Research Article

Related Fixed Point Theorems via General Approach of Simulations Functions

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In this work, we extend and complement some results in view of general and wider structures, such as $b$–metric spaces. By considering existing classes of $\mathcal{Z}$–contractions and $\Psi$–simulating functions with a solid impact in database results of fixed point theory, we introduce a new general class of simulating functions, called as $\Psi$–$s$ simulation functions, and also types of $\kappa_{\Psi,s}$–contractions in a more general framework. This approach covers, extends, and unifies several published works in the early and late literature.

1. Introduction

Some of the significant generalizations of metric fixed point theory are related with the well-known Banach Contraction Principle [1] and classical contractions such as Boyd and Wong, Geraghty, Browder, and Cirić. In recent years, the theory of fixed points has attracted widespread attention and has been rapidly growing. It was massively studied by many researchers giving new results by using classes of implicit functions defining new and large contractive conditions. Recently, Khojasteh et al. [2] presented the notion of $\mathcal{Z}$–contractions involving a new class of simulation functions that has been used and improved by many authors in various spaces, see [3–30]. Authors in [19] proposed new notion $\Psi$–simulation functions and established the type of $\mathcal{Z}_{\Psi}$–contractions.

Inspired by the above works, in this paper we introduce a new class of general type of $\mathcal{Z}$–$s$ simulation functions, defined in the setting of $b$–metric-like spaces. This class generalizes further and complements some results given in the framework of $b$–metric spaces.

2. Preliminaries

Definition 1 (see [6]). Let $X$ be a nonempty set and $s \geq 1$ be a given real number. A mapping $d: X \times X \rightarrow [0, +\infty)$ is called a $b$–metric-like if for all $x, y, z \in X$, the following conditions are satisfied:

$$
\begin{align*}
d(x, y) = 0 & \text{ implies } x = y, \\
d(x, y) &= d(y, x), \\
d(x, y) \leq s[d(x, z) + d(z, y)].
\end{align*}
$$

The pair $(X, d)$ is called a $b$–metric-like space.

In a $b$–metric-like space $(X, d)$, if $x, y \in X$, and $d(x, y) = 0$, then $x = y$; however, the converse need not be true, and $d(x, x)$ may be positive for $x \in X$.

Definition 2 (see [6]). Let $(X, d)$ be a $b$–metric-like space with parameter $s \geq 1$ and let $\{x_n\}$ be any sequence in $X$ and $x \in X$. Then, we have the following:
(a) \( \{ x_q \} \) is said to be convergent to \( x \) if
\[
\lim_{q \to \infty} d(x_q, x) = d(x, x)
\]
(b) \( \{ x_q \} \) is said to be a Cauchy sequence in \((X, d)\) if
\[
\lim_{p, q \to \infty} d(x_p, x_q) \text{ exists and is finite}
\]
(c) The pair \((X, d)\) is called a complete \(b\)-metric-like space if, for every Cauchy sequence \( \{ x_q \} \) in \( X \), there is \( x \in X \) such that
\[
\lim_{q \to \infty} d(x_q, x) = d(x, x)
\]
\[
d(x_q, x_p) = \lim_{q \to \infty} d(x_q, x_p)
\]

**Lemma 1** (see [6, 29, 30]). Let \( \{ x_q \} \) and \( \{ y_q \} \) be two sequences in \((X, d)\) that converge to \( x \) and \( y \), respectively. Then, we have
\[
s^{-2}d(x, y) - s^{-1}d(x, x) - d(y, y) \leq \liminf_{q \to \infty} d(x_q, y_q) \leq \limsup_{q \to \infty} d(x_q, y_q) \leq s d(x, x) + s^2 d(y, y) + s^2 d(x, y).
\]

In particular, \( d(x, y) = 0 \iff \lim_{q \to \infty} d(x_q, y_q) = 0 \).

\[
s^{-1}d(x, z) - d(x, x) \leq \liminf_{q \to \infty} d(x_q, z) \leq \limsup_{q \to \infty} d(x_q, z) \leq s d(x, z) + s d(x, x).
\]

In particular, if \( d(x, x) = 0 \), then
\[
s^{-1}d(x, z) \leq \liminf_{q \to \infty} d(x_q, z) \leq \limsup_{q \to \infty} d(x_q, z) \leq s d(x, z).
\]

**Lemma 2** (see [23]). Let \( \{ x_q \} \) be a sequence in the \(b\)-metric-like space \((X, d)\) with parameter \( s \geq 1 \), such that
\[
\lim_{q \to \infty} d(x_q, x_{q+1}) = 0.
\]

If \( \lim_{q \to \infty} d(x_q, x_p) \neq 0 \), then there are \( \varepsilon > 0 \) and two sequences of natural numbers \( p(k), q(k) \) with \( q(k) > p(k) > k \), (positive integers) such that
\[
d(x_{p(k)}, x_{q(k)}) \geq \varepsilon,
\]
\[
d(x_{p(k-1)}, x_{q(k-1)}) \leq \varepsilon,
\]
\[
\frac{\varepsilon}{s} \leq \limsup_{k \to \infty} d(x_{p(k-1)}, x_{q(k-1)}) \leq s \varepsilon,
\]
\[
\frac{\varepsilon}{s} \leq \limsup_{k \to \infty} d(x_{q(k-1)}, x_{p(k-1)}) \leq s \varepsilon.
\]

