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Research Article

On the Study of Trigonometric Polynomials Using Strum Sequence

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This article constructs trigonometric polynomials of the sine and cosine whose sums are nonnegative. As an application, those nonnegative trigonometric sums are used to study the geometric properties of complex polynomials in the unit disk. The Strum sequences are used to prove the main outcome.

1. Introduction

The trigonometric polynomials with nonnegative sums and its enormous application in various branches of mathematics and science are well known. There are several theories in the literature to find necessary (or/and sufficient) conditions by which

$$\begin{aligned} C(\theta) &= \sum_{n=0}^{m} a_n \cos(n\theta), \\ S(\theta) &= \sum_{n=1}^{m} a_n \sin(n\theta), \quad \theta \in (0,\pi), \end{aligned} \tag{1}$$

are nonnegative or positive. This is a century old problem and many mathematicians [1–10] have contributed numerous results. The geometric function theory (univalent functions) has a very close association with the positive trigonometric sums. Both areas have taken and given to each other as evident in the work [11–22]. Here, we intend to present few more results of this interplay.

An analytic function f is univalent in the unit disk $\mathbb{D} = \{z \in \mathbb{C}: |z| < 1\}$ if for $z_1 \neq z_2$ implies $f(z_1) \neq f(z_2)$. Let $\mathcal{S}$ be the class of all univalent functions with normalization $f(0) = 0 = f'(0) - 1$ and have the series form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{2}$$

Denote by $\mathcal{A}$, the class of all analytic functions of form (2). A domain $D \in \mathbb{C}$ is said to be starlike with respect to a point $z_0 \in D$ if the line segment joining any arbitrary chosen points in D with z_0 lies completely in D. The domain D will be convex if it is star-like with respect to all of its points. Analytically, these classes are characterized as

$$\begin{aligned} f \in \mathcal{S}^* &\Longleftrightarrow \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0, \\ f \in \mathcal{C} &\Longleftrightarrow \operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \end{aligned} \tag{3}$$

where $\mathcal{S}^*$ and $\mathcal{C}$, respectively, denote the class of star-like and convex functions.

The aim of this article is to construct the star-like polynomials. It is known that partial sum of a univalent function in $\mathbb{D}$ need not be univalent. For example, the Koebe

function $k(z) = z(1-z)^{-2}$ is an extremal function of many classes in the univalent functions theory defined on $\mathbb{D}$. However, it can be observed that $S_2(k,z) = z + 2z^2$, the 2nd partial sum of the Taylor series for $k(z)$ is univalent only on the disk $\{z: |z| < 1/4\}$. Thus, the geometric properties of an analytic function do not inherit to its partial sum. These motivate us to construct the polynomial which are star-like on $\mathbb{D}$.

The following result related to the starlikeness of f is required in sequel.

Lemma 1 (see [23]). *If f' is typically real in $\mathbb{D}$, that is, $(\mathrm{Im} z)(\mathrm{Im} f'(z)) > 0$ for $z \in \mathbb{D}$ and satisfies $\mathrm{Re} f'(z) > 0$, $z \in \mathbb{D}$, then f is star-like in $\mathbb{D}$.*

It is clear from Lemma 1 that the positivity of cosine and sine sums has a significant role to study the star-like polynomials. During this investigation, we observed that, along with other conditions, the monotonicity of a_n is pivotal for the positivity or nonnegativity for the sine and cosine sums. This makes us think about what will be the case if we chose a_k randomly (especially nonmonotone)? How we can prove the positivity of those trigonometric sums with arbitrary coefficients? Can those positive sums have an application in the study of geometric function theory?

Here, we will provide some affirmative answers for the above questions. The work is motivated from the recent work by Kwong [24], where several trigonometric polynomials with positive sums are given.

2. Statements of Main Results and Their Consequences

Now, we will state our main results. The first result will give the positivity of cosine and sine sum with nonmonotonic coefficients.

