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Research Article

Multivalued Problems, Orthogonal Mappings, and Fractional Integro-Differential Equation

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In this manuscript, we propose some sufficient conditions for the existence of solution for the multivalued orthogonal $\mathcal{F}$-contraction mappings in the framework of orthogonal metric spaces. As a consequence of results, we obtain some interesting results. Also as application of the results obtained, we investigate Ulam's stability of fixed point problem and present a solution for the Caputo-type nonlinear fractional integro-differential equation. An example is also provided to illustrate the usability of the obtained results.

1. Introduction and Preliminaries

The theory of multivalued mappings has an important role in mathematics and allied sciences because of its many applications, for instance, in real and complex analysis as well as in optimal control problems. Over the years, this theory has increased its significance, and hence in the literature, there are many papers focusing on the discussion of abstract and practical problems involving multivalued mappings. As a matter of fact, amongst the various approaches utilized to develop this theory, one of the most interesting approaches is based on methods of fixed point theory.

Acknowledging the work of Nadler [1], Gordji et al. [2], and Wardowski [3–5], the aim of this paper is to introduce the notion of multivalued orthogonal $\mathcal{F}$-contraction mappings in the framework of orthogonal metric space and to establish some sufficient conditions for the existence of fixed points for such class of mappings. Many researchers [6–11] proved the existence of fixed points using the concept of $\mathcal{F}$-contraction introduced by Wardowski [3–5]. In 1974, Reich [12, 13] asked whether we can take into account nonempty closed and bounded set instead of nonempty compact set. Although a lot of fixed point theorists studied this problem, it has not been completely solved. There are some partial affirmative answers to this problem, for instance, Mizoguchi et al. [14] and Olgun et al. [15]. We provide a partial solution to Reich's original problem using multivalued orthogonal $\mathcal{F}$-contraction mappings in the setting of orthogonal metric spaces. Also, as application of the interesting and new results obtained, we investigate Ulam's stability of fixed point problem and present a solution for a Caputo-type nonlinear fractional integro-differential equation.

Recently, Gordji et al. [2] introduced the concept of an orthogonal set (briefly, O-set) and presented some fixed point theorems in orthogonal metric spaces.

Definition 1. Let $\mathcal{X} \neq \varnothing$ and $\perp \subset \mathcal{X} \times \mathcal{X}$ be a binary relation. If $\perp$ satisfies the following condition: there exists $x_0 \in \mathcal{X}$ such that for all $y \in \mathcal{X}$, $y \perp x_0$, or for all $y \in \mathcal{X}$, $x_0 \perp y$, then it is called an orthogonal set (briefly O-set). We denote this O-set by $(\mathcal{X}, \perp)$.

Example 1. Let $\mathcal{X} = \mathbb{Z}$. Define $m \perp n$ if there exists $k \in \mathbb{Z}$ such that $m = kn$. It is easy to see that $0 \perp n$ for all $n \in \mathbb{Z}$. Hence, $(\mathcal{X}, \perp)$ is an O-set [2].

Example 2. Let $(\mathcal{X}, d)$ be a metric space and $\mathcal{T}\colon \mathcal{X} \longrightarrow \mathcal{X}$ be a Picard operator, that is, $\mathcal{T}$ has a unique fixed point

$x^* \in \mathcal{X}$ and $\lim_{n\longrightarrow+\infty} \mathcal{T}_n(y) = x^*$ for all $y \in \mathcal{X}$. We define the binary relation $\perp$ on $\mathcal{X}$ by $x \perp y$ if

$$\lim_{n\longrightarrow\infty} d(x, \mathcal{T}_n(y)) = 0. \tag{1}$$

Then, $(\mathcal{X}, \perp)$ is an O-set [2].

Example 3. Let $\mathcal{X}$ be an inner product space with the inner product $\langle .,. \rangle$. Define the binary relation $\perp$ on $\mathcal{X}$ by $x \perp y$ if $\langle x, y \rangle = 0$. It is easy to see that $0 \perp x$ for all $x \in \mathcal{X}$. Hence, $(\mathcal{X}, \perp)$ is an O-set [2].

For more interesting examples for an O-set, see [2].

Definition 2. Let $(\mathcal{X}, \perp)$ be an O-set. A sequence $\{x_n\}_{n\in\mathbb{N}}$ is called an orthogonal sequence (briefly, O-sequence) if for all $n, x_n \perp x_{n+1}$, or for all $n, x_{n+1} \perp x_n$.

Definition 3. Let $(\mathcal{X}, \perp, d)$ be an orthogonal metric space ($(\mathcal{X}, \perp)$ is an O-set, and $(\mathcal{X}, d)$ is a metric space). Then $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be orthogonally continuous (or $\perp$-continuous) at $a \in \mathcal{X}$ if, for each O-sequence $\{a_n\}$ in $\mathcal{X}$ with $a_n \longrightarrow a$, we have $\mathcal{T}(a_n) \longrightarrow \mathcal{T}(a)$. Also, $\mathcal{T}$ is said to be $\perp$-continuous on $\mathcal{X}$ if $\mathcal{T}$ is $\perp$-continuous for each $a \in \mathcal{X}$.

It is easy to see that every continuous mapping is $\perp$-continuous, but the converse is not true [2].