**Definition 3.** A function \( \kappa: [0, +\infty)^{2} \to R \) is a \( \Psi - s \) simulation function if there are \( \psi \in \Psi \) and a coefficient \( \lambda \geq 1 \) so that
\[
\kappa_1: \kappa(t, \psi(t)) < \psi(t) \land \kappa(t) < \psi(t) \land \kappa(t) = \kappa(t) \land \kappa(t) = \kappa(t)
\]
\[
\kappa_2: \text{If} \ \{ t_n \}, \ \{ v_n \} \text{ are sequences in } (0, \infty) \text{ such that} \ \lim_{n \to \infty} t_n = \lim_{n \to \infty} v_n = l > 0, \text{ and} \ \lim_{n \to \infty} t_n \leq v_n \text{ then} \ \lim_{n \to \infty} \sup \kappa(t_n, v_n) < 0
\]

**Remark 1**
If in the definition above we take \( s = 1 \), then we obtain the definition of a \( \Psi \) - simulation function.

If we take \( \psi \) as the identity function, then we get a definition of an \( s \) - simulation function.

If we take \( \kappa \) as \( \psi \) and \( \phi \) as \( \psi \), then we get the definition of a \( \Psi \) - simulation function.

We denote by \( K_{\Psi, s} \) the set of all \( \Psi - s \) simulation functions. In the following example, we give such a kind of functions.

**Example 1.** Let \( \kappa: [0, +\infty)^{2} \to R \) be defined by
\[
\kappa_1: \kappa(t, \psi(t)) = c \psi(t) \land \kappa(t) = \kappa(t) \land \kappa(t) = \kappa(t)
\]
\[
\kappa_2: \phi(t) = \psi(t) - \phi(t) \land \phi(t) = \phi(t) \land \phi(t) = \phi(t)
\]

3. Main Results
Let \((X, d, s)\) be a \(b\)-metric-like space and \( \Psi([0, +\infty)) \) represent the collection of continuous functions \( \psi: [0, +\infty) \to [0, +\infty) \) with the following properties:
(3) \( \kappa(t, v) = \phi(v) - \psi(s^t) \) for all \( t, v \in (0, +\infty) \), where \( \psi, \phi : [0, +\infty) \to [0, +\infty) \) are continuous and \( \psi \) is increasing such that \( \phi(v) < \psi(v) \) for all \( v > 0 \).

(4) \( \kappa(t, v) = F(\psi(v), \phi(v)) - \psi(s^t) \) for all \( t, v \in (0, +\infty) \), where \( F : R^* \times R^* \to R \) is a C-class function where \( F \) is continuous such that \( F(t, v) < t \) for all \( t > 0 \).

For a self-mapping \( f : X \to X \), we denote by \( A(x, y) \) the following:

\[
A(x, y) = \max \left\{ d(x, y), d(x, fx), d(y, fy), \frac{1}{4s} (d(x, f y) + d(y, f x)) \right\}, \quad \text{for all } x, y \in X. \tag{8}
\]

**Theorem 1.** Let \( f : X \to X \) be a self-map on a \( b \)-metric-like space \( (X, d, s) \) with parameter \( s \geq 1 \). Suppose that there is \( \kappa \in K_{\psi, \phi} \) such that

\[
\kappa(d(fx, fy), A(x, y)) \geq 0, \tag{9}
\]

for all \( x, y \in X \), where \( A(x, y) \) is defined as in (8), then the self-map \( f \) has a unique fixed point in \( X \).

\[
A(x_{q-1}, x_q) = \max \left\{ d(x_{q-1}, x_q), d(x_{q-1}, fx_{q-1}), d(x_q, fx_q), \frac{1}{4s} (d(x_{q-1}, fx_q) + d(x_q, fx_{q-1})) \right\} = \max \left\{ d(x_{q-1}, x_q), d(x_{q-1}, x_q), d(x_q, x_{q+1}), \frac{1}{4s} (d(x_{q-1}, x_{q+1}) + d(x_q, x_q)) \right\}. \tag{10}
\]

Since

\[
\frac{1}{4s} (d(x_{q-1}, x_{q+1}) + d(x_q, x_q)) \leq \frac{1}{4s} (d(x_{q-1}, x_q) + d(x_q, x_{q+1})) + \frac{1}{2s} d(x_{q-1}, x_q) = \frac{1}{4} (3d(x_{q-1}, x_q) + d(x_q, x_{q+1})) \leq \max \{d(x_{q-1}, x_q), d(x_q, x_{q+1})\}. \tag{11}
\]

we obtain using (10),

\[
A(x_{q-1}, x_q) = \max \{d(x_{q-1}, x_q), d(x_q, x_{q+1})\}. \tag{12}
\]

By the supposition \( d(x_{q}, x_{q+1}) > 0 \) and (12), we get \( A(x_{q-1}, x_q) > 0 \). Assume that \( A(x_{q-1}, x_q) = d(x_{q}, x_{q+1}) \).