Theorem 1. *For $\theta \in (0,\pi)$, the cosine sum*

$$C_1(\theta) = 1 + \frac{\cos(\theta)}{2} + \frac{\cos(2\theta)}{6} + \frac{\cos(3\theta)}{4} + \frac{\cos(4\theta)}{9} + \frac{\cos(5\theta)}{8} + \frac{\cos(6\theta)}{7}, \tag{4}$$

and the sine sum

$$S_1(\theta) = \frac{\sin(\theta)}{2} + \frac{\sin(2\theta)}{6} + \frac{\sin(3\theta)}{4} + \frac{\sin(4\theta)}{9} + \frac{\sin(5\theta)}{8} + \frac{\sin(6\theta)}{7}, \tag{5}$$

are positive.

The basic identity $\cos(2\pi - \theta) = \cos(\theta)$ implies that $C_1(\theta) > 0$ for all $\theta \in (0, 2\pi)$.

In the sense of complex function theory, the above two positive trigonometric sums can be described as follows. On the boundary of the unit circle $|z| = 1$, that is, for $z = e^{i\theta}$, $0 \le \theta \le 2\pi$, $\mathrm{Re}(f(e^{i\theta})) > 0$; while $\mathrm{Im}(f(e^{i\theta})) > 0$ when $\theta \in (0,\pi)$. Now, by reflection principal, it follows that $\mathrm{Im}(f(e^{i\theta})) < 0$ when $\theta \in (\pi, 2\pi)$. Here,

$$f(z) \sqsubseteq 1 + \frac{z}{2} + \frac{z^2}{6} + \frac{z^3}{4} + \frac{z^4}{9} + \frac{z^5}{8} + \frac{z^6}{7}. \tag{6}$$

Denote $g(z) := \int_0^z f(t)\mathrm{d}t$. Then, clearly $\mathrm{Re} g'(z) = \mathrm{Re} f(re^{i\theta})$ and $\mathrm{Im} g'(z) = \mathrm{Im} f(re^{i\theta})$, $0 \le r < 1$. Since $\mathrm{Re}(f(e^{i\theta})) > 0$, the minimum principal for harmonic functions yield $\mathrm{Re}(f(e^{i\theta})) > 0$; similarly, $\mathrm{Im} g'(z) > 0$ when $\mathrm{Im} z > 0$.

The above fact together with Lemma 1 gives the following result.

Theorem 2. *The polynomial*

$$g(z) = z + \frac{z^2}{4} + \frac{z^3}{18} + \frac{z^4}{16} + \frac{z^5}{45} + \frac{z^6}{48} + \frac{z^7}{49}, \tag{7}$$

is star-like with respect to the origin in $\mathbb{D}$.

Theorem 1 can also be validated from Figures 1(a) and 1(b), while starlikeness of g can be seen in Figure 2.

Now, by the same argument as in Theorem 1, several other examples can be constructed by suitable positive cosine and sine sums. Now, we state our second theorem.

Theorem 3. *The polynomial*

$$g_1(z) = z + \frac{8z^2}{31} + \frac{1}{2400}z^3 + \frac{5}{3}\left(\frac{\sqrt{3}}{2} - \frac{4}{5}\right)z^4, \tag{8}$$

maps the unit disk $\mathbb{D}$ to a star-like domain.

Our third result can be stated as follows.

Theorem 4. *The polynomial*

$$h_n(z) = z + \sum_{k=1}^{n} \frac{2z^{2k+1}}{3(2k+1)(3k+1)} + \sum_{k=0}^{n} \frac{2z^{2k+2}}{3(2k+2)(2k+1)}, \tag{9}$$

maps the unit disk to a star-like domain for $n = 1, 2$.

Before we give analytical Proof of Theorem 4 in Section 3, the starlikeness of h_1 and h_2 can be seen in Figures 3 and 4, respectively.

