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Research Article

Entire Solutions of the Second-Order Fermat-Type Differential-Difference Equation

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In this paper, the entire solutions of finite order of the Fermat-type differential-difference equation $[f''(z)]^2 + [\triangle_c^k f(z)]^2 = 1$ and the system of equations $[f_1''(z)]^2 + [\triangle_c^k f_2(z)]^2 = 1$ and $[f_2''(z)]^2 + [\triangle_c^k f_1(z)]^2 = 1$ have been studied. We give the necessary and sufficient conditions of existence of the entire solutions of finite order.

1. Introduction

In this work, we assume that the readers are familiar with general definitions and fundamental theories of Nevanlinna theory [1–3]. f is a meromorphic function which means f is meromorphic in the finite complex plane $\mathbb{C}$. If f has no poles, we call f is an entire function. We denote by $S(r, f)$, any function satisfying$S(r, f) = o(T(r, f))$, $r \longrightarrow \infty$, outside of a possible exceptional set of finite logarithmic measure. For a meromorphic function $f(z)$, we define its shift by $f_c(z) = f(z + c)$ and its difference operators by

$$\begin{aligned} \triangle_c f(z) &= f(z+c) - f(z), \\ \triangle_c^n f(z) &= \triangle_c^{n-1}(\triangle_c f(z)). \end{aligned} \tag{1}$$

The classic Fermat-type functional equation

$$f(z)^n + g(z)^n = 1, \tag{2}$$

has been intensively studied in recent years. If $n \geq 4$, equation (2) has no transcendental meromorphic solutions [4]. If $n \geq 3$, equation (2) has no transcendental entire solutions [4]. If $n = 2$, all entire solutions are the forms of $f(z) = \sin(h(z))$ and $g(z) = \cos(h(z))$, where $h(z)$ is any entire function [5].

Yang [6] studied the following generalized Fermat-type functional equation:

$$f(z)^n + g(z)^m = 1, \tag{3}$$

where n and m are positive integers and obtained the following theorem.

Theorem 1. *If $(1/n) + (1/m) < 1$, then equation (3) has no nonconstant entire solutions $f(z)$ and $g(z)$.*

For further research solutions of the Fermat-type functional equation, Yang and Li [7] considered the following special Fermat-type functional equation:

$$f(z)^2 + f'(z)^2 = 1, \tag{4}$$

and they obtained the following theorem.

Theorem 2. *The transcendental meromorphic solutions of (4) must satisfy $f(z) = (1/2)Pe^{\lambda z} + (1/2P)e^{-\lambda z}$, where P and λ are nonzero constants.*

In the following, c is a nonzero constant, unless otherwise specified.

Liu [8] investigated the following special Fermat-type functional equation:

$$f(z)^2 + f(z+c)^2 = 1. \tag{5}$$

He obtained that each transcendental entire solution of (5) with the finite order must satisfy $f(z) = (h_1(z) + h_2(z))/2$, where $h_1(z+c)/h_1(z) = i, h_2(z+c)/h_2(z) = -i$,

and $h_1(z)h_2(z) = 1$. Liu et al. [9] proved that the nonconstant finite-order entire solutions of (5) must have order one. Then, Liu et al. [10] considered the entire solutions of the following difference equations:

$$f(z)^n + f(z+c)^m = 1, \tag{6}$$

$$f'(z)^n + f(z+c)^m = 1, \tag{7}$$

and obtained the following results.

Theorem 3. *Equation (6) has no nonconstant entire solution if $n > m > 1$ or $n = m > 2$; has no transcendental entire solution with finite order if $n > m$; and has the general solutions $f(z) = (1/2) + e^{\pi z i/c} h(z)$, where $h(z)$ is any entire function periodic with period c.*

Theorem 4. *Equation (7) has no transcendental entire solution with finite order, provided that $m \neq n$, where n and m are positive integers.*

Liu et al. [10] obtained that $f(z) = e^z + 1$ satisfies the equation $f'(z) + f(z+c) = 1$ when $m = n = 1$; $f(z) = e^z + 1$ satisfies the equation $f'(z) + f(z+c) = 1$ when $m = n = 1$, where $e^c = -1$; and $f(z) = \sin(z \pm Bi)$ satisfies the equation $f'(z)^2 + f(z+c)^2 = 1$ when $m = n = 2$, where $c = 2k\pi$ or $c = 2k\pi + \pi$ and k is an integer.

Liu et al. [10] also investigated the following special Fermat-type functional equation:

$$f'(z)^n + [f(z+c) - f(z)]^m = 1, \tag{8}$$

and obtained the following theorem.