Then, applying condition (9) and property \( \kappa_1 \), we have for all \( q \in N \)

\[
0 \leq \kappa(d(x_q, x_{q+1}), A(x_{q-1}, x_q)) = \kappa(d(f x_{q-1}, f x_q), A(x_{q-1}, x_q)) = \kappa(d(x_{q-1}, x_q), d(x_q, x_{q+1})) < \psi(d(x_q, x_{q+1})) - \psi(s^d x_{q-1}, x_{q+1}). \tag{13}
\]

That is, a contradiction. Therefore,

\[
A(x_{q-1}, x_q) = d(x_{q-1}, x_q). \tag{14}
\]

From (9) and using (14), we obtain

Proof. Let \( x_0 \in X \) be an arbitrary element. Define a sequence \( \{x_q\} \) in \( X \) such that \( \forall q \in N \cup \{0\}, x_{q+1} = f(x_q) \).

If \( d(x_q, x_{q+1}) = 0 \) for some \( q \in N \cup \{0\} \), that is, \( x_q = x_q \) and \( x_q = x_{q+1} = f(x_q) \); therefore, \( x_q \) is a fixed point of \( f \). Thus, suppose that \( d(x_q, x_{q+1}) > 0 \) for all \( q \in N \cup \{0\} \). Considering the set \( A(x, y) \), we have

\[
0 \leq \kappa(d(x_q, x_{q+1}), A(x_{q-1}, x_q)) = \kappa(d(f x_{q-1}, f x_q), A(x_{q-1}, x_q)) = \kappa(d(x_{q-1}, x_q), d(x_q, x_{q+1})) < \psi(d(x_q, x_{q+1})) + \frac{1}{2s} d(x_{q-1}, x_q) = \psi s^d(x_q, x_{q+1}). \tag{15}
\]

In view of property of \( (\psi_1) \), the above inequality gives \( d(x_q, x_{q+1}) \leq d(x_q, x_{q+1}) \) for all \( q \in N \). Hence, \( \{d(x_q, x_{q+1})\} \) is a decreasing sequence of nonnegative reals, so there is \( l \geq 0 \) such that \( d(x_q, x_{q+1}) \to l \). Also, by (14),

\[
\lim_{q \to +\infty} d(x_q, x_{q+1}) = \lim_{q \to +\infty} A(x_{q-1}, x_q) = l. \tag{16}
\]

Suppose that \( l > 0 \), then \( \lim_{q \to +\infty} d(x_q, x_{q+1}) = \lim_{q \to +\infty} A(x_{q-1}, x_q) = l > 0 \). By property \( (\kappa_2) \), we have
\[ 0 \leq \lim_{q \to +\infty} \kappa(\mathbf{d}(x_{q,q'}, A(x_{q-1}, x_q))) < 0, \]  
(17)

which is a contradiction. Therefore, \( l = 0 \). Hence,

\[ \lim_{q \to +\infty} \mathbf{d}(x_{q,q'}, x_{q+1}) = \lim_{q \to +\infty} A(x_{q-1}, x_q) = 0. \]  
(18)

Next, we show that \( \lim_{q,p \to +\infty} \mathbf{d}(x_{q,q'}, x_p) = 0 \). Suppose, to the contrary, that is, \( \lim_{q,p \to +\infty} \mathbf{d}(x_{q,q'}, x_p) > 0 \), then by Lemma 2, there are \( \varepsilon > 0 \) and sequences \( \{p_k\} \) and \( \{q_k\} \) of positive integers with \( q_k > p_k > k \) such that

\[ \mathbf{d}(x_{p_k}, x_{q_k}) \geq \varepsilon, \mathbf{d}(x_{p_k}, x_{q_k}) < \varepsilon, \frac{\varepsilon}{s} \leq \lim_{k \to +\infty} \mathbf{d}(x_{q_{k-1}}, x_{p_k}) \leq \varepsilon, \]

\[ \frac{\varepsilon}{s} \leq \lim_{k \to +\infty} \mathbf{d}(x_{p_{k-1}}, x_{q_k}) \leq \varepsilon^2. \]  
(19)

From the definition of \( A(x, y) \), we have

\[ A(x_{p_{k-1}}, x_{q_{k-1}}) = \max \left\{ \mathbf{d}(x_{p_{k-1}}, x_{q_{k-1}}), \mathbf{d}(x_{p_{k-1}}, x_{p_k}), \mathbf{d}(x_{q_{k-1}}, x_{q_k}), \mathbf{d}(x_{q_{k-1}}, x_{p_k}), \frac{1}{4s} \mathbf{d}(x_{p_{k-1}}, x_{q_{k-1}}) + \mathbf{d}(x_{q_{k-1}}, x_{p_k}) \right\}. \]  
(20)