3. Proof of the Main Results

Proof of Theorem 1. To prove identity (1), we need to recall the classical identity:

$$\cos(\alpha) + \cos(\beta) = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right),$$

$$1 \pm \cos(n\alpha) \ge 0, \quad \text{for all } \alpha \in [0, 2\pi], n = 0, 1, 2, \ldots. \tag{10}$$

This implies

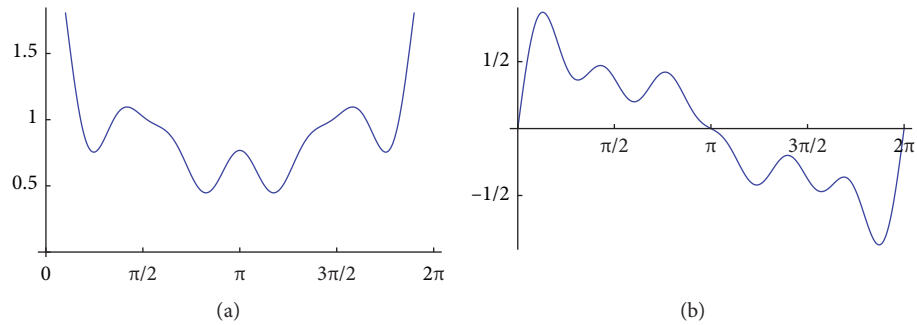

Figure 1: Image of (a) $C_1(\theta)$ and (b) $S_1(\theta)$.

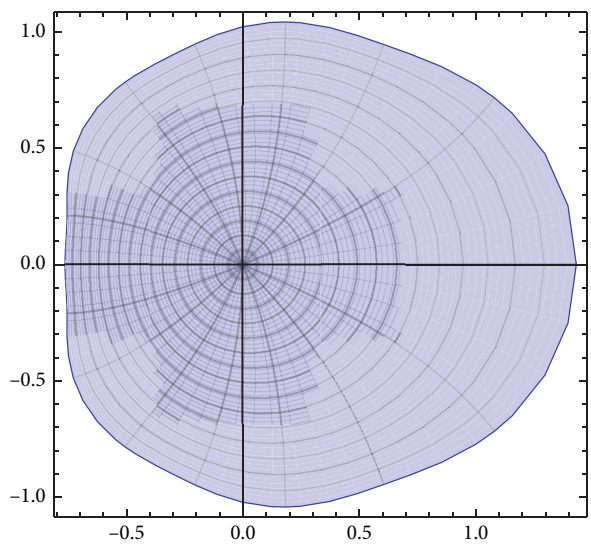

Figure 2: Image of $\mathbb{D}$ under the mapping g.

$$\begin{aligned}\frac{\cos(\theta)}{2}+\frac{\cos(3\theta)}{4}+\frac{\cos(5\theta)}{8}&=\frac{1}{8}(4\cos(\theta)+2\cos(3\theta)+\cos(5\theta))=\frac{1}{8}(3\cos(\theta)+(\cos(\theta)+\cos(3\theta))\\&\quad+(\cos(3\theta)+\cos(5\theta)))\\&=\frac{\cos(\theta)}{8}(3+2\cos(2\theta)+2\cos(4\theta)).\end{aligned}\tag{11}$$

Now, we can rewrite (1) as

$$\begin{aligned}&1+\frac{\cos(\theta)}{2}+\frac{\cos(2\theta)}{6}+\frac{\cos(3\theta)}{4}+\frac{\cos(4\theta)}{9}+\frac{\cos(5\theta)}{8}+\frac{\cos(6\theta)}{7}\\&=\frac{1}{8}(1+\cos(\theta))(3+2\cos(2\theta)+2\cos(4\theta))+\left(\frac{5}{8}-\frac{1}{12}\cos(2\theta)-\frac{5}{36}\cos(4\theta)+\frac{1}{7}\cos(6\theta)\right)\\&=\frac{1}{8}(1+\cos(\theta))\left((1+\cos(2\theta))^2+3\cos^2(2\theta)\right)+\left(\frac{131}{504}+\frac{1}{12}(1-\cos(2\theta))+\frac{5}{36}(1-\cos(4\theta))+\frac{1}{7}(1+\cos(6\theta))\right)>0,\end{aligned}\tag{12}$$

hence proved.

