + a \geq 1$ and $t > a$, then $M(x, y, t) + a \geq M(x, y, a) + a \geq 1$. Therefore,

$$\begin{aligned} \{a > 0: M(x, y, a) + a \geq 1\} \\ \subseteq \{s > 0: \text{if } t > s, \text{ then } M(x, y, t) + s \geq 1\}. \end{aligned} \quad (31)$$

From (29) and (30), we get $\rho(x, y) \geq d(x, y)$. On the other hand, from (30), for any $\varepsilon > 0$, there exists $s > 0$ such that $s < d(x, y) + \varepsilon$ and $M(x, y, t) + s \geq 1$ when $t > s$. Thus,

$$M(x, y, d(x, y) + \varepsilon) + s \geq 1, \quad (32)$$

and hence

$$M(x, y, d(x, y) + \varepsilon) + d(x, y) + \varepsilon > 1. \quad (33)$$

From (29), we get $\rho(x, y) \leq d(x, y) + \varepsilon$. From the arbitrariness of $\varepsilon > 0$, we have $\rho(x, y) \leq d(x, y)$. Thus, $d(x, y) = \rho(x, y)$.

Let $r \in (0, 1), x \in X$. We put $U(x, r) = \{y \in X: \rho(x, y) < r\}$. It is easy to show that

$$U(x, r) \subseteq B_M(x, r, r) \subseteq U(x, 2r). \quad (34)$$

In fact, for any $y \in U(x, r), \inf\{a > 0: M(x, y, a) + a \geq 1\} < r$. Thus, there is $a > 0$ such that $a < r$ and $M(x, y, a) + a \geq 1$. So, $M(x, y, r) + r > 1$, that is, $y \in B_M(x, r, r)$. Hence,

$$U(x, r) \subseteq B_M(x, r, r). \quad (35)$$

On the other hand, for any $z \in B_M(x, r, r), M(x, z, r) + r > 1$. From (29), we get $\rho(x, z) \leq r < 2r$. So, $z \in U(x, 2r)$. Hence,

$$B_M(x, r, r) \subseteq U(x, 2r). \quad (36)$$

And (34) holds, which implies that $\tau_\rho = \tau_M$ directly. \square

Lemma 3. $(X, M, *)$ is fuzzy metric space; the functions K and G satisfy the conditions C_1 and C_2 , respectively. Then, the function d defined by (20) can be represented as follows, $\forall x, y \in X$:

$$d(x, y) = \sup\{s > 0: \text{if } G(t) > s, \text{ then } M(x, y, t) + K(s) < 1\}. \quad (37)$$

Proof. In fact, we only need to show that $\inf A = \sup B$, where

$$\begin{aligned} A &= \{a > 0: \text{if } G(t) > a, \text{ then } M(x, y, t) + K(a) \geq 1\}, \\ B &= \{b > 0: \text{if } G(t) > b, \text{ then } M(x, y, t) + K(b) < 1\}. \end{aligned} \quad (38)$$

Take $a \in A, b \in B$ arbitrarily. Suppose $b > a$. If $G(t) > b$, then $G(t) > a$. From the definitions of A and B , we obtain

$$M(x, y, t) + K(a) \geq 1 > M(x, y, t) + K(b). \quad (39)$$

This is in direct contradiction to the condition that K is increasing. So, $a \geq b$, and hence, $\inf A \geq \sup B$.

Now, let $\inf A = \alpha, \sup B = \beta$. For any $\delta > 0$, we know $\beta + \delta \notin B$, that is, $\beta + \delta \in A$, which implies that $\inf A \leq \beta + \delta$.

By the arbitrariness of δ , we know that $\inf A \leq \beta = \sup B$. This completes the proof. \square

Theorem 5. *(X, M, *) is a fuzzy metric space. If M satisfies condition (15) or $* \geq \Delta_1$, then $\forall x, y \in X$:*

$$\rho(x, y) = \sup\{t > 0: M(x, y, t) + t < 1\}, \quad (40)$$

where ρ is defined by (29).

Proof. Let $G(t) = t, K(t) = t, \forall t \geq 0$. Then, (40) follows from Lemma 3 directly. \square

4. Conclusions

In the present paper, we investigate some metric structures in a fuzzy metric space. Especially, we give the concrete form of metric function with respect to the metrizable topology for a fuzzy metric under two special cases. Based on the results in the paper, interesting future research studies about the related topics may be prospective. Moreover, the technique used in this paper is suggestive to discuss the related problem in the general case.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Jianrong Wu was responsible for conceptualization, methodology, funding acquisition, and review and editing. Hao Yang contributed to formal analysis, investigation, and original draft preparation. All authors have read and agreed to the published version of the manuscript.

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