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Research Article

On Generalized Schur Numbers of the Equation $x + ay = z$

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Let a and r be positive integers. By definition, $s_a(r)$ is the least positive integer such that, for any r-coloring of the interval $[1, s_a(r)]$, there exists a monochromatic solution to $x + ay = z$. For $a = 1$, the numbers $s(r) = s_1(r)$ are classical Schur numbers. In this paper, we study the numbers $s_a(r)$ for $a \geq 2$.

1. Introduction

Ramsey theory, an area that has seen a remarkable burst of research activity during the past twenty years, is the study of the preservation of properties under set partitions. In other words, given a particular set S that has a property P, is it true that whenever S is partitioned into finitely many subsets, one of the subsets must also have property P? Ramsey theory is named after Frank Plumpton Ramsey and his theorem, which he proved in 1928. Ramsey stated his fundamental theorem in a general setting and applied it to formal logic. Schur [1] and van der Waerden [2] obtained similar results in number theory. Dilworth's theorem [3] for partially ordered sets is another typical example. Ramsey's theorem was applied to geometry by Erdős and Szekeres [4]. They also defined the Ramsey numbers and gave some upper and lower bounds for them. Ramsey-type theorems have applications in different branches of mathematics such as number theory, set theory, geometry, ergodic theory, and theoretical computer science. In [5], connections between the computation of van der Waerden numbers and propositional theories have been shown. Using these connections, they serve as benchmarks for solvers of satisfiability problems. For more on Ramsey theory and its applications, we refer to the book of Graham et al., [6], and the surveys of Nešetřil [7] and Rosta [8]. Somewhat surprisingly, Ramsey's theorem was not the first theorem in the area now known as Ramsey theory. The result that is generally accepted to be the first Ramsey-type theorem is due to Schur [1] and it deals with colorings of the integers: if $\mathbb{N}$ is partitioned into a finite number of classes, at least one partition class contains a solution to the equation $x + y = z$.

Let us now go over the details of some definitions and notations from [9]. An interval $[a, b]$ is a set of the form $\{a, a + 1, \ldots, b\}$, where $a < b$ are integers.

Definition 1. An r-coloring of a set S is a function $\chi\colon S \longrightarrow C$, where $|C| = r$.

In fact, an r-coloring χ of a set S is a partition of S into r subsets $S_1, \ldots, S_r$, by associating the subset S_i with the set $\{x \in S\colon \chi(x) = i\}$.

Definition 2. A coloring χ is monochromatic on a set S if χ is constant on S.

It is often convenient to represent a particular 2-coloring of an interval as a string of 0s and 1s. For example, the coloring $\chi\colon [1, 5] \longrightarrow \{0, 1\}$ with $\chi(1) = \chi(2) = \chi(3) = 1$ and $\chi(4) = \chi(5) = 0$ could be represented by the string 11100 or $1^3 0^2$. As we mentioned before, one of the earliest results in Ramsey theory was proved by Issai Schur in 1916.

Theorem 1 (Schur's Theorem). *For any $r \geq 1$, there exists a least positive integer $s(r)$ such that, for any r-coloring of $[1, s(r)]$, there exists a monochromatic solution to $x + y = z$.*

The numbers $s(r)$ are called the Schur numbers. The known Schur numbers are $s(1) = 2$, $s(2) = 5$, $s(3) = 14$, and $s(4) = 45$. Note that x and y in Schur's theorem need not be distinct. If we require that x and y be distinct, the resulting Ramsey-type statement is also true.

Theorem 2. *For $r \geq 1$, there exists a minimal integer ${}^{s}(r)$ such that every r-coloring of $[1,{}^{s}(r)]$ admits a monochromatic solution to $x + y = z$ with x and y being distinct.*

It is known that Schur's theorem is a consequence of Ramsey's theorem. There are a number of interesting results proved during the last years concerning Schur's theorem and generalizations. A triple x, y, z of natural numbers is called a Schur triple if $x \neq y$ and $x + y = z$. Let $S(N)$ be the minimum number of monochromatic Schur triples in any 2-coloring of $[N] = \{1, 2, \ldots, N\}$. Graham et al. [10] found the lower bound $S(N) \geq (1/38)N^2 + O(N)$. They used the Ramsey multiplicity result [11, 12], which says that in every 2-coloring of the edges of a complete graph on N vertices, there are at least $N^3/24 + O(N^2)$ monochromatic triangles. Answering a question raised in [10], Robertson and Zeilberger [13] and, independently, Schoen [14] showed that $S(N) = (1/22)N^2 + O(N)$. Robertson and Zeilberger found a 2-coloring with $N^2/22$ monochromatic Schur triples and formulated a conjecture on the minimum number of triples. Schoen showed that every extremal coloring looks like the Robertson–Zeilberger construction, and he used this result to find the exact number $S(N) = (1/22)N^2 - (7/22)N$, for $N \equiv 0 \pmod{22}$.