Theorem 5. *Equation (8) has no transcendental entire solutions with finite order, provided that $m \neq n > 1$, where n and m are positive integers.*

Liu et al. built exact solutions for equation (8); for $m = n = 1$, the equation $f'(z) + f(z+c) - f(z) = 1$ has a transcendental entire solution $f(z) = (e^{-z} + Az)/\ln 2$, where $A = \ln 2/(1+c)$ and $e^{-c} = 2$; for $m = 2, n = 1$, the equation $f'(z) + [f(z+c) - f(z)]^2 = 1$ admits a transcendental entire solution $f(z) = B(e^{Ez} + e^{(E/2)z} + Az)$, where $B = -E/4$, $A = -E/4ki\pi, c = 2ki\pi/E$, and $E^2 = 64ki\pi/ki\pi + 4$, where k is odd; and for $m = n = 2$, the equation $f'(z)^2 + [f(z+c) - f(z)]^2 = 1$ admits a transcendental entire solution $f(z) = (1/2)\sin(2z + Bi)$, where $c = k\pi + (\pi/2), k$ is an integer, and B is a constant.

In 2019, Liu et al. [11] researched the entire solutions with finite order of the Fermat-type differential-difference equation

$$\left[f^{(k)}(z)\right]^2 + \left[\triangle_c f(z)\right]^2 = 1, \tag{9}$$

and the following system of differential-difference equations:

$$\begin{cases} \left[f_1^{(k)}(z)\right]^2 + \left[\triangle_c f_2(z)\right]^2 = 1, \\ \left[f_2^{(k)}(z)\right]^2 + \left[\triangle_c f_1(z)\right]^2 = 1. \end{cases} \tag{10}$$

In 2019, Dang and Chen [12] obtained the meromorphic solutions of the following special Fermat-type functional equation:

$$af^n + b(f')^m \equiv 1. \tag{11}$$

In the following, we will consider the entire solutions with finite order of the Fermat-type differential-difference equation

$$\left[f''(z)\right]^2 + \left[\triangle_c^k f(z)\right]^2 = 1, \tag{12}$$

and obtain the following result.

Theorem 6. *$f(z)$ be entire solutions of finite order of differential-difference equation (12) if and only if $f(z)$ be the following forms:*

$$f(z) = \frac{1}{i^k (e^{ac} - 1)^k} \cos\left[i^{1+(k/2)} (e^{ac} - 1)^{k/2} z + ib\right] + ez + d, \tag{13}$$

where a, b, d, and e are constants, $e^{ac} \neq 1$, $(e^{ac})^k + (-1)^k = 0$, and $a^2 = i(e^{ac} - 1)^k$; $f(z) = az^2 + bz + d$, where $a = 0, b = \pm(1/c)$ for $k = 1$, $4a^2c^4 + 4a^2 = 1$ for $k = 2$, and $a = \pm(1/2)$ for $k \geq 3$.

Obviously, from Theorem 6, we immediately get the following example.

Example 1. The transcendental entire function solutions with finite order of the differential-difference equation

$$\left[f''(z)\right]^2 + \left[\triangle_c^2 f(z)\right]^2 = 1, \tag{14}$$

must satisfy

$$f(z) = \frac{1}{-(e^{ac} - 1)^2} \cos\left[(e^{ac} - 1)z - ib\right] + ez + d, \tag{15}$$

where a, b, d, and e are constants, $e^{ac} \neq 1$, $e^{2ac} + 1 = 0$, and $a^2 = i(e^{ac} - 1)^2$; $f(z) = az^2 + bz + d$, where $4a^2c^4 + 4a^2 = 1$.

Then, we will study the following system of differential-difference equations:

$$\begin{cases} \left[f_1''(z)\right]^2 + \left[\triangle_c^k f_2(z)\right]^2 = 1, \\ \left[f_2''(z)\right]^2 + \left[\triangle_c^k f_1(z)\right]^2 = 1, \end{cases} \tag{16}$$

and obtain the next result.

Theorem 7. *$(f_1(z), f_2(z))$ be the transcendental entire solutions of finite order of the system of differential-difference equation (16) if and only if $(f_1(z), f_2(z))$ be the following forms:*

$$f_1(z) = \frac{\cos\left(i\left[i(e^{ac} - 1)^k\right]^{1/2} z + ib_1\right)}{i(e^{ac} - 1)^k} + e_1 z + d_1, \tag{17}$$

$$f_2(z) = f_1(z) + ez + d,$$

where $e^{kac} = (-1)^k$, $e^{ac} \neq 1$, and $e^{b_1 - b_2} = 1$;

$$f_1(z) = \frac{\cos\left(i\left[-i(e^{ac} - 1)^k\right]^{1/2} z + ib_1\right)}{-i(e^{ac} - 1)^k} + e_1 z + d_1, \tag{18}$$

$$f_2(z) = -f_1(z) + ez + d,$$

where $e^{kac}+(-1)^k=0$, $e^{ac}\neq 1$, *and* $e^{b_1-b_2}=-1$; $f_j(z)=\pm(1/2)z^2+b_jz+d_j$ *for* $k\geq 3$; $f_1(z)=a_1z^2+b_1z+d_1$, $f_2(z)=\pm a_1z^2+b_2z+d_2$, *where* $a_2^2=a_1^2$ *for* $k=2$; $f_1(z)=a_1z^2+b_1z+d_1$, $f_2(z)=a_2z^2+b_2z+d_2$, *where* $c^4=1$ *for* $k=2$; *and* $f_j(z)=\pm(1/c)z+d$ *for* $k=1$, *where* a_j,b_j,d_j, *and* d *are constants,* $j=1,2$.