By the upper limit \( k \to +\infty \) in (20) and keeping in mind (18–19), we obtain

\[ \lim_{k \to +\infty} A(x_{p_{k-1}}, x_{q_{k-1}}) = \lim_{k \to +\infty} \max \left\{ \mathbf{d}(x_{p_{k-1}}, x_{q_{k-1}}), \mathbf{d}(x_{p_{k-1}}, x_{p_k}), \mathbf{d}(x_{q_{k-1}}, x_{q_k}), \mathbf{d}(x_{q_{k-1}}, x_{p_k}), \frac{1}{4s} \mathbf{d}(x_{p_{k-1}}, x_{q_{k-1}}) + \mathbf{d}(x_{q_{k-1}}, x_{p_k}) \right\} \]

\[ \leq \max \left\{ \varepsilon s, 0, \frac{1}{4s} (\varepsilon^2 + \varepsilon) \right\} \leq \varepsilon. \]  
(21)

Also, from condition \( \kappa_1 \), we have

\[ 0 \leq \kappa(\mathbf{d}(x_{p_k}, x_{q_k}), A(x_{p_{k-1}}, x_{q_{k-1}})) = \kappa(\mathbf{d}(x_{p_k}, x_{q_k}), A(x_{p_{k-1}}, x_{q_{k-1}})) \leq \psi(\mathbf{d}(x_{p_{k-1}}, x_{q_{k-1}})) - \psi(\frac{s}{4} \mathbf{d}(x_{p_k}, x_{q_k})), \]  
(22)

which by property of \( \psi_1 \) implies

\[ \frac{s}{4} \mathbf{d}(x_{p_k}, x_{q_k}) < A(x_{p_{k-1}}, x_{q_{k-1}}). \]  
(23)

By taking upper limit on both sides of (23) in view of (19) and (21), it follows that

\[ \varepsilon s^4 < \varepsilon s, \]  
(24)

which contradicts \( \varepsilon > 0 \). Thus, \( \lim_{q,p \to +\infty} \mathbf{d}(x_{q,q'}, x_p) = 0 \) and the sequence \( \{x_q\} \) is Cauchy in \( (X, \mathbf{d}, \mathbf{s}) \). So, there is \( \omega \in X \), such that

\[ \lim_{q \to +\infty} \mathbf{d}(x_{q,q'}, \omega) = \mathbf{d}(\omega, \omega) = \lim_{q,p \to +\infty} \mathbf{d}(x_{q,q'}, x_p) = 0. \]  
(25)

For elements \( \omega \) and \( x_{q'} \), we consider
\[ A(x,\omega) = \max \left\{ d(x,\omega), d(x, f x), d(\omega, f \omega), \frac{1}{4s} \left( d(x, f \omega) + d(\omega, f x) \right) \right\} = \max \left\{ d(x,\omega), d(x, x_{q+1}), d(\omega, f \omega), \frac{1}{4s} \left( d(x, f \omega) + d(\omega, x_{q+1}) \right) \right\}. \]  

(26)

By Lemma 1 together with (18) and (25), it follows by passing in the upper limit of (26):

\[ \lim_{q \to +\infty} \sup A(x,\omega) \leq \max \left\{ 0, 0, d(\omega, f \omega), \frac{sd(\omega, f \omega)}{4s} \right\} = d(\omega, f \omega). \]  

(27)

Now, using the \( \kappa_1 \) condition, we have

\[ 0 \leq \kappa \left( d(x, f \omega), A(x, \omega) \right) = \kappa \left( d(f x, f \omega), A(x, \omega) \right) < \psi \left( A(x, \omega) \right) - \psi \left( s^3 d(x_{q+1}, f \omega) \right), \]  

(28)

which implies

\[ s^3 d(x_{q+1}, f \omega) < A(x, \omega). \]  

(29)

Taking the limit superior in (29) and by Lemma 1 and inequality (27), we obtain

\[ A(\omega, y) = \max \left\{ d(\omega, y), d(\omega, \omega), d(y, y), \frac{1}{4s} \left( d(\omega, y) + d(y, y) \right) \right\} = \max \left\{ d(\omega, y), 0, 0, \frac{1}{2s} d(\omega, y) \right\} = d(\omega, y) > 0. \]  

(31)

From condition (9) and property \( \kappa_1 \), we have

\[ 0 \leq \kappa \left( d(f \omega, f y), A(\omega, y) \right) = \kappa \left( d(\omega, y), d(\omega, y) < \psi \left( d(\omega, y) \right) - \psi \left( s^3 d(\omega, y) \right) \right), \]  

(32)

which is a contradiction. Therefore, \( d(\omega, y) = 0 \) and \( \omega = y \). Thus, there is a unique fixed point of \( f \). \( \square \)

**Example 2.** Let \( X = [0,1] \) with the \( b \)-metric-like \( d(x, y) = (x + y)^2 \). Define \( f: X \to X \) as \( f x = \begin{cases} 1/6x & \text{if } x \neq 1 \\ 1/8 & \text{if } x = 1 \end{cases} \)

Also, we take the functions \( \phi(x) = x; \psi(x) = 2x \) and \( \kappa(t, v) = \phi(v) - \psi(z^t), \) where \( \lambda = 2 \) for all \( t, v \in (0, +\infty) \), where \( \psi; \phi: [0, +\infty) \to [0, +\infty) \) are continuous and \( \psi \) is increasing such that \( \phi(v) < \psi(v) \) for all \( v > 0 \).