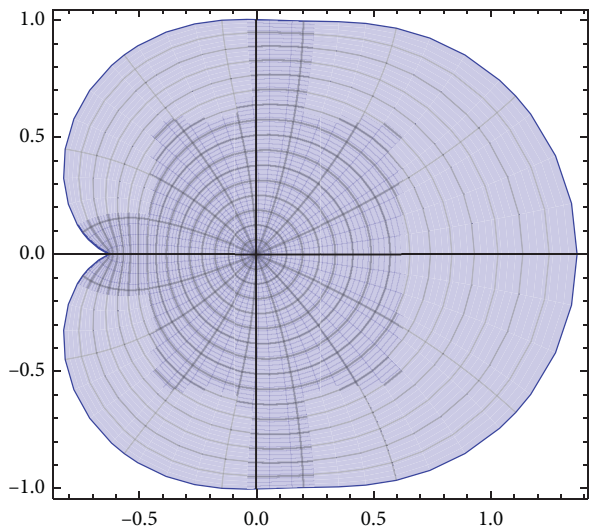

Figure 3: Image of $\mathbb{D}$ under the mapping h_1.

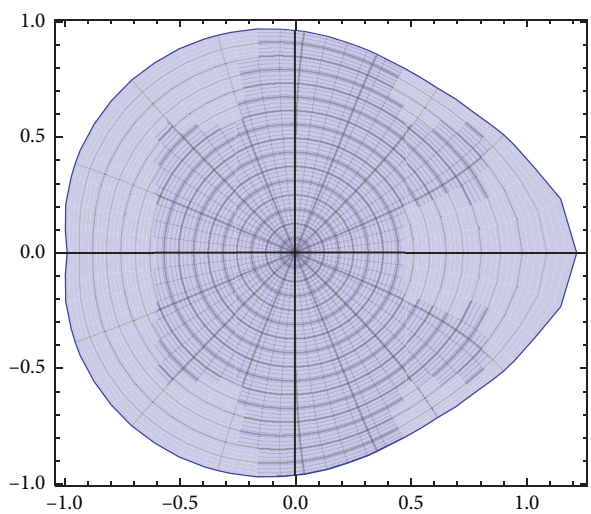

Figure 4: Image of $\mathbb{D}$ under the mapping h_2.

Next, we will prove the positivity of the sine sum in (5). From the well-known trigonometric identities, one can obtain the following identities:

$$S_1(\theta) = \frac{\sin(\theta)}{504}(441 + 424\cos(\theta) + 378\cos(2\theta) + 256\cos(3\theta) + 126\cos(4\theta) + 144\cos(5\theta)). \tag{13}$$

It is known that $\sin(\theta)$ is positive for $0 < \theta < \pi$, and hence $S_1(\theta) > 0$ if

$$\begin{aligned} &441 + 424\cos(\theta) + 378\cos(2\theta) + 256\cos(3\theta) + 126\cos(4\theta) + 144\cos(5\theta) \\ &\quad = 189 + 376\cos(\theta) - 252\cos^2(\theta) - 1856\cos^3(\theta) + 1008\cos^4(\theta) + 2304\cos^5(\theta) > 0. \end{aligned} \tag{14}$$

Denote $y := \cos(\theta)$. Then, clearly $y \in (-1,1)$ and (14) is equivalent to

$$P_0(y) = 189 + 376y - 252y^2 - 1856y^3 + 1008y^4 + 2304y^5 > 0. \tag{15}$$

Since $P_0(0) = 189 > 0$, inequality (15) holds if P_0 have no zero in $(-1,1)$. We will prove these facts by constructing Strum sequences to determine the number of zeroes of a polynomial in an open interval.