Obviously, from Theorem 7, we immediately get the following example.

Example 2. Let $k=2$. Then, the transcendental entire function solutions with finite order of the system of differential-difference equations

$$\begin{cases}\left[f_1''(z)\right]^2+\left[\triangle_c^2 f_2(z)\right]^2=1,\\ \left[f_2''(z)\right]^2+\left[\triangle_c^2 f_1(z)\right]^2=1,\end{cases}\tag{19}$$

must be in the following forms:

$$\begin{aligned}f_1(z)&=\frac{\cos\left(i\left[i(e^{ac}-1)^2\right]^{1/2}z+ib_1\right)}{i(e^{ac}-1)^2}+e_1z+d_1,\\ f_2(z)&=f_1(z)+ez+d,\end{aligned}\tag{20}$$

where $e^{kac}=(-1)^k$, $e^{ac}\neq 1$, and $e^{b_1-b_2}=1$;

$$\begin{aligned}f_1(z)&=\frac{\cos\left(i\left[-i(e^{ac}-1)^2+(-1)^2\right]^{1/2}z+ib_1\right)}{-i(e^{ac}-1)^2}+e_1z+d_1,\\ f_2(z)&=-f_1(z)+ez+d,\end{aligned}\tag{21}$$

where $e^{kac}+(-1)^k=0$, $e^{ac}\neq 1$, and $e^{b_1-b_2}=-1$.

2. Lemmas

Lemma 1 (see [1]). *Let* $f_j(z)$ *be a meromorphic function,* $f_j(z)$: $j=1,2,\ldots,n-1$, *being nonconstant, satisfying* $\sum_{j=1}^n f_j(z)=1$ *and* $n\geq 3$. *If* $f_n(z)\equiv 0$ *and*

$$\sum_{j=1}^n N\left(r,\frac{1}{f_j}\right)+(n-1)\sum_{j=1}^n\overline{N}\left(r,f_j\right)<(\lambda+o(1))T\left(r,f_k\right),\tag{22}$$

where $\lambda<1$ *and* $k=1,2,\ldots,n-1$, *then* $f_n\equiv 1$.

Lemma 2 (see [1]). *Suppose that* $f_1(z),f_2(z),\ldots,f_n(z)(n\geq 2)$ *are meromorphic functions and* $g_1(z)$, $g_{(z)},\ldots,g_n(z)$ *are entire functions satisfying the following conditions:*

(1) $\sum_{j=1}^n f_j(z)e^{g_j(z)}\equiv 0$

(2) The orders of f_j *are less than those of* $e^{g_k-g_l}$ *for* $1\leq j\leq n, 1\leq k<l\leq n$

Then, $f_j(z)\equiv 0\,(i=1,2,\ldots,n)$.

Lemma 3 (Hadamard's factorization theorem; see [1]). *Let* f *be an entire function of finite order* $\rho(f)$ *with zeros* $z_1,z_2,\ldots\subset\mathbb{C}\backslash\{0\}$ *and k-fold zero at the origin. Then,*

$$f(z)=z^kP(z)e^{Q(z)},\tag{23}$$

where $P(z)$ *is the canonical product of* f *formed with nonnull of* f *and* $Q(z)$ *is a polynomial of degree less than* $\rho(f)$.

3. Proof of Theorem 6

Suppose that $f(z)$ is an entire solution with finite order which satisfies (12). We rewrite (12) as follows:

$$\left[f''(z)+i\triangle_c^k f(z)\right]\left[f''(z)-i\triangle_c^k f(z)\right]=1.\tag{24}$$

It follows that $f''(z)+i\triangle_c^k f(z)$ and $f''(z)-i\triangle_c^k f(z)$ have no zeros. By Lemma 3, we have

$$\begin{cases}f''(z)+i\triangle_c^k f(z)=e^{p}(z),\\ f''(z)-i\triangle_c^k f(z)=e^{-p(z)},\end{cases}\tag{25}$$

where $p(z)$ is a polynomial.