The pair \((X, d)\) is a \( b \)-metric-like space with coefficient \( s = 2 \). We claim that the mapping \( f \) satisfies the contraction type condition (8):

**Case 1.** For \( x \neq y \neq 1 \), we have
\[
A(x, y) = \max \left\{ \frac{d(x, y)}{4s}, \frac{d(x, f_1(x, y))}{4s}, \frac{d(x, f_1(x, y))}{4s} \left( d(x, f_1(x, y)) + d(y, f_1(x, y)) \right) \right\}
\]

\[
= \max \left\{ (x + y)^2, (x + \frac{y}{6})^2, \left( y + \frac{x}{6} \right)^2, \frac{1}{4s} \left( (x + \frac{y}{6})^2 + \left( y + \frac{x}{6} \right)^2 \right) \right\}.
\]  

(33)

And \(d(f_1(x, y), A(x, y)) = d(x/6, y/6) = (x/6, y/6)^2 = 1/36(x + y)^2 = (1/36)d(x, y). \) Then,

\[
\kappa(d(f_1(x, y), A(x, y)) = \phi(A(x, y)) - \psi\left( s^2d(f_1(x, y)) \right) = A(x, y) - 2s^2d(f_1(x, y)) = A(x, y) - 8d(f_1(x, y))
\]

\[= A(x, y) - 8 \frac{1}{36}d(x, y) = A(x, y) - \frac{2}{9}d(x, y) \geq 0.
\]  

(34)

Case 2. For \(x = y = 1\), we note

\[
A(1, 1) = \max \left\{ d(1, 1), d(1, f_1), d(1, f_1), \frac{1}{4s} \left( d(1, f_1) + d(1, f_1) \right) \right\} = \max\{d(1, 1), d(1, f_1)\}
\]

\[
= \max \left\{ (1 + 1)^2, \left( 1 + \frac{1}{8} \right)^2 \right\} = 4 = d(1, 1).
\]  

(35)

And \(d(f_1, f_1) = d(1/8, 1/8) = (1/8, + 1/8)^2 = 4/64 = (1/64)d(1, 1) < A(1, 1). \) Then,

\[
\kappa(d(f_1, f_1), A(1, 1)) = \phi(A(1, 1)) - \psi\left( s^2d(f_1, f_1) \right) = A(1, 1) - 2s^2d(f_1, f_1) = A(1, 1) - 8d(f_1, f_1) = A(1, 1) - 8 \frac{1}{64}d(1, 1)
\]

\[= A(1, 1) - \frac{2}{9}d(1, 1) \geq 0.
\]  

(36)

Case 3. \(x < y = 1\) we note

\[
d(f_1, f_1) = d\left( \frac{x}{6}, \frac{1}{8} \right) = \left( \frac{x}{6} + \frac{1}{8} \right)^2 < \frac{1}{36}(x + 1)^2 = \frac{1}{36}d(x, 1)
\]

\[\implies d(f_1, f_1) < \frac{1}{36}d(x, 1)
\]

\[\implies 8d(f_1, f_1) < 8 \frac{1}{36}d(x, 1)
\]

\[\implies 2s^2d(f_1, f_1) < \frac{2}{9}d(x, 1) < A(x, 1).
\]

Then,

\[
\kappa(d(f_1, f_1), A(x, 1)) = \phi(A(x, 1)) - \psi\left( s^2d(f_1, f_1) \right) = A(x, 1) - 2s^2d(f_1, f_1) \geq 0.
\]

(38)
Here, 0 is the unique fixed point of f.

Some applications of Theorem 1 are the following corollaries.

**Corollary 1.** Let \( f : X \to X \) be a mapping on a \( b \)-metric-like space \((X, d, s)\). Suppose that there are \( \psi \in \Psi \) and \( \lambda \geq 1 \) such that

\[
\psi(s^t d(fx, fy)) \leq \frac{\psi(A(x, y))}{1 + \psi(A(x, y))}
\]

for all \( x, y \in X \), where \( A(x, y) \) is defined as in (8). Then, the self-map \( f \) has a unique fixed point in \( X \).

**Proof.** In Theorem 1, take into account the function \( \kappa(t, v) = \psi(v)/(1 + \psi(v)) - \psi(s^t) \) for all \( t, v \in (0, +\infty) \).