The Strum sequence for the polynomial P_0 is given as follows:

$$\begin{aligned}
P_1(y) &= P_0'(y) = 376 - 504y - 5568y^2 + 4032y^3 + 11520y^4,\\
P_2(y) &= -\mathrm{rem}(P_0(y), P_1(y)) = -\frac{9121}{50} - \frac{15481y}{50} + \frac{1344y^2}{25} + \frac{20324y^3}{25},\\
P_3(y) &= -\mathrm{rem}(P_1(y), P_2(y)) = -\frac{28651198000}{25816561} - \frac{85877316000y}{25816561} + \frac{36060640800y^2}{25816561},\\
P_4(y) &= -\mathrm{rem}(P_2(y), P_3(y)) = -\frac{15786533307854936851}{11287932421064450} - \frac{171856954494801497753y}{33863797263193350},\\
P_5(y) &= -\mathrm{rem}(P_3(y), P_4(y)) = \frac{995719081958136788394316950594748 00}{11440258370674650555418814652583 69}.
\end{aligned} \tag{16}$$

Here, by $\mathrm{rem}(P(y), Q(y))$, we mean the remainder of the long division of P by the polynomial Q. Now, by Strum theorem the number of roots of P_0 in $(-1,1)$ is equal to $\sigma(-1) - \sigma(1)$, where $\sigma(a)$ is the number of change of sign by the Strum sequence at the point a. From Table 1, it follows that $\sigma(-1) - \sigma(1) = 2 - 2 = 0$. Thus, P_0 does not have any zero on $(-1,1)$. □

Proof of Theorem 2. Clearly,

$$g_1'(z) = 1 + \frac{16}{31}z + \frac{1}{400}z^2 + \frac{20}{3}\left(\frac{\sqrt{3}}{2} - \frac{4}{5}\right)z^3. \tag{17}$$

Now, for $z = e^{i\theta}$, it follows that

$$\begin{aligned}
\mathrm{Re}\, g_1'\left(e^{i\theta}\right) &= 1 + \frac{16}{31}\cos(\theta) + \frac{1}{400}\cos(2\theta) + \frac{20}{3}\left(\frac{\sqrt{3}}{2} - \frac{4}{5}\right)\cos(3\theta)\\
&> 1 - \frac{16}{31} - \frac{1}{400} - \frac{20}{3}\left(\frac{\sqrt{3}}{2} - \frac{4}{5}\right) \approx 0.0412016,
\end{aligned} \tag{18}$$

and for $0 < \theta < \pi$,

$$\begin{aligned}
\mathrm{Im}\, g_1'\left(e^{i\theta}\right) &= \frac{16}{31}\sin(\theta) + \frac{1}{400}\sin(2\theta) + \frac{20}{3}\left(\frac{\sqrt{3}}{2} - \frac{4}{5}\right)\sin(3\theta)\\
&= \sin(\theta)\left(\frac{16}{31} + \frac{1}{200}\cos(\theta) + t\frac{20}{3}n\left(\frac{\sqrt{3}}{2} - \frac{4}{5}\right)q\left(4\cos^2(\theta) - 1\right)\right) > \sin(\theta)\left(\frac{16}{31} - \frac{1}{200} - \frac{20}{3}\left(\frac{\sqrt{3}}{2} - \frac{4}{5}\right)\right) \approx 0.0709597\sin(\theta) > 0.
\end{aligned} \tag{19}$$

The result follows from Lemma 1. □

Proof of Theorem. 3. For $z = e^{i\theta}$, it follows from (3) that

$$\mathrm{Re}\left(h_n'\left(e^{i\theta}\right)\right) = 1 + \frac{2}{3}\sum_{k=1}^{n}\frac{\cos(2k\theta)}{3k+1} + \frac{2}{3}\sum_{k=0}^{n}\frac{\cos((2k+1)\theta)}{2k+1}, \tag{20}$$

for $\theta \in (0, 2\pi)$.

Clearly, for $n = 1$, identity (20) gives

$$\begin{aligned}
\mathrm{Re}\left(h_1'\left(e^{i\theta}\right)\right) &= 1 + \frac{2}{3}\cos(\theta) + \frac{1}{6}\cos(2\theta) + \frac{2}{9}\cos(3\theta)\\
&= \frac{5}{6} + \frac{1}{3}\cos^2(\theta) + \frac{8}{9}\cos^3(\theta)\\
&= \frac{5}{18} + \frac{\cos^2(\theta)}{3}(1 + \cos(\theta))\\
&\quad + \frac{5}{9}\left(1 + \cos^3(\theta)\right) > 0.
\end{aligned} \tag{21}$$

Similarly, for $n = 1$, we have

Table 1: Change the signs of Strum sequence for P_0.