From (25), we get

$$f''(z)=\frac{e^{p(z)}+e^{-p(z)}}{2},\tag{26}$$

$$\triangle_c^k f(z)=\frac{e^{p(z)}-e^{-p(z)}}{2i}.\tag{27}$$

Case 1. Suppose that f is a transcendental entire function. Then, it follows from (26) that $p(z)$ is a nonconstant polynomial. Let $\deg p(z)=m$; then, $m\geq 1$. It follows from (26) that

$$f''(z+jc)=\frac{e^{p(z+jc)}+e^{-p(z+jc)}}{2}.\tag{28}$$

It follows from (27) that

$$\begin{aligned}\left[\triangle_c^k f(z)\right]'&=\sum_{j=0}^k(-1)^{k-j}C_k^j f'(z+jc)\\ &=\frac{p'(z)}{2i}\left[e^{p(z)}+e^{-p(z)}\right].\end{aligned}\tag{29}$$

Then,

$$\begin{aligned}\left[\triangle_c^k f(z)\right]''&=\sum_{j=0}^k(-1)^{k-j}C_k^j f''(z+jc)\\ &=\frac{p''(z)+\left[p'(z)\right]^2}{2i}e^{p(z)}+\frac{p''(z)-\left[p'(z)\right]^2}{2i}e^{-p(z)}.\end{aligned}\tag{30}$$

Combining (28) and (30), we get

$$\begin{aligned}&\sum_{j=0}^k(-1)^{k-j}C_k^j\frac{e^{p(z+jc)}+e^{-p(z+jc)}}{2}\\ &=\frac{p''(z)+\left[p'(z)\right]^2}{2i}e^{p(z)}+\frac{p''(z)-\left[p'(z)\right]^2}{2i}e^{-p(z)}.\end{aligned}\tag{31}$$

This implies

$$\sum_{j=1}^{k}(-1)^{k-j}C_k^j\left[e^{p(z+jc)}+e^{-p(z+jc)}\right]e^{p(z)}+i\left[p''(z)+p'(z)^2\right]e^{2p(z)}+i\left[p''(z)-p'(z)^2\right]+(-1)^k\left(e^{p(z)}+e^{-p(z)}\right)e^{p(z)}=0. \tag{32}$$

Therefore,

$$\begin{aligned}&\sum_{j=1}^{k}(-1)^{k-j}C_k^j\left[e^{p(z+jc)}+e^{-p(z+jc)}\right]\\&+\left[(-1)^k+ip''(z)+ip'(z)^2\right]e^{p(z)}\\&+\left[(-1)^k+ip''(z)-ip'(z)^2\right]e^{-p(z)}=0.\end{aligned} \tag{33}$$

If $m\geq 2$, then for $0\leq j<l\leq k$, we have

$$\begin{aligned}\rho\left(e^{p(z+jc)-p(z+lc)}\right)&=m-1\geq 1,\\ \rho\left(e^{p(z+jc)+p(z+lc)}\right)&=m\geq 2,\end{aligned} \tag{34}$$

and by (33) and Lemma 2, we obtain $(-1)^{k-k}C_k^k=0$, which is contradicting.

Hence, $m=1$. Let $p(z)=az+b$, where $a\neq 0$; then, $p(z+kc)=az+b+akc=p(z)+akc$, $p(z+jc)=az+b+ajc=p(z)+ajc$, $p(z+kc)-p(z+jc)=(k-j)ac$, and $p(z+kc)+p(z+jc)=2p(z)+(k+j)ac$. Then, by (33), we have

$$\begin{aligned}&\sum_{j=1}^{k}(-1)^{k-j}C_k^j\left[e^{p(z+kc)+p(z+jc)}+e^{p(z+kc)-p(z+jc)}\right]\\&+\left[(-1)^k+ip''(z)+ip'(z)^2\right]e^{p(z+kc)+p(z)}\\&+\left[(-1)^k+ip''(z)-ip'(z)^2\right]e^{p(z+kc)-p(z)}=0.\end{aligned} \tag{35}$$

This implies

$$\begin{aligned}&\sum_{j=1}^{k}(-1)^{k-j}C_k^j\left[e^{2p(z)+(k+j)ac}+e^{(k-j)ac}\right]\\&+\left[(-1)^k+ip''(z)+ip'(z)^2\right]e^{2p(z)+kac}\\&+\left[(-1)^k+ip''(z)-ip'(z)^2\right]e^{kac}=0.\end{aligned} \tag{36}$$

Then,

$$\begin{aligned}&\sum_{j=1}^{k}(-1)^{k-j}C_k^je^{2p(z)+(k+j)ac}+(-1)^ke^{2p(z)+kac}\\&+\sum_{j=1}^{k}(-1)^{k-j}C_k^je^{(k-j)ac}\\&+(-1)^ke^{kac}+\left[ip''(z)+ip'(z)^2\right]e^{2p(z)+kac}\\&+\left[ip''(z)-ip'(z)^2\right]e^{kac}=0.\end{aligned} \tag{37}$$

This implies

$$\begin{aligned}&\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{2p(z)+(k+j)ac}+\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{(k-j)ac}\\&+\left[ip''(z)+ip'(z)^2\right]e^{2p(z)+kac}+\left[ip''(z)-ip'(z)^2\right]e^{kac}=0.\end{aligned} \tag{38}$$

Then,

$$\begin{aligned}&\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{2p(z)+(k+j)ac}+\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{(k-j)ac}\\&+ip'(z)^2e^{2p(z)+kac}-ip'(z)^2e^{kac}=0.\end{aligned} \tag{39}$$