**Corollary 2.** Let \( f : X \to X \) be a mapping on a \( b \)-metric-like space \((X, d, s)\). Suppose that there are \( \psi \in \Psi \), \( \varphi : [0, +\infty) \to [0, +\infty) \) a lower semicontinuous function with \( \varphi(v) = 0 \) if \( v = 0 \), and \( \lambda \geq 1 \) such that

\[
\psi(s^t d(fx, fy)) \leq \frac{\psi(A(x, y))}{1 + \varphi(A(x, y))}
\]

for all \( x, y \in X \), where \( A(x, y) \) is defined as in (8). Then, the self-map \( f \) admits a unique fixed point in \( X \).

**Proof.** In Theorem 1, take into account the function \( \kappa(t, v) = \psi(v)/(1 + \varphi(v)) - \psi(s^t) \) for all \( t, v \in (0, +\infty) \).

**Corollary 3.** Let \( f : X \to X \) be a mapping on a \( b \)-metric-like space \((X, d, s)\). Suppose that there are \( \psi \in \Psi \), \( \alpha \in (0, 1) \) and \( \lambda \geq 1 \) such that

\[
\psi(s^t d(fx, fy)) \leq \alpha \psi(A(x, y)),
\]

for all \( x, y \in X \), where \( A(x, y) \) is defined as in (8). Then, the self-map \( f \) has a unique fixed point in \( X \).

**Proof.** In Theorem 1, take into account the function \( \kappa(t, v) = \alpha \psi(v) - \psi(s^t) \) for all \( t, v \in (0, +\infty) \), and \( \alpha \in (0, 1) \).

**Corollary 4.** Let \( f : X \to X \) be a mapping on a \( b \)-metric-like space \((X, d, s)\). Suppose that there are \( \psi \in \Psi \), \( \lambda \geq 1 \), and \( \phi : R^+ \to R^+ \) continuous with \( \phi(v) < \psi(v) \) for \( v > 0 \), such that

\[
\psi(s^t d(fx, fy)) \leq \phi(A(x, y)),
\]

for all \( x, y \in X \), where \( A(x, y) \) is defined as in (8). Then, the self-map \( f \) has a unique fixed point in \( X \).

**Proof.** In Theorem 1, take into account the function \( \kappa(t, v) = \phi(v) - \psi(s^t) \) for all \( t, v \in (0, +\infty) \), and take \( \psi(v) = v \) (it corresponds to Theorem 3.16 in [23]).

In the following result, we include two mappings \( f \) and \( g \) in the set

\[
E(x, y) = \max \left\{ \frac{d(x, y)d(y, gy) - d(x, fx)d(y, gy)}{1 + d(x, y)}, \frac{d(x, fx)d(y, gy)}{1 + d(fx, gy)} \right\}.
\]
Theorem 2. Let \((X, d, s)\) be a \(b\)-metric-like space and \(f, g: X \rightarrow X\) be two given mappings. Suppose that there exists \(\kappa \in K_{b-s}\) such that
\[
\kappa (d(f(x, g(y)), E(x, y)) \geq 0, \quad (46)
\]
for all \(x, y \in X\), where \(E(x, y)\) is denoted by (45); then, the mappings \(f\) and \(g\) have a unique common fixed point in \(X\).

Proof. Let \(x_0 \in X\) be an arbitrary element. Define a sequence \(\{x_n\}\) in \(X\) such that \(\forall q \in N \cup \{0\}, x_{2q} = f x_{2q}\) and \(x_{2q+2} = g x_{2q+1}\).

Let for some \(q \in N \cup \{0\}\) \(x_{2q+1} = x_{2q}\). Since

\[
E(x_{2q+1}, x_{2q+2}) = \max \left\{ d(x_{2q+1}, x_{2q+2}), \frac{d(x_{2q+1}, f x_{2q+2})d(x_{2q+2}, g x_{2q+1})}{1 + d(x_{2q+1}, g x_{2q+1})}, \frac{d(x_{2q+1}, f x_{2q+2})d(x_{2q+2}, g x_{2q+1})}{1 + d(x_{2q+1}, g x_{2q+1})} \right\}
\]

\[
= \max \left\{ d(x_{2q+1}, x_{2q+2}), \frac{d(x_{2q+1}, x_{2q+2})d(x_{2q+2}, x_{2q+1})}{1 + d(x_{2q+1}, x_{2q+2})}, \frac{d(x_{2q+1}, x_{2q+2})d(x_{2q+2}, x_{2q+1})}{1 + d(x_{2q+1}, x_{2q+2})} \right\}
\]

\[
= \max \left\{ 0, d(x_{2q+1}, x_{2q+2}), 0, 0 \right\}
\]

\[
= d(x_{2q+1}, x_{2q+2}).
\]

Then, by (46) and \((\kappa_1)\), we have

\[
0 \leq \kappa (d(f x_{2q}, g x_{2q+1}), E(x_{2q}, x_{2q+1})) = \kappa (d(x_{2q+1}, x_{2q+2}), d(x_{2q+1}, x_{2q+2})) < \psi (d(x_{2q+1}, x_{2q+2})) - \psi (d(x_{2q+1}, x_{2q+2})�)
\]

By property \((\psi_1)\), we get \(d(x_{2q+1}, x_{2q+2}) = 0\), that is, \(x_{2q+1} = x_{2q+2}\). We deduce that \(x_{2q} = x_{2q+1} = f x_{2q}\) and \(g x_{2q} = f x_{2q} = g x_{2q+1} = x_{2q+2}\). Hence, \(x_{2q}\) is a common fixed point of \(f\) and \(g\).