	$y = -1$	$y = 1$
$P_0(y)$	+	+
$P_1(y)$	−	+
$P_2(y)$	+	−
$P_3(y)$	+	−
$P_4(y)$	+	+
$P_5(y)$	+	+

Table 2: Change the signs of Strum sequence for Q.

	$y = -1$	$y = 1$
$Q(y)$	+	+
$Q_1(y)$	−	+
$Q_2(y)$	−	−
$Q_3(y)$	+	−
$Q_4(y)$	+	+

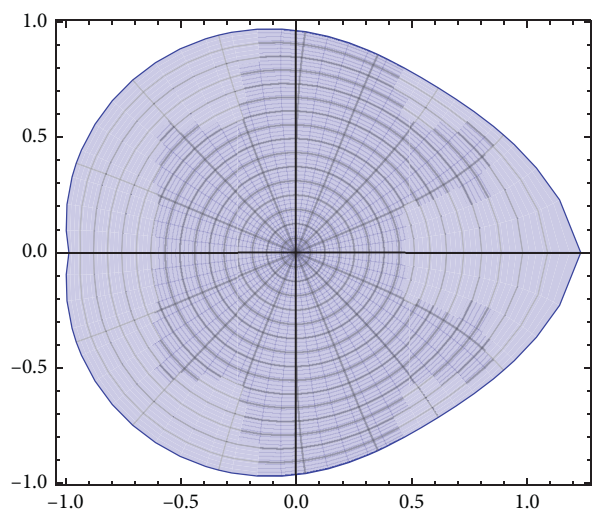

Figure 5: Image of $\mathbb{D}$ under the mapping h_{28}.

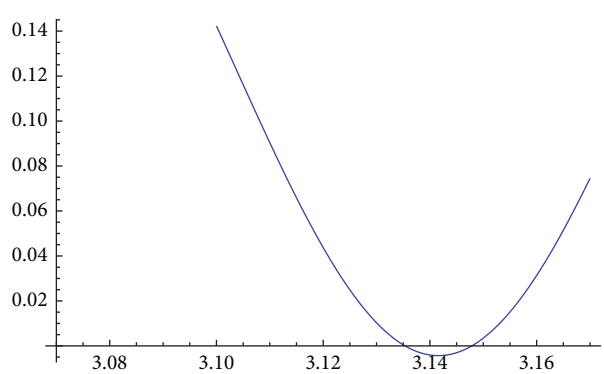

Figure 6: Image of $\mathrm{Re}(h'_{28}(e^{i\theta}))$, $3.1 \le \theta \le 3.17$.

$$\begin{aligned}\operatorname{Im}\left(h_1'\left(e^{i\theta}\right)\right) &= \frac{2}{3}\sin(\theta) + \frac{1}{6}\sin(2\theta) + \frac{2}{9}\sin(3\theta) \\ &= \frac{\sin(\theta)}{9}\left(4 + 3\cos(\theta) + 8\cos^2(\theta)\right).\end{aligned} \tag{22}$$

If $\theta \in (0, \pi)$, then (22) implies that $\operatorname{Im}(h_2'(e^{i\theta})) > 0$, while if $\theta \in (\pi, 2\pi)$, then $\operatorname{Im}(h_2'(e^{i\theta})) < 0$. Thus, we can conclude that $\operatorname{Im}(z)\operatorname{Im}(h_2'(z)) > 0$. Finally, the conclusion follows from Lemma 1.