This implies

$$\begin{aligned}&\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{2p(z)+(k+j)ac}+\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{(k-j)ac}\\&+ia^2e^{2p(z)+kac}-ia^2e^{kac}=0.\end{aligned} \tag{40}$$

Then,

$$\begin{aligned}&\left[\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{jac}+ia^2\right]e^{2p(z)+kac}\\&+\left[\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{-jac}-ia^2\right]e^{kac}=0.\end{aligned} \tag{41}$$

Obviously, $\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{jac}=(e^{ac}-1)^k$, and $\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{-jac}=(e^{-ac}-1)^k$. From (41) and Lemma 2, we obtain that

$$\begin{cases}(e^{ac}-1)^k+ia^2=0,\\(e^{-ac}-1)^k-ia^2=0.\end{cases} \tag{42}$$

By (42), it is easy to get $e^{ac}\neq 1$ and $(e^{ac})^k+(-1)^k=0$. It follows from (42) that $a^2=i(e^{ac}-1)^k$. Thus, it follows from (26) that

$$\begin{aligned}f(z)&=\frac{e^{(ie^{ac}-i)(k/2)z+b}+e^{-(ie^{ac}-i)(k/2)z-b}}{2(ie^{ac}-i)^k}+ez+d\\&=\frac{e^{i(-i)\left[(ie^{ac}-i)^{k/2}z+b\right]}+e^{-i(-i)\left[(ie^{ac}-i)^{k/2}z+b\right]}}{2(ie^{ac}-i)^k}+ez+d\\&=\frac{1}{(ie^{ac}-i)^k}\cos\left((-i)\left[(ie^{ac}-i)^{k/2}z+b\right]\right)+ez+d\\&=\frac{1}{(ie^{ac}-i)^k}\cos\left[(ie^{ac}-i)^{k/2}iz+ib\right]+ez+d\\&=\frac{1}{(ie^{ac}-i)^k}\cos\left[i^{1+(k/2)}(e^{ac}-1)^{k/2}z+ib\right]+ez+d.\end{aligned} \tag{43}$$

Case 2. Suppose that f is a polynomial. Then, it follows from (26) that $p(z)$ is a constant, and $f(z)=az^2+bz+d$.

If $k \geq 3$, then $\triangle_c^k f(z) = 0$. It follows from (12) that $a = \pm(1/2)$.

If $k = 2$, then $\triangle_c^k f(z) = 2ac^2$. It follows from (12) that $4a^2c^4 + 4a^2 = 1$.

If $k = 1$, then $\triangle_c^k f(z) = 2acz + ac^2 + bc$. It follows from (12) that $f(z) = \pm(1/c)z + d$.

Thus, Theorem 6 is proved.

4. Proof of Theorem 7

Suppose that $(f_1(z), f_2(z))$ be an entire solution with finite order which satisfies equation (16). We rewrite (16) as follows:

$$\begin{cases} \left[f_1''(z) + i\triangle_c^k f_2(z)\right]\left[f_1''(z) - i\triangle_c^k f_2(z)\right] = 1, \\ \left[f_2''(z) + i\triangle_c^k f_1(z)\right]\left[f_2''(z) - i\triangle_c^k f_1(z)\right] = 1. \end{cases} \tag{44}$$

It follows that

$$\begin{cases} f_1''(z) + i\triangle_c^k f_2(z) \neq 0, \\ f_1''(z) - i\triangle_c^k f_2(z) \neq 0, \\ f_2''(z) + i\triangle_c^k f_1(z) \neq 0, \\ f_2''(z) - i\triangle_c^k f_1(z) \neq 0. \end{cases} \tag{45}$$

By Lemma 3, we have

$$\begin{cases} f_1''(z) + i\triangle_c^k f_2(z) = e^{p(z)}, \\ f_1''(z) - i\triangle_c^k f_2(z) = e^{-p(z)}, \\ f_2''(z) + i\triangle_c^k f_1(z) = e^{q(z)}, \\ f_2''(z) - i\triangle_c^k f_1(z) = e^{-q(z)}, \end{cases} \tag{46}$$

where $p(z)$ and $q(z)$ are polynomials.

From (46), we get

$$f_1''(z) = \frac{e^{p(z)} + e^{-p(z)}}{2}, \tag{47}$$

$$\triangle_c^k f_2(z) = \frac{e^{p(z)} - e^{-p(z)}}{2i}, \tag{48}$$

$$f_2''(z) = \frac{e^{q(z)} + e^{-q(z)}}{2}, \tag{49}$$

$$\triangle_c^k f_1(z) = \frac{e^{q(z)} - e^{-q(z)}}{2i}. \tag{50}$$

Case 3. Suppose that f_1 is a transcendental entire function with finite order. Then, it follows from (47)–(50) that $p(z)$ and $q(z)$ are two nonconstant polynomials. Let $\deg p(z) = m_1$ and $\deg q(z) = m_2$; then, $m_1 \geq 1$ and $m_2 \geq 1$.