Assume the general case that \(d(x_{2q}, x_{2q+1}) > 0\) for all \(q \in N \cup \{0\}\), then

\[
E(x_{2q-1}, x_{2q}) = d(x_{2q-1}, x_{2q}) > 0.
\]

From (50), applying \((\psi_1)\), (46), and \((\kappa_1)\), we have

\[
0 \leq \kappa (d(x_{2q+1}, x_{2q}), E(x_{2q}, x_{2q+1})) = \kappa (d(f x_{2q}, g x_{2q+1}), E(x_{2q}, x_{2q+1})) < \psi (E(x_{2q}, x_{2q+1})) - \psi (d(f x_{2q}, g x_{2q+1})�)
\]

\[
= \psi (d(x_{2q+1}, x_{2q})) - \psi (d(x_{2q+1}, x_{2q})) \leq 0.
\]
That is a contradiction. So, we have $d(x_{2q}, x_{2q+1}) < d(x_{2q-1}, x_{2q})$ for all $q \in \mathbb{N}$. Hence, \[
\lim_{q \to \infty} d(x_{q}, x_{q+1}) = l \quad \text{and also} \quad \lim_{q \to \infty} d(x_{q}, x_{q+1}) = \lim_{q \to \infty} E(x_{q-1}, x_{q}) = l.
\] (52)

Assume that $l > 0$; then, by applying $\kappa_2$, we have \[
\limsup_{q \to \infty} \kappa(d(x_{2q+1}, x_{2q}), E(x_{2q}, x_{2q-1})) \leq 0, \tag{53}
\] a contradiction. Therefore, \[
\lim_{q \to \infty} d(x_{q}, x_{q+1}) = \lim_{q \to \infty} E(x_{q-1}, x_{q}) = l > 0. \tag{54}
\]

\[
\epsilon \leq \limsup_{k \to \infty} d(x_{2q}, x_{2q+1}) \leq \epsilon s, \quad \epsilon \leq \limsup_{k \to \infty} d(x_{2q+1}, x_{2q-1}) \leq \epsilon s, \quad \epsilon \leq \limsup_{k \to \infty} d(x_{2q+1}, x_{2q}) \leq \epsilon s^2. \tag{55}
\]

From (45), we note
\[
E(x_{2p+1}, x_{2q-1}) = \max\left\{ d(x_{2p+1}, x_{2q-1}), \frac{d(x_{2p+1}, x_{2q-1})}{1 + d(x_{2p+1}, x_{2q-1})} \right\} \leq \max\left\{ d(x_{2p+1}, x_{2q-1}), \frac{d(x_{2p+1}, x_{2q-1})}{1 + d(x_{2p+1}, x_{2q-1})} \right\}.
\]

Hence, by (54)–(56), and Lemma 2, we have
\[
\limsup_{k \to \infty} E(x_{2p+1}, x_{2q-1}) = \limsup_{k \to \infty} \max\left\{ d(x_{2p+1}, x_{2q-1}), \frac{d(x_{2p+1}, x_{2q-1})}{1 + d(x_{2p+1}, x_{2q-1})} \right\} \leq \max\{\epsilon s, 0, 0, 0\} = \epsilon s. \tag{57}
\]

By (46) and using properties $(\psi_1)$, $(\kappa_1)$, we have
\[
0 \leq \kappa(d(x_{2p+1}, x_{2q-1}), E(x_{2p+1}, x_{2q-1})) = \kappa(d(x_{2p+1}, x_{2q-1}), E(x_{2p+1}, x_{2q-1})) < \psi(E(x_{2p+1}, x_{2q-1})) - \psi(s^3 d(x_{2p+1}, x_{2q-1})). \tag{58}
\]
which leads to
\[ s^4 \delta(x_{2q+1}, x_{2q+2}) < E(x_{2q+1}, x_{2q+1}). \] (59)

Hence, by (55), (57), and (58) and taking the upper limit, we obtain
\[ \varepsilon \delta^{-1} < \varepsilon s, \] (60)
which implies that \( \varepsilon = 0 \), a contradiction with \( \varepsilon > 0 \). It remains to show that \( \lim_{q \to \infty} d(x_q, x_p) = 0 \); therefore, \( \{x_q\} \) is a Cauchy sequence in \( X \). Since \( (X_\ast, d, s) \) is a complete \( \beta \)-metric-like space, there is \( \omega \in X \) such that \( \{x_q\} \) is convergent to \( \omega \), that is,
\[ \lim_{q \to \infty} d(x_q, \omega) = \lim_{q \to \infty} d(x_q, x_p) = d(\omega, \omega) = 0. \] (61)