Definition 4. Let $(\mathcal{X}, \perp, d)$ be an orthogonal set with the metric d. Then $\mathcal{X}$ is said to be orthogonally complete (briefly, O-complete) if every Cauchy O-sequence is convergent.

It is easy to see that every complete metric space is O-complete, but the converse is not true [2].

Definition 5. Let $(\mathcal{X}, \perp)$ be an O-set. A mapping $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be $\perp$-preserving if $\mathcal{T}(x) \perp \mathcal{T}(y)$, whenever $x \perp y$. Also, $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$ is said to be weakly $\perp$-preserving if $\mathcal{T}(x) \perp \mathcal{T}(y)$ or $\mathcal{T}(y) \perp \mathcal{T}(x)$, whenever $x \perp y$.

It is easy to see that every $\perp$-preserving mapping is weakly $\perp$-preserving. But the converse is not true [2].

Definition 6. (see [3, 5]). Let $F: (0, +\infty) \longrightarrow \mathbb{R}$ be a mapping satisfying the following:

(F1) For all $a, b > 0$, $a > b$ implies $F(a) > F(b)$

(F2) For every sequence $\{a_n\}$ in $\mathbb{R}^+$, we have $\lim_{n\longrightarrow+\infty} a_n = 0$ if and only if $\lim_{n\longrightarrow+\infty} F(a_n) = -\infty$

(F3) There exists a number $k \in (0,1)$ such that $\lim_{a\longrightarrow 0^+} a^k F(a) = 0$

If $\lim_{t\longrightarrow 0^+} F(t) = -\infty$, then using (F1), we have $F(t_n) \longrightarrow -\infty \Rightarrow t_n \longrightarrow 0$ [5, 11].

Inspired by the work of Wardowski [3–5], we denote $\mathcal{F}$ be the family of all the functions $F: (0, +\infty) \longrightarrow \mathbb{R}$ satisfying (F1) and (F3)

We denote $\mathcal{F}1$ be the family of all the functions $F: (0, +\infty) \longrightarrow \mathbb{R}$ satisfying (F1), (F3), and (F4) $F(\inf A) = \inf F(A)$ for all $A \subset (0, \infty)$ with $\inf A > 0$

Here, $\lim_{c\longrightarrow d^-} F(c) = F(d-0) = \lim_{\varepsilon\longrightarrow 0^+} F(d-\varepsilon)$ (left limit at d) and $\lim_{c\longrightarrow d^+} F(c) = F(d+0) = \lim_{\varepsilon\longrightarrow 0^+} F(d+\varepsilon)$ (right limit at d) for all $d \in (0, +\infty)$. From mathematical analysis, the following is true for all $d \in (0, +\infty)$:

$$F(d-0) \leq F(d) \leq F(d+0). \tag{2}$$

Example 4. Let functions F_1, F_2, F_3: $(0, +\infty) \longrightarrow \mathbb{R}$ defined as follows:

(1) $F_1(a) = (-1/\sqrt{a})$, for all $a > 0$.

(2) $F_2(a) = \ln a$, for all $a > 0$.

(3) $F_3(a) = a + \ln a$, for all $a > 0$.

Then $F_1, F_2, F_3 \in \mathcal{F}$.

Let $(\mathcal{X}, d)$ be a metric space and H be a Hausdorff–Pompeiu metric induced by metric d on a set $\mathcal{X}$. Denote $\mathcal{P}(\mathcal{X})$ the family of all nonempty subsets of $\mathcal{X}$, $\mathcal{CB}(\mathcal{X})$ the family of all nonempty, and closed and bounded subsets of $\mathcal{X}$ and $\mathcal{K}(\mathcal{X})$ the family of all nonempty compact subsets of $\mathcal{X}$. H: $\mathcal{CB}(\mathcal{X}) \times \mathcal{CB}(\mathcal{X}) \longrightarrow \mathbb{R}$ defined by, for every $A, B \in \mathcal{CB}(\mathcal{X})$:

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \tag{3}$$

where $d(x, A) = \inf\{d(x, y): y \in A\}$.

2. Multivalued Results

In this section, we establish some results on the existence of fixed point for weak orthogonal multivalued contraction mappings using conditions of Wardowski [3–5].

Now, we define the following orthogonal relation between two nonempty subsets of an orthogonal set.

Definition 7. Let A and B be two nonempty subsets of an orthogonal set $(\mathcal{X}, \perp)$. The set A is orthogonal to set B is denoted by $\perp_1$ and defined as follows: $A \perp_1 B$, if for every $a \in A$ and $b \in B$, $a \perp b$.

It is easy to observe the following results.

Lemma 1. *Let $(\mathcal{X}, \perp, d)$ be an orthogonal metric space, $x \in \mathcal{X}$ and $A \in \mathcal{K}(\mathcal{X})$. Then there exists $a \in A$ such that $d(x, A) = d(x, a)$.*

Lemma 2. *Let $(\mathcal{X}, \perp, d)$ be an orthogonal metric space, and $A, B \in \mathcal{K}(\mathcal{X})$, $a \in A$. Then there exists $b \in B$ such that $d(a, b) \leq H(A, B)$.*

Now, we are ready to present our first result.