It follows from (47)–(50) that

$$f_1''(z + jc) = \frac{e^{p(z+jc)} + e^{-p(z+jc)}}{2}, \tag{51}$$

$$f_2''(z + jc) = \frac{e^{q(z+jc)} + e^{-q(z+jc)}}{2}, \tag{52}$$

$$\begin{aligned} \left[\triangle_c^k f_1(z)\right]' &= \sum_{j=0}^{k} (-1)^{k-j} C_k^j f_1'(z + jc) \\ &= \frac{q'(z)}{2i}\left[e^{q(z)} + e^{-q(z)}\right], \end{aligned} \tag{53}$$

$$\begin{aligned} \left[\triangle_c^k f_2(z)\right]' &= \sum_{j=0}^{k} (-1)^{k-j} C_k^j f_2'(z + jc) \\ &= \frac{p'(z)}{2i}\left[e^{p(z)} + e^{-p(z)}\right]. \end{aligned} \tag{54}$$

Then,

$$\begin{aligned} \left[\triangle_c^k f_1(z)\right]'' &= \sum_{j=0}^{k} (-1)^{k-j} C_k^j f_1''(z + jc) \\ &= \frac{q''(z) + [q'(z)]^2}{2i} e^{q(z)} + \frac{q''(z) - [q'(z)]^2}{2i} e^{-q(z)}, \end{aligned} \tag{55}$$

$$\begin{aligned} \left[\triangle_c^k f_2(z)\right]'' &= \sum_{j=0}^{k} (-1)^{k-j} C_k^j f_2''(z + jc) \\ &= \frac{p''(z) + [p'(z)]^2}{2i} e^{p(z)} \\ &\quad + \frac{p''(z) - [p'(z)]^2}{2i} e^{-p(z)}. \end{aligned} \tag{56}$$

From (51) and (55), we get

$$\begin{aligned} &\sum_{j=0}^{k} (-1)^{k-j} C_k^j \frac{e^{p(z+jc)} + e^{-p(z+jc)}}{2} \\ &= \frac{q''(z) + [q'(z)]^2}{2i} e^{q(z)} + \frac{q''(z) - [q'(z)]^2}{2i} e^{-q(z)}. \end{aligned} \tag{57}$$

From (52) and (56), we get

$$\begin{aligned} &\sum_{j=0}^{k} (-1)^{k-j} C_k^j \frac{e^{q(z+jc)} + e^{-q(z+jc)}}{2} \\ &= \frac{p''(z) + [p'(z)]^2}{2i} e^{p(z)} + \frac{p''(z) - [p'(z)]^2}{2i} e^{-p(z)}. \end{aligned} \tag{58}$$

It follows that

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j\left[e^{p(z+jc)}+e^{-p(z+jc)}\right]+i\left[q''(z)+q'(z)^2\right]e^{q(z)}+i\left[q''(z)-q'(z)^2\right]e^{-q(z)}=0, \tag{59}$$

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j\left[e^{q(z+jc)}+e^{-q(z+jc)}\right]+i\left[p''(z)+p'(z)^2\right]e^{p(z)}+i\left[p''(z)-p'(z)^2\right]e^{-p(z)}=0. \tag{60}$$

Then,

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j\left[e^{p(z+jc)+q(z)}+e^{-p(z+jc)+q(z)}\right]+\left[iq''(z)+iq'(z)^2\right]e^{2q(z)}+\left[iq''(z)-iq'(z)^2\right]=0, \tag{61}$$

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j\left[e^{q(z+jc)+p(z)}+e^{-q(z+jc)+p(z)}\right]+\left[ip''(z)+ip'(z)^2\right]e^{2p(z)}+\left[ip''(z)-ip'(z)^2\right]=0. \tag{62}$$

If $p(z+jc)+q(z)$ and $-p(z+jc)+q(z)$ are nonconstant polynomials for any $j(0\le j\le k)$, then we obtain that $e^{q(z)\pm p(z+jc)}$ are nonconstants for any $j(0\le j\le k)$. Assume $m_2\ge 1$; then, $(-1)^k+iq''(z)-iq'(z)^2$ does not vanish; then, we multiply $-(1/(iq''(z)-iq'(z)^2))$ on both sides of (61), and we have

$$-\frac{1}{iq''(z)-iq'(z)^2}\sum_{j=0}^{k}(-1)^{k-j}C_k^j\left[e^{p(z+jc)+q(z)}+e^{-p(z+jc)+q(z)}\right]-\frac{iq''(z)+iq'(z)^2}{iq''(z)-iq'(z)^2}e^{2q(z)}=1. \tag{63}$$

Then, by Lemma 1, we have $-((ip''(z)+ip'(z)^2)/(iq''(z)-iq'(z)^2))e^{2q(z)}\equiv 1$; this contradicts with $m_2\ge 1$.