Next, we will consider the case for $n = 2$. In this case,

$$\begin{aligned}\operatorname{Re}\left(h_2'\left(e^{i\theta}\right)\right) &= 1 + \frac{2}{3}\cos(\theta) + \frac{1}{6}\cos(2\theta) + \frac{2}{9}\cos(3\theta) + \frac{2}{21}\cos(4\theta) + \frac{2}{15}\cos(5\theta) = \frac{5}{6} + \frac{1}{3}\cos^2(\theta) + \frac{8}{9}\cos^3(\theta) \\ &\quad + \frac{2}{21}\cos(4\theta) + \frac{2}{15}\cos(5\theta) \\ &= \frac{31}{630} + \frac{\cos^2(\theta)}{3}(1 + \cos(\theta)) + \frac{5}{9}\left(1 + \cos^3(\theta)\right) + \frac{2}{21}(1 + \cos(4\theta)) + \frac{2}{15}(1 + \cos(5\theta)) > 0.\end{aligned} \tag{23}$$

Similarly,

$$\begin{aligned}\operatorname{Im}\left(h_1'\left(e^{i\theta}\right)\right) &= \frac{2}{3}\sin(\theta) + \frac{1}{6}\sin(2\theta) + \frac{2}{9}\sin(3\theta) + \frac{2}{21}\sin(4\theta) + \frac{2}{15}\sin(5\theta) \\ &= \frac{\sin(\theta)}{315}(322 + 165\cos(\theta) + 224\cos(2\theta) + 60\cos(3\theta) + 84\cos(4\theta)) \\ &= \frac{\sin(\theta)}{315}\left(182 - 15\cos(\theta) - 224\cos^2(\theta) + 240\cos^3(\theta) + 672\cos^4(\theta)\right).\end{aligned} \tag{24}$$

It is well known that $\sin(\theta)$ is positive for $\theta \in (0, \pi)$. Thus, it is enough to show that

$$\begin{aligned}&182 - 15\cos(\theta) - 224\cos^2(\theta) + 240\cos^3(\theta) \\ &\quad + 672\cos^4(\theta) > 0,\end{aligned} \tag{25}$$

which is equivalent to prove

$$Q(y) = 182 - 15y - 224y^2 + 240y^3 + 672y^4 > 0. \tag{26}$$

Since $Q(0) = 182 > 0$, we need to prove that $Q(y)$ does not have any zero in $[-1, 1]$. We apply the Strum theorem for this purpose. A differentiation of (26) gives

$$Q_1(y) = Q'(y) = -15 - 448y + 720y^2 + 2688y^3. \tag{27}$$

Next to form the Strum sequence, we need to find the remainder of $Q(y)$ and $Q_1(y)$ as

$$Q_2(y) = -\operatorname{rem}\left(Q(y), Q_1(y)\right) = -\frac{40843}{224} + \frac{5y}{4} + \frac{1793y^2}{14}, \tag{28}$$

and continuing, similar to the Strum theorem, we have

$$\begin{aligned}Q_3(y) &= -\operatorname{rem}\left(Q_1(y), Q_2(y)\right) = -\frac{284283300}{292259} - \frac{10840870880}{3214849}y, \\ Q_4(y) &= -\operatorname{rem}\left(Q_2(y), Q_3(y)\right) = \frac{14441992188069699847}{83946058169165696}.\end{aligned} \tag{29}$$

The change of sign of the Strum sequence $\{Q, Q_1, Q_2, Q_3, Q_4\}$, respectively, at $y = -1$ and $y = 1$ can be seen in Table 2. Since $\sigma(-1) - \sigma(1) = 2 - 2 = 0$, the polynomial Q does not have any zero on $(-1, 1)$.

Finally, we have $\operatorname{Im}(h_2'(e^{i\theta})) > 0$ for all $\theta \in (0, \pi)$, and by Lemma 1, it can be concluded that $h_2(\mathbb{D})$ is star-like with respect to the origin. □

4. Concluding Remark

From the above three examples, it can be concluded that the trigonometric sine and cosine sum can also be positive with some nonmonotone coefficients.

In Theorem 4, though we prove that h_n is star-like for $n = 1, 2$, but the graphical experiment Figure 5 shows that h_n

is star-like for all $n \in \mathbb{N}$. However, $\mathrm{Re}\, h_n'(e^{i\theta})$ is negative for some range of θ when n is larger (greater than 28, see Figure 6). Thus, our method is not applicable on h_n for larger n.

The results in this article indicate that several polynomials can be constructed which maps unit disk to a star-like domain by using positivity of cosine and sine sum.

Data Availability

No data were used in this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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