Theorem 1. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{K}(\mathcal{X})$ be a multivalued mapping on $\mathcal{X}$. Assume that the following conditions are satisfied:*

(i) There exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \perp_1 \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp_1 \{x_0\}$

(ii) For all $x, y \in \mathcal{X}$, $x \perp y$ implies $Tx \perp_1 Ty$

(iii) *If* $\{x_n\}$ *is an orthogonal sequence in* $\mathscr{X}$ *such that* $x_n \longrightarrow x^* \in \mathscr{X}$*, then* $x_n \perp x^*$ *or* $x^* \perp x_n$ *for all* $n \in \mathbb{N}$

(iv) *If* $F \in \mathscr{F}$*, there exists* $\tau > 0$ *such that for all* $x, y \in \mathscr{X}$ *with* $x \perp y$ *satisfying the following:*

$$H(\mathscr{T}x, \mathscr{T}y) > 0,\ \tau + F(H(\mathscr{T}x, \mathscr{T}y)) \leq F(d(x, y)). \tag{4}$$

Then $\mathscr{T}$ *has at least a fixed point.*

Proof. By assumption (i), there exists $x_1 \in \mathscr{T}x_0$ such that $x_0 \perp x_1$ or $x_1 \perp x_0$. By assumption (ii), we get $\mathscr{T}x_0 \perp_1 \mathscr{T}x_1$; that is, there exists $x_2 \in \mathscr{T}x_1$ such that $x_1 \perp x_2$ or $x_2 \perp x_1$. If $x_1 \in \mathscr{T}x_1$, then x_1 is a fixed point of $\mathscr{T}$. Suppose that $x_1 \notin \mathscr{T}x_1$. Since $\mathscr{T}x_1$ is compact, $d(x_1, \mathscr{T}x_1) > 0$. As $d(x_1, \mathscr{T}x_1) \leq H(\mathscr{T}x_0, \mathscr{T}x_1)$, using (F1), we have $F(d(x_1, \mathscr{T}x_1)) \leq F(H(\mathscr{T}x_0, \mathscr{T}x_1))$. Therefore, using (iv), we get

$$F(d(x_1, \mathscr{T}x_1)) \leq F(H(\mathscr{T}x_0, \mathscr{T}x_1)) \leq F(d(x_0, x_1)) - \tau. \tag{5}$$

Continuing this process inductively, we can construct an orthogonal sequence $\{x_n\}$ in $\mathscr{X}$ such that $x_{n+1} \in \mathscr{T}x_n$, for all $n \in \mathbb{N} \cup \{0\}$. Thus we have $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

If $x_k \in \mathscr{T}x_k$ for some $k \in \mathbb{N} \cup \{0\}$, then x_k is a fixed point of $\mathscr{T}$.

So, we may assume that $x_n \notin \mathscr{T}x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathscr{T}x_n$ is closed, we have $d(x_n, \mathscr{T}x_n) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Also $d(x_n, \mathscr{T}x_n) \leq H(\mathscr{T}x_{n-1}, \mathscr{T}x_n)$. So using (F1), we have $F(d(x_n, \mathscr{T}x_n)) \leq F(H(\mathscr{T}x_{n-1}, \mathscr{T}x_n))$. Further from (iv) and for every $n \geq 1$, we have

$$F(d(x_n, \mathscr{T}x_n)) \leq F(H(\mathscr{T}x_{n-1}, \mathscr{T}x_n)) \leq F(d(x_{n-1}, x_n)) - \tau. \tag{6}$$

Hence from the strictly increasing property of F, we get $H(\mathscr{T}x_n, \mathscr{T}x_{n-1}) < d(x_n, x_{n-1})$. We know that $x_{n+1} \in \mathscr{T}x_n$, $d(x_n, x_{n+1}) = d(x_n, \mathscr{T}x_n) \leq H(\mathscr{T}x_{n-1}, \mathscr{T}x_n) < d(x_{n-1}, x_n)$. Therefore, the sequence $\{d(x_{n+1}, x_n)\}$ is strictly decreasing sequence. Suppose that $t_n = d(x_{n+1}, x_n) \longrightarrow t$, for some $t \geq 0$.

Furthermore, for all $n \geq n_0$, we have

$$\begin{aligned} \tau + F(d(x_{n+1}, x_n)) &\leq \tau + F(H(\mathscr{T}x_n, \mathscr{T}x_{n-1})) \\ &\leq F(d(x_n, x_{n-1})). \end{aligned} \tag{7}$$

Taking $n \longrightarrow +\infty$ in (7), we get $\tau + F(t+0) \leq F(t+0)$, which is contradiction, and hence $t_n = d(x_{n+1}, x_n) \longrightarrow 0$. By (F3), there exists $k \in (0,1)$ such that

$$\lim_{n \longrightarrow +\infty} t_n^k F(t_n) = 0. \tag{8}$$

Using (6), we get

$$F(t_n) \leq F(t_{n-1}) - \tau \leq F(t_{n-2}) - 2\tau \leq \ldots \leq F(t_0) - n\tau. \tag{9}$$