Hence, there exist $j_0(0\le j_0\le k)$ such that $p(z+j_0c)+q(z)=A$ or $-p(z+j_0c)+q(z)=A$, where A is a constant.

Suppose that $-p(z+j_0c)+q(z)=A$. Obviously, $m_1=m_2$. We assert that $m_1=m_2=1$. Otherwise, we assume that $m_1=m_2\ge 2$; then, $\deg(-p(z+jc)+q(z))=\deg(-p(z+jc)+p(z+j_0c)-p(z+j_0c)+q(z))=\deg(p(z))-1\ge 1$ for any $j(0\le j(\ne j_0)\le k)$ and $\deg(p(z+jc)+q(z))=m_1\ge 2$ for any $j(0\le j(\ne j_0)\le k)$.

From (63), we have

$$-\frac{1}{iq''(z)-iq'(z)^2}\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{p(z+jc)+q(z)}-\frac{1}{iq''(z)-iq'(z)^2}\sum_{j=0,j\ne j_0}^{k}(-1)^{k-j}C_k^je^{-p(z+jc)+q(z)}-\frac{iq''(z)+iq'(z)^2}{iq''(z)-iq'(z)^2}e^{2q(z)}-\frac{(-1)^{k-j_0}C_k^{j_0}e^{-p(z+j_0c)+q(z)}}{iq''(z)-iq'(z)^2}=1. \tag{64}$$

Then, it follows from (64) and Lemma 2 that $-((-1)^{k-j_0}C_k^{j_0}e^A)/(iq''(z)-iq'(z)^2)=1$. This implies $iq''(z)-iq'(z)^2$ is a constant, and $m_2=m_1=1$, which is contradicting; therefore, the assertion is proved.

Accordingly, $-p(z+j_0c)+q(z)=A$ and $m_1=m_2$. Thus, we can assume that $p(z)=az+b_1, q(z)=az+b_2$; then,

$$\begin{cases} p(z+jc)=az+ajc+b_1,\\ q(z+jc)=az+ajc+b_2,\end{cases} \tag{65}$$

for any $0\le j\le k$.

Combining (61) and (62), we get

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j\left[e^{az+ajc+b_1}+e^{-az-ajc-b_1}\right]e^{az+b_2}+ia^2e^{2az+2b_2}-ia^2=0, \tag{66}$$

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j\left[e^{az+ajc+b_2}+e^{-az-ajc-b_2}\right]e^{az+b_1}+ia^2e^{2az+2b_1}-ia^2=0. \tag{67}$$

Furthermore,

$$\left[\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{ajc}+ia^2e^{b_2-b_1}\right]e^{2az+b_1+b_2}+\left[\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{-ajc-b_1+b_2}-ia^2\right]=0, \tag{68}$$

$$\left[\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{ajc}+ia^2e^{b_1-b_2}\right]e^{2az+b_1+b_2}+\left[\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{-ajc+b_1-b_2}-ia^2\right]=0. \tag{69}$$

It follows from (68) and (69) that

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{ajc}+ia^2e^{b_2-b_1}=0, \tag{70}$$

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^je^{-ajc-b_1+b_2}-ia^2=0, \tag{71}$$

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j e^{ajc}+ia^2e^{b_1-b_2}=0, \tag{72}$$

$$\sum_{j=0}^{k}(-1)^{k-j}C_k^j e^{-ajc+b_1-b_2}-ia^2=0. \tag{73}$$

By the binomial formula and (70)–(73), we have

$$\left(e^{ac}-1\right)^k+ia^2e^{b_2-b_1}=0, \tag{74}$$

$$e^{b_2-b_1}\left(e^{-ac}-1\right)^k-ia^2=0, \tag{75}$$

$$\left(e^{ac}-1\right)^k+ia^2e^{b_1-b_2}=0, \tag{76}$$

$$e^{b_1-b_2}\left(e^{-ac}-1\right)^k-ia^2=0. \tag{77}$$

From (75) and (77), we get $e^{b_1-b_2}=e^{b_2-b_1}$; hence, $e^{b_1-b_2}=\pm 1$. This implies $b_2=b_1+in\pi$, where n is an integer. From (74) and (75) and $e^{b_1-b_2}=\pm 1$, we obtain that $e^{kac}=(-1)^k$ and $e^{ac}\neq 1$, where $e^{b_1-b_2}=1$; $e^{kac}+(-1)^k=0$ and $e^{ac}\neq 1$, where $e^{b_1-b_2}=-1$.

It follows from (47) and (49) that

$$\begin{cases} f_1''(z)=\dfrac{e^{p(z)}+e^{-p(z)}}{2}=\dfrac{e^{az+b_1}+e^{-az-b_1}}{2}, \\ f_2''(z)=\dfrac{e^{q(z)}+e^{-q(z)}}{2}=\dfrac{e^{az+b_2}+e^{-az-b_2}}{2}. \end{cases} \tag{78}$$

Therefore,

$$\begin{cases} f_1(z)=\dfrac{e^{az+b_1}+e^{-az-b_1}}{2a^2}+e_1z+d_1, \\ f_2(z)=\dfrac{e^{az+b_2}+e^{-az-b_2}}{2a^2}+e_2z+d_2, \end{cases} \tag{79}$$

where e_i and d_i are constants.