Also, the subsequences \( \{f x_{2q}\}, \{g x_{2q+1}\} \) are convergent, so
\[ \lim_{q \to \infty} d(x_{2q+1}, \omega) = \lim_{q \to \infty} d(f x_{2q}, \omega) = d(\omega, \omega) = 0, \]
\[ \lim_{q \to \infty} d(x_{2q+2}, \omega) = \lim_{q \to \infty} d(g x_{2q+1}, \omega) = d(\omega, \omega) = 0. \] (62)

Consider
\[ E(x_{2q}, \omega) = \max \left\{ \frac{d(x_{2q}, \omega) d(\omega, g \omega)}{1 + d(x_{2q}, \omega)}, \frac{d(x_{2q}, f x_{2q}) d(\omega, g \omega)}{1 + d(f x_{2q}, g \omega)} \right\} \] (63)

Taking the limit superior in (63) and applying Lemma 1 and (62), it follows
\[ \limsup_{q \to \infty} E(x_{2q}, \omega) \leq \max\{0, d(\omega, g \omega), 0, 0\} = d(\omega, g \omega). \] (64)

which implies
\[ s^4 \delta(x_{2q+1}, g \omega) < E(x_{2q}, \omega). \] (66)

Taking the upper limit as \( q \to +\infty \) and using Lemma 1 and (64), we have
\[ E(\omega, \delta) = \max \left\{ d(\omega, \delta), d(\delta, g \delta), \frac{d(\omega, \omega) d(\delta, g \delta)}{1 + d(\omega, \delta)}, \frac{d(\omega, \omega) d(\delta, g \delta)}{1 + d(f \omega, g \delta)} \right\} = \max \left\{ d(\omega, \delta), d(\delta, \delta), \frac{d(\omega, \omega) d(\delta, \delta)}{1 + d(\omega, \delta)}, \frac{d(\omega, \omega) d(\delta, \delta)}{1 + d(\omega, \delta)} \right\} \]
\[ = d(\omega, \delta) > 0. \] (67)

From (\( \psi_1 \)), (\( \kappa_1 \)), (67), and (46), we have
\[ 0 \leq \kappa(d(f \omega, g \delta), E(\omega, \delta)) = \kappa(d(\omega, \delta), d(\omega, \delta)) < \psi(d(\omega, \delta)) - \psi(s^4 d(\omega, \delta)) \leq 0, \] (68)
which contradicts the supposition \( d(\omega, \delta) > 0 \). Hence, 
\( d(\omega, \delta) = 0 \) and the common fixed point is unique. \( \square \)

**Corollary 7.** Let \( f, g: X \rightarrow X \) be two self-mappings given in a \( b \)-metric-like space \((X, d, s)\). Suppose that there exist \( \psi \in \Psi \) and \( \alpha: [0, \infty) \rightarrow [0, 1) \) with \( \lim_{t \to \infty} \alpha(t) < 1 \) for all \( r > 0 \), such that
\[
\psi(s^{-1}d(fx, gy)) \leq \alpha(E(x, y))\psi(E(x, y)),
\]
for all \( x, y \in X \), where \( E(x, y) \) is defined as in (45).

Then, the self-mappings \( f \) and \( g \) have a unique common fixed point in \( X \).

**Proof.** In Theorem 2, take the \( \Psi - s \) simulation function \( \kappa(t, v) = \alpha(v)\psi(v) - \psi(s^{-1}t) \) for all \( t, v \in \left(0, +\infty\right) \). \( \square \)

**Remark 3.** The above theorem reduces to a one mapping if we put \( g = f \). Further corollaries can be stated for \( s = 1 \), either by taking the function \( \psi \) as an identity function or by taking different functions \( \kappa \in K_{\psi-s} \) as listed in Corollary 1–6.

**4. Conclusion**

In this work, we established common fixed point results for one and two mappings on a \( b \)-metric-like space which overcomes and unifies classical and previous results developed in papers [19–28]. The considered set of generalized contractive mappings contains the families of many contractions as a proper subset. We remark based on Example 2/(4) which are functions of \( C \)-class used by many researchers and taken as a special case of \( \Psi - s \) simulation functions.

By using additional set of functions \( \Psi, \phi \), coefficient \( \lambda \), and parameter \( s \), the rich class of \( \Psi - s \) simulation functions make it possible to collect, extend, and complement previously existing results related to a variety types of contractions.

In terms of \( \Psi - s \) simulating functions, many classical and still recent contractions take a simple form as \( \kappa(d(fx, fy), A(x, y)) \geq 0 \) not including other additional symbols and long formulas.

This wide approach reflects a wide work and an unifying power for more general theorems made on the theory of fixed points.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no competing interests.

**Authors’ Contributions**

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript. All contributed equally to the writing of this paper.

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