From (9), the following holds for all $n \in \mathbb{N}$:

$$t_n^k F(t_n) - t_n^k F(t_0) \leq -t_n^k n\tau \leq 0. \tag{10}$$

Letting $n \longrightarrow \infty$ in (10), we get $\lim_{n \longrightarrow +\infty} n t_n^k = 0$. Hence there exists $n_1 \in \mathbb{N}$ such that $n t_n^k \leq 1$ for all $n \geq n_1$. So, we have all for all $n \geq n_1$:

$$t_n \leq \frac{1}{n^{(1/k)}}. \tag{11}$$

Now, we have to show that $\{x_n\}$ is a Cauchy orthogonal sequence. Consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using the triangle inequality and (11), we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots + d(x_{m-1}, x_m), \\ &= t_n + t_{n+1} + \cdots + t_{m-1} = \sum_{i=n}^{m-1} t_i \leq \sum_{i=n}^{+\infty} t_i \leq \sum_{i=n}^{+\infty} \frac{1}{i^{(1/k)}}. \end{aligned} \tag{12}$$

By the convergence of series, $\sum_{i=n}^{+\infty} (1/i^{1/k})$, passing to limit $n \longrightarrow +\infty$, we get $d(x_n, x_m) \longrightarrow 0$.

This shows that $\{x_n\}$ is a Cauchy orthogonal sequence. Since $\mathscr{X}$ is O-complete, there exists $x^* \in \mathscr{X}$ such that $\lim_{n \longrightarrow +\infty} x_n = x^*$.

Now, we claim that $x^* \in \mathscr{T}x^*$. Assume the contrary that $x^* \notin \mathscr{T}x^*$. Hence there exists $n_1 \in \mathbb{N}$ such that $x^* \notin \{x_n\}_{n \geq n_1}$, $H(\mathscr{T}x_n, \mathscr{T}x^*) > 0$. Therefore, further by our assumption, $x_n \perp x^*$ or $x^* \perp x_n$, and using (iv), we get

$$F(d(x_{n+1}, \mathscr{T}x^*)) \leq \tau + F(H(\mathscr{T}x_n, \mathscr{T}x^*)) \leq F(d(x_n, x^*)). \tag{13}$$

Now using strict increasing property of F and $\tau > 0$, we get $d(x_{n+1}, \mathscr{T}x^*) < d(x_n, x^*)$. Taking $n \longrightarrow +\infty$, we get $x^* \in \overline{\mathscr{T}x^*} = \mathscr{T}x^*$. Hence, the result is obtained.

Here it should be noted that in Theorem 1, $\mathscr{T}x$ is compact for all $x \in \mathscr{X}$. Now, we have the following result in which we give a partial answer to Reich's problem for a closed and bounded set. □

Theorem 2. *Let* $(\mathscr{X}, \perp, d)$ *be an O-complete orthogonal metric space and* $\mathscr{T}: \mathscr{X} \longrightarrow \mathscr{CB}(\mathscr{X})$ *be a multivalued mapping on* $\mathscr{X}$*. Assume that the following conditions are satisfied:*

(i) *There exists* $x_0 \in \mathscr{X}$ *such that* $\{x_0\} \perp_1 \mathscr{T}x_0$ *or* $\mathscr{T}x_0 \perp_1 \{x_0\}$

(ii) *For all* $x, y \in \mathscr{X}$*,* $x \perp y$ *implies* $\mathscr{T}x \perp_1 \mathscr{T}y$

(iii) *If* $\{x_n\}$ *is an orthogonal sequence in* $\mathscr{X}$ *such that* $x_n \longrightarrow x^* \in \mathscr{X}$*, then* $x_n \perp x^*$ *or* $x^* \perp x_n$ *for all* $n \in \mathbb{N}$.

(iv) *If* $F \in \mathscr{F}1$*, there exists* $\tau > 0$ *such that for all* $x, y \in \mathscr{X}$ *with* $x \perp y$ *satisfying the following:*

$$H(\mathscr{T}x, \mathscr{T}y) > 0, \tau + F(H(\mathscr{T}x, \mathscr{T}y)) \leq F(d(x, y)). \tag{14}$$

Then $\mathscr{T}$ *has at least a fixed point.*

Proof. Let $x_0 \in \mathscr{X}$. Since $\mathscr{T}x$ is nonempty for all $x \in \mathscr{X}$, by assumption (i), we can choose $x_1 \in \mathscr{T}x_0$ such that $x_0 \perp x_1$ or $x_1 \perp x_0$. If $x_1 \in \mathscr{T}x_1$, then x_1 is a fixed point of $\mathscr{T}$. Let

$x_1 \notin \mathcal{T}x_1$. Then $d(x_1, \mathcal{T}x_1) > 0$ since $\mathcal{T}x_1$ is closed. Since $d(x_1, \mathcal{T}x_1) \le H(\mathcal{T}x_0, \mathcal{T}x_1)$, then from (F1), we get

$$F(d(x_1, \mathcal{T}x_1)) \le F(H(\mathcal{T}x_0, \mathcal{T}x_1)). \tag{15}$$

Using (iv), we get

$$F(d(x_1, \mathcal{T}x_1)) \le F(H(\mathcal{T}x_0, \mathcal{T}x_1)) \le F(d(x_0, x_1)) - \tau. \tag{16}$$