Combining (79) and $e^{b_1-b_2}=\pm 1$, we get that $f_2(z)=\pm f_1(z)+ez+d$, where e and d are constants.

If $b_2=b_1+2ni\pi$, then it follows from (74) and (79) that $a=[i(e^{ac}-1)^k]^{1/2}$ and

$$\begin{cases} f_1(z)=\dfrac{e^{\left[i(e^{ac}-1)^k\right]^{1/2}z+b_1}+e^{-\left[i(e^{ac}-1)^k\right]^{1/2}z-b_1}}{2i(e^{ac}-1)^k}+e_1z+d_1 \\ =\dfrac{e^{i(-i)\left[i(e^{ac}-1)^k\right]^{1/2}z-iib_1}+e^{ii\left[i(e^{ac}-1)^k\right]^{1/2}z+iib_1}}{2i(e^{ac}-1)^k}+e_1z+d_1 \\ =\dfrac{\cos\left(-i\left[i(e^{ac}-1)^k\right]^{1/2}z-ib_1\right)}{i(e^{ac}-1)^k}+e_1z+d_1 \\ =\dfrac{\cos\left(i\left[i(e^{ac}-1)^k\right]^{1/2}z+ib_1\right)}{i(e^{ac}-1)^k}+e_1z+d_1, \\ f_2(z)=f_1(z)+ez+d. \end{cases} \tag{80}$$

If $b_2=b_1+(2n+1)i\pi$, then it follows from (74) and (79) that $a=[-i(e^{ac}-1)^k]^{1/2}$ and

$$\begin{cases} f_1(z)=\dfrac{e^{\left[-i(e^{ac}-1)^k\right]^{1/2}z+b_1}+e^{-\left[i(e^{ac}-1)^k\right]^{1/2}z-b_1}}{2\left[-i(e^{ac}-1)^k\right]}+e_1z+d_1 \\ =\dfrac{e^{i(-i)\left[-i(e^{ac}-1)^k\right]^{1/2}z-iib_1}+e^{ii\left[-i(e^{ac}-1)^k\right]^{1/2}z+iib_1}}{2\left[-i(e^{ac}-1)^k\right]}+e_1z+d_1 \\ =\dfrac{\cos\left(i\left[-i(e^{ac}-1)^k\right]^{1/2}z+ib_1\right)}{-i(e^{ac}-1)^k}+e_1z+d_1, \\ f_2(z)=-f_1(z)+ez+d. \end{cases} \tag{81}$$

Case 4. Suppose that f is a polynomial. Then, it follows from (47)–(50) that $p(z), q(z)$ are two constants. Hence, $f_1(z)=a_1z^2+b_1z+d_1$ and $f_2(z)=a_2z^2+b_2z+d_2$, where a_j, b_j, and d_j are constants.

If $k\geq 3$, then $\triangle_c^k f_j(z)=0$, $j=1,2$. It follows from (16) that $a_j=\pm(1/2)$, and $f_j(z)=\pm(1/2)z^2+b_jz+d_j$.

If $k=2$, then $\triangle_c^k f_j(z)=2a_jc^2$, $j=1,2$. It follows from (16) that

$$\begin{cases} a_1^2c^4+a_2^2=\dfrac{1}{4}, \\ a_2^2c^4+a_1^2=\dfrac{1}{4}. \end{cases} \tag{82}$$

Therefore, $a_2^2=a_1^2$, or $c^4=1$. Therefore,

$$\begin{cases} f_1(z)=a_1z^2+b_1z+d_1, \\ f_2(z)=\pm a_1z^2+b_2z+d_2, \end{cases} \tag{83}$$

where $a_2^2=a_1^2$, and

$$\begin{cases} f_1(z)=a_1z^2+b_1z+d_1, \\ f_2(z)=a_2z^2+b_2z+d_2, \end{cases} \tag{84}$$

where $c^4=1$.

If $k=1$, we have $\triangle_c^k f_j(z)=2a_jcz+a_jc^2+b_jc$, $j=1,2$. It follows from (16) that $f_j(z)=\pm(1/c)z+d$, where d is a constant.

Thus, Theorem 7 is proved.

5. Discussion

Consider the differential-difference equation

$$\left[f''(z)\right]^n+\left[\triangle_c^k f(z)\right]^m=1, \tag{85}$$

where n and m are positive integers. If $n>m>1$ or $m>n>1$ or $n=m>2$, it follows easily that (85) has no nonconstant entire solution by Theorem 1. For the case $n=m=2$, the entire solutions with finite order have been illustrated in Theorem 6.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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