From (F4), we get $F(d(x_1, \mathcal{T}x_1)) = \inf_{y \in \mathcal{T}x_1} F(d(x_1, y))$. So from (16), we have

$$\begin{aligned} F(d(x_1,\mathcal{T}x_1)) &= \inf_{y\in\mathcal{T}x_1} F(d(x_1,y)) \le F(H(\mathcal{T}x_0,\mathcal{T}x_1)),\\ &\le F(d(x_0,x_1)) - \tau,\\ &< F(d(x_0,x_1)) - \frac{\tau}{2}. \end{aligned} \tag{17}$$

By assumption (ii), we get $\mathcal{T}x_0 \perp_1 \mathcal{T}x_1$. Continuing this process, we construct an orthogonal sequence $\{x_n\}$ in $\mathcal{X}$ such that $x_{n+1} \in \mathcal{T}x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Thus we have $x_n \perp x_{n+1}$ or $x_{n+1} \perp x_n$ for all $n \in \mathbb{N} \cup \{0\}$.

If $x_k \in \mathcal{T}x_k$ for some $k \in \mathbb{N} \cup \{0\}$, then x_k is a fixed point of $\mathcal{T}$, and so the proof is completed.

So, we may assume that $x_n \notin \mathcal{T}x_n$ for all $n \in \mathbb{N} \cup \{0\}$. Since $\mathcal{T}x_n$ is closed, we have $d(x_n, \mathcal{T}x_n) > 0$, for all $n \in \mathbb{N} \cup \{0\}$. Also $d(x_n, \mathcal{T}x_n) \le H(\mathcal{T}x_{n-1}, \mathcal{T}x_n)$, and from (F1), we get $F(d(x_n, \mathcal{T}x_n)) \le F(H(\mathcal{T}x_{n-1}, \mathcal{T}x_n))$.

Furthermore, using (iv), we have

$$\begin{aligned} F(d(x_n, \mathcal{T}x_n)) &\le F(H(\mathcal{T}x_n, \mathcal{T}x_{n+1}))\\ &\le F(d(x_n, x_{n+1})) - \tau < F(d(x_n, x_{n+1})) - \frac{\tau}{2}. \end{aligned} \tag{18}$$

Since $F(d(x_n, \mathcal{T}x_n)) = \inf_{y \in \mathcal{T}x_n} F(d(x_n, y))$. Therefore, using (18), we get

$$\begin{aligned} F(d(x_n,\mathcal{T}x_n)) &= \inf_{y\in\mathcal{T}x_n} F(d(x_n,y)) \le F(H((\mathcal{T}x_{n-1},\mathcal{T}x_n))\\ &< F(d(x_{n-1},x_n)) - \frac{\tau}{2}. \end{aligned} \tag{19}$$

So from (19), we can get a sequence $\{x_n\}$ in $\mathcal{X}$ such that there exists $x_{n+1} \in \mathcal{T}x_n$ and $F(d(x_n, x_{n+1})) < F(d(x_{n-1}, x_n))$ for all $n \in \mathbb{N}$. Now, proceeding on the same lines of Theorem 1, we get the result. □

3. Consequences

In this section, we give some interesting consequences of the results proved in the previous section.

The following result is an immediate consequence of Theorem 1.

Corollary 1. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{K}(\mathcal{X})$. Assume that the following conditions are satisfied:*

(i) *There exists $x_0 \in \mathcal{X}$ such that $\{x_0\} \perp_1 \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp_1 \{x_0\}$*

(ii) *For all $x, y \in \mathcal{X}$, $x \perp y$ implies $\mathcal{T}x \perp_1 \mathcal{T}y$*

(iii) *If $\{x_n\}$ is an orthogonal sequence in $\mathcal{X}$ such that $x_n \longrightarrow x^* \in \mathcal{X}$, then $x_n \perp x^*$ or $x^* \perp x_n$ for all $n \in \mathbb{N}$.*

(iv) *There exists some $\tau_i > 0$, $i = 1, 2, 3$ such that for all $x, y \in \mathcal{X}$ with $x \perp y$, $H(\mathcal{T}x, \mathcal{T}y) > 0$, either of the following contractive conditions hold:*

$$\begin{aligned} &\tau_1 + H(\mathcal{T}x, \mathcal{T}y) \le d(x, y);\\ &\tau_2 - \frac{1}{H(\mathcal{T}x, \mathcal{T}y)} \le -\frac{1}{d(x, y)};\\ &\tau_3 + \frac{1}{1 - e^{H(\mathcal{T}x, \mathcal{T}y)}} \le \frac{1}{1 - e^{d(x, y)}}. \end{aligned} \tag{20}$$

Then $\mathcal{T}$ has at least a fixed point in each of these cases.

Proof. As each functions $F_1(r) = r$, $F_2(r) = (-1/r)$, and $F_3(r) = (1/1 - e^r)$, where $r = d(x, y) > 0$, is strictly increasing on $(0, +\infty)$, so the proof immediately follows from Theorem 1.

As a consequence of Theorem 1, we have the following result for single-valued mapping by replacing condition (iii) with $\mathcal{T}$ is $\perp$-continuous. □

Corollary 2. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$. Assume that the following conditions are satisfied:*

(i) *There exists some $\tau > 0$, such that for all $x, y \in \mathcal{X}$ with $x \perp y$, $d(\mathcal{T}x, \mathcal{T}y) > 0$:*

$$\tau + F(d(\mathcal{T}x, \mathcal{T}y)) \le F(d(x, y)), \tag{21}$$

where $F \in \mathcal{F}$.

(ii) *There exists $x_0 \in \mathcal{X}$ such that $x_0 \perp \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp x_0$.*

(iii) *For all $x, y \in \mathcal{X}$, $x \perp y$ implies $\mathcal{T}x \perp \mathcal{T}y$*

(iv) *$\mathcal{T}$ is $\perp$-continuous*

Then, $\mathcal{T}$ has a fixed point.

Proof. Here, we can choose $\mathcal{T}$ as a multivalued mapping by considering $\mathcal{T}x$ is a singleton set for every $x \in \mathcal{X}$. Arguing on the same lines of Theorem 1, we consider $\{x_n\}$ is a Cauchy orthogonal sequence and $\lim_{n\longrightarrow\infty} x_n = x^*$. As $\mathcal{T}$ is $\perp$-continuous, we have

$$d(x^*, Tx^*) = \lim_{n\longrightarrow+\infty} d(\mathcal{T}x_n, \mathcal{T}x^*) = 0, \tag{22}$$

i.e., x^* is a fixed point of $\mathcal{T}$.

As a consequence of Corollary 2, we have the following result by taking $F(r) = \ln r$, $r > 0$. □

Corollary 3. *Let $(\mathcal{X}, \perp, d)$ be an O-complete orthogonal metric space and $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$. Assume that the following conditions are satisfied:*

(i) *There exists some $\tau > 0$, such that for all $x, y \in \mathcal{X}$ with $x \perp y$, $d(\mathcal{T}x, \mathcal{T}y) > 0$:*

$$d(\mathcal{T}x, \mathcal{T}y) \leq e^{-\tau} d(x, y), \tag{23}$$

where $F \in \mathcal{F}$.

(ii) *There exists $x_0 \in \mathcal{X}$ such that $x_0 \perp \mathcal{T}x_0$ or $\mathcal{T}x_0 \perp x_0$.*

(iii) *For all $x, y \in \mathcal{X}$, $x \perp y$ implies $\mathcal{T}x \perp \mathcal{T}y$.*

(iv) *$\mathcal{T}$ is $\perp$-continuous.*

Then $\mathcal{T}$ has a fixed point.

4. Illustration

In this section, we illustrate an example which shows that $\mathcal{T}$ is a multivalued orthogonal mapping and satisfies the condition (iv) of Theorem 1, but it is not multivalued orthogonal contraction ($H(\mathcal{T}x, \mathcal{T}y) \leq k\, d(x, y)$, for $k \in [0, 1)$ with $x \perp y$).

Example 5. Let $\mathcal{X} = \{S_n = (n(n+1)/2): n \in \mathbb{N}\}$ and $d: \mathcal{X} \times \mathcal{X} \longrightarrow [0, \infty)$ be a mapping defined by $d(x, y) = |x - y|$ for all $x, y \in \mathcal{X}$.

Define a relation $\perp$ on $\mathcal{X}$ by $x \perp y$ if and only if $xy \in \{x, y\} \subseteq \mathcal{X} = \{S_n\}$.

Thus $(\mathcal{X}, \perp, d)$ is an O-complete orthogonal metric space. Now, we define a mapping $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{K}(\mathcal{X})$ by

$$\mathcal{T}x = \begin{cases} \{x_1\}, & x = x_1, \\ \{x_1, \ldots, x_{n-1}\}, & x = x_n, n \geq 1. \end{cases} \tag{24}$$

We claim that $\mathcal{T}$ is a multivalued orthogonal mapping satisfying condition (iv) of Theorem 1 with respect to $F(\alpha) = \alpha + \ln(\alpha)$, $\alpha > 0$ and $\tau = 1$. To see this, we have the following cases.

First, we observe that for all $m, n \in \mathbb{N}$, $H(\mathcal{T}x, \mathcal{T}y) > 0$ if and only if $m > 2$ and $n = 1$ or $m > n > 1$.

Case 1. For $m > 2$ and $n = 1$, we have

$$\begin{aligned} &\frac{H(\mathcal{T}x_m, \mathcal{T}x_1)}{d(x_m, x_1)} e^{H(\mathcal{T}x_m, \mathcal{T}x_1) - d(x_m, x_1)} \\ &= \frac{x_{m-1} - x_1}{x_m - x_1} e^{x_{m-1} - x_1} = \frac{m^2 - m - 2}{m^2 + m - 2} e^{-m} < e^{-m} < e^{-1}. \end{aligned} \tag{25}$$

Case 2. For $m > n > 1$, we get

$$\begin{aligned} &\frac{H(\mathcal{T}x_m, \mathcal{T}x_n)}{d(x_m, x_n)} e^{H(\mathcal{T}x_m, \mathcal{T}x_n) - d(x_m, x_n)} \\ &= \frac{x_{m-1} - x_{n-1}}{x_m - x_n} e^{x_{m-1} - x_{n-1} - x_m + x_n} \\ &= \frac{m + n - 1}{m + n + 1} e^{n-m} < e^{n-m} \leq e^{-1}. \end{aligned} \tag{26}$$

This shows that $\mathcal{T}$ satisfies (iv) of Theorem 1. Hence, $\mathcal{T}$ has a fixed point.

On the contrary, $\mathcal{T}$ is not multivalued orthogonal contraction ($H(\mathcal{T}x, \mathcal{T}y) \leq k\, d(x, y)$, $k \in [0, 1)$), as

$$\lim_{n \longrightarrow +\infty} \frac{H(\mathcal{T}x_n, \mathcal{T}x_1)}{d(x_n, x_1)} = \lim_{n \longrightarrow +\infty} \frac{x_{n-1} - 1}{x_n - 1} = 1. \tag{27}$$

5. Applications

In this section, we present the Ulam stability and solve a nonlinear fractional differential-type equation using Corollary 3.

5.1. Ulam Stability. The Ulam [16, 17] stability has attracted attention of several authors in fixed point theory [18]. On orthogonal metric space $(\mathcal{X}, \perp, d)$, $\mathcal{T}: \mathcal{X} \longrightarrow \mathcal{X}$, we investigate the fixed point equation:

$$\mathcal{T}v = v, \tag{28}$$

and the inequality (for $\varepsilon > 0$):

$$d(\mathcal{T}x, x) \leq \varepsilon. \tag{29}$$

Equation (28) is called the Ulam stable if it satisfies the following condition:

(A) There is a constant $\delta > 0$, for each $\varepsilon > 0$, and for every solution x^* of the inequality (29), there is a solution $v^* \in X$ for equation (28) such that

$$d(v^*, x^*) \leq \delta\varepsilon. \tag{30}$$

Theorem 3. *Under the hypothesis of Corollary 3, the fixed point equation (28) is Ulam stable.*

Proof. On account of Corollary 3, we guarantee a unique $v^* \in X$ such that $v^* = \mathcal{T}v^*$, that is, $v^* \in \mathcal{X}$ forms a solution of (28). Let $\varepsilon > 0$ and $x^* \in \mathcal{X}$ be an ε-solution, that is,

$$d(\mathcal{T}x^*, x^*) \leq \varepsilon. \tag{31}$$

We have

$$\begin{aligned} d(v^*, x^*) &= d(\mathcal{T}v^*, x^*) \\ &\leq d(\mathcal{T}v^*, \mathcal{T}x^*) + d(\mathcal{T}x^*, x^*) \\ &\leq e^{-\tau} d(v^*, x^*) + \varepsilon. \end{aligned} \tag{32}$$

Hence, $d(v^*, x^*) \leq (1/1 - e^{-\tau})\varepsilon = k\varepsilon$, where $k = (1/1 - e^{-\tau}) > 0$. Therefore, equation (28) is Ulam stable. □

5.2. Application to Nonlinear Fractional Integro-Differential Equation. Here, we give a solution for a Caputo-type nonlinear fractional integro-differential equation. For more details on fractional calculus, see [19–25] and references cited therein.

The Caputo derivative of a continuous mapping $g: [0, \infty) \longrightarrow \mathbb{R}$ (order $\delta > 0$) is given by

$$
{}^{C}D^{\delta}g(t) := \frac{1}{\Gamma(n-\delta)}\int_0^t \frac{g^{(n)}(s)\mathrm{d}s}{(t-s)^{\delta-n+1}},\quad n-1\le\delta<n, n=[\delta]+1, \tag{33}
$$

where Γ represents the gamma function and $[\delta]$ refers to the integer part of the positive real number δ.

In this section, we examine the nonlinear fractional integro-differential equation of the Caputo type:

$$
\begin{cases} {}^{C}D^{\delta}u(t) = \mathcal{G}(t,u(t)), & t\in I=[0,1], 1<\delta\le 2,\\ u(0)=0, u(1)=\displaystyle\int_0^{\theta}u(s)\mathrm{d}s, \end{cases} \tag{34}
$$

where $u\in(C[0,1],\mathbb{R})$, $\theta\in(0,1)$, and $\mathcal{G}\colon I\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function (for more details, see [20]).

We consider $\mathcal{X}=\{u\colon u\in(C[0,1],\mathbb{R})\}$ with supremum norm $\|u\|=\sup_{t\in[0,1]}|u(t)|$. So $(X,\|.\|)$ is a Banach space.

The space $\mathcal{X}\colon = C([0,1],\mathbb{R})$ endowed with the metric $d\colon\mathcal{X}\times\mathcal{X}\longrightarrow[0,\infty)$ defined as $d(u,v)=\|u-v\|=\sup_{t\in[0,1]}(t)|u(t)-v(t)|$ and define an orthogonal relation $u\perp v$ if and only if $uv\ge 0$, for all $u,v\in\mathcal{X}$. Then $(\mathcal{X},\perp,d)$ is an orthogonal metric space.

Clearly, a solution of equation (34) is a fixed point of the integral equation:

$$
\begin{aligned}
\mathcal{T}u(t) &= \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}((s,u(s)))\mathrm{d}s,\\
&\quad -\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s,\\
&\quad +\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,u(m))\mathrm{d}m\right)\mathrm{d}s.
\end{aligned} \tag{35}
$$

Theorem 4. *Assume that $\mathcal{G}\colon I\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function satisfying*

$$
|\mathcal{G}(s,u(s))-\mathcal{G}(s,v(s))|\le\frac{\Gamma(\delta+1)}{5}e^{-\tau}|u(s)-v(s)|, \tag{36}
$$

for each $s\in[0,1]$, for some $\tau>0$ and for all $u,v\in C([0,1],\mathbb{R})$. Then the fractional differential equation (34) with given boundary conditions has a solution.

Proof. The space $\mathcal{X}\colon = C([0,1],\mathbb{R})$ endowed with the metric $d\colon\mathcal{X}\times\mathcal{X}\longrightarrow[0,\infty)$ defined as $d(u,v)=\sup_{t\in[0,1]}|u(t)-v(t)|$, for all $u,v\in\mathcal{X}$. Define an orthogonal relation $u\perp v$ if and only if $uv\ge 0$, for all $u,v\in\mathcal{X}$. Then $(\mathcal{X},\perp,d)$ is an orthogonal metric space. Define $\mathcal{T}\colon\mathcal{X}\longrightarrow\mathcal{X}$ as in (35). So $\mathcal{T}$ is $\perp$-continuous. First, we show that $\mathcal{T}$ is $\perp$-preserving, let $u(t)\perp v(t)$ for all $t\in[0,1]$. Now, from (35). we have

$$
\begin{aligned}
\mathcal{T}u(t) &= \frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s,\\
&\quad -\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s,\\
&\quad +\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,u(m))\mathrm{d}m\right)\mathrm{d}s>0,
\end{aligned} \tag{37}
$$

which implies that $\mathcal{T}u\perp\mathcal{T}v$.

Now, we have to show that $\mathcal{T}$ satisfies (i) of Corollary 2 for $F(r)=\ln r$, $r>0$. For all $t\in[0,1]$, $u(t)\perp v(t)$, we have

$$
\begin{aligned}
|\mathcal{T}u(t)-\mathcal{T}v(t)| &= \Bigg|\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s-\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,u(s))\mathrm{d}s\\
&\quad +\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,u(m))\mathrm{d}m\right)\mathrm{d}s,\\
&\quad -\left(\frac{1}{\Gamma(\delta)}\int_0^t (t-s)^{\delta-1}\mathcal{G}(s,v(s))\mathrm{d}s-\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^1 (1-s)^{\delta-1}\mathcal{G}(s,v(s))\mathrm{d}s\right.\\
&\quad \left.+\frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)}\int_0^{\theta}\left(\int_0^s (s-m)^{\delta-1}\mathcal{G}(s,v(m))\mathrm{d}m\right)\mathrm{d}s\right)\Bigg|,
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} |\mathcal{G}(s,u(s)) - \mathcal{G}(s,v(s))| \mathrm{d}s - \frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} |\mathcal{G}(s,u(s)) - \mathcal{G}(s,v(s))| \mathrm{d}s \\
&\quad + \frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)} \int_0^\theta \left(\int_0^s (s-m)^{\delta-1} |\mathcal{G}(s,u(m)) - \mathcal{G}(s,v(m))| \mathrm{d}m \right) \mathrm{d}s, \\
&\leq \frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \left[\frac{\Gamma(\delta+1)}{5} e^{-\tau} \sup_{s\in[0,1]} |u(s)-v(s)| \right] \mathrm{d}s \\
&\quad - \frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} \left[\frac{\Gamma(\delta+1)}{5} e^{-\tau} \sup_{s\in[0,1]} |u(s)-v(s)| \right] \mathrm{d}s \\
&\quad + \frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)} \int_0^\theta \left(\int_0^s (s-m)^{\delta-1} \left[\frac{\Gamma(\delta+1)}{5} e^{-\tau} \sup_{s\in[0,1]} |u(s)-v(s)| \right] \mathrm{d}m \right) \mathrm{d}s, \\
&\leq \left[\frac{\Gamma(\delta+1)}{5} e^{-\tau} \sup_{s\in[0,1]} |u(s)-v(s)| \right] \times \sup_{t\in[0,1]} \left(\frac{1}{\Gamma(\delta)} \int_0^t (t-s)^{\delta-1} \mathrm{d}s - \frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)} \int_0^1 (1-s)^{\delta-1} \mathrm{d}s \right. \\
&\quad \left. + \frac{2t}{\left(2-\theta^2\right)\Gamma(\delta)} \int_0^\theta \left(\int_0^s (s-m)^{\delta-1} \mathrm{d}m \right) \mathrm{d}s \right) \leq e^{-\tau} \sup_{s\in[0,1]} |u(s)-v(s)| = e^{-\tau} d(u,v),
\end{aligned}
\tag{38}
$$

for all $u, v \in \mathcal{X}$. Therefore, the condition (i) of Corollary 2 holds. Accordingly, all axioms of Corollary 2 are verified, and $\mathcal{T}$ has a fixed point. The Caputo-type nonlinear fractional differential equation (34) possesses a solution is yielded.

6. Conclusions

In this manuscript, we prove some existence results for the multivalued orthogonal mappings using the conditions (F1) and (F2) of Wardowski's and obtain the stability of a fixed point problem and a solution for the Caputo-type nonlinear fractional differential equation.

Now, we have an open question, whether we can obtain Theorems 1 and 2 with condition (F1) only of Wardowski in the setting of orthogonal metric space?

Data Availability

No data are used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no known conflicts of financial interest or personal relationships that could have appeared to influence the work reported in this paper.

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