One can look at Schur's theorem in terms of sum-free sets. A set $A \subseteq \mathbb{N}$ is called sum-free if $x, y \in A$ implies $x + y \notin A$. The Schur function s_t is defined as the maximum $m \in \mathbb{N}$ such that $\{1, 2, \ldots, m\}$ can be partitioned into t sum-free sets. We mention here a generalization of Schur's theorem for sum-free sets: if $\mathbb{N}$ is finitely colored, there exists arbitrarily large finite set $A \subseteq \mathbb{N}$ such that the sum-free set of A, $\{\sum_{a \in B} a\colon B \subseteq A, 1 \leq |B| < \infty\}$ is monochromatic. Note that Hindman's theorem [15] gives the same result when A is an infinite set. On the other hand, Alekseev and Savchev [16] considered a similar problem and proved that for every equinumerous 3-coloring of $[3n]$ (i.e., a coloring in which different color classes have the same cardinality), the equation $x + y = z$ has a solution, with x, y, and z belonging to different color classes. Such solutions will be called rainbow solutions. Moreover, Schönheim [17] proved that for every 3-coloring of $[n]$, every color class has cardinality greater than $n/4$ and the equation $x + y = z$ has rainbow solutions. Moreover, he showed that $n/4$ is optimal.

We now recall another generalization of Schur's theorem. Let $\mathscr{L}(t)$ represent the equation $x_1 + x_2 + \cdots + x_{t-1} = x_t$, where $x_1, \ldots, x_t$ are variables.

Theorem 3. *Let $r \geq 1$ and, for $1 \leq i \leq r$, assume that $k_i \geq 3$. Then, there exists a least positive integer $s(k_1, k_2, \ldots, k_r)$ such that for every r-coloring of $[1, s(k_1, k_2, \ldots, k_r)]$, there is a solution to $\mathscr{L}(k_j)$ of color j for some $j \in \{1, 2, \ldots, r\}$.*

Just as Schur's theorem, Theorem 3 follows from Ramsey's theorem. The numbers $s(k_1, k_2, \ldots, k_r)$ are called the generalized Schur numbers. In [18], the authors determine 26 previously unknown values of $s(k_1, k_2, \ldots, k_r)$ and conjecture that for $4 \leq k_1 \leq k_2 \leq k_3$, $s(k_1, k_2, k_3) = k_1k_2k_3 - k_3k_2 - k_3 - 1$.

Following a problem proposed in [9], we consider the monochromatic solutions to $x + ay = z$. For abbreviation, we write s_a instead of $s_a(2)$.

Definition 3. For integers a and $r \geq 1$, let $s_a(r)$ be the least positive integer such that, for any r-coloring of $[1, s_a(r)]$, there exists a monochromatic solution to $x + ay = z$.

The existence of such monochromatic solutions is implied by Rado's theorem.

Theorem 4 (Rado's Theorem). *For any $r \geq 1$, there exists n such that for every r-coloring of $[1, n]$, there is a monochromatic solution to the linear equation $\sum_{i=1}^{n} c_i x_i = 0$, where $c_i \in \mathbb{Z} - \{0\}$ for $1 \leq i \leq n$, if and only if some nonempty subset of the c_i's sums to 0.*

2. Exact Value of s_a

In this section, we find the exact value of s_a. The standard methodology for finding the exact value of any particular Ramsey-type number is to show that some number serves both as a lower bound and an upper bound. We may illustrate this phenomenon by a simple example. We will establish that $s_2 = 11$ by using the above method to prove that $s_2 \geq 11$ and $s_2 \leq 11$.

To show that $s_2 \geq 11$, it suffices to exhibit a 2-coloring of $[1, 10]$ with no monochromatic solution to $x + 2y = z$. One such coloring is the following: color the intervals $[1, 2]$ and $[9, 10]$ red and color the interval $[3, 8]$ blue. It is easy to see that this coloring avoids any monochromatic solution to $x + 2y = z$. For all possible solutions, see Table 1.

To show that $s_2 \leq 11$, we must show that every 2-coloring of $[1, 11]$ admits a monochromatic solution to $x + 2y = z$. Using red and blue as the colors, assume, for a contradiction, that there exists a 2-coloring χ of $[1, 11]$ with no monochromatic solution to $x + 2y = z$. Since $(1, 1, 3)$ and $(3, 3, 9)$ cannot be monochromatic, without loss of generality, we can assume that $\chi(1) = \chi(9) =$ red and $\chi(3) =$ blue. Since neither of the triples $(1, 4, 9)$, $(7, 1, 9)$, and $(9, 1, 11)$ can be red, 4, 7, and 11 must be blue. From this, we have that $(3, 4, 11)$ is blue, contradicting our assumption.

The method to obtain the lower and upper bounds in general is similar to the one used in the above example. We first establish the lower bound.

Theorem 5. *For $a \geq 1$, $s_a > a^2 + 3a$.*

Proof. Let $a \geq 1$. To show that $s_a > a^2 + 3a$, it suffices to find some 2-coloring of $[1, a^2 + 3a]$ that yields no monochromatic solution to $x + ay = z$. Using 0 and 1 as the colors, it is an easy task to check that the coloring $1^a 0^{a(a+1)} 1^a$ contains no monochromatic solution to $x + ay = z$. Hence, $s_a > a + a(a + 1) + a = a^2 + 3a$. □

Theorem 6. *For $a \geq 1$, $s_a \leq a^2 + 3a + 1$.*

Proof. To show that $s_a \leq a^2 + 3a + 1$, we must show that every 2-coloring of $[1, a^2 + 3a + 1]$ admits a monochromatic

Table 1: All solutions to $x + 2y = z$ in the interval $[1, 10]$.

Variables	Values																			
x	1	2	1	3	2	4	1	3	5	2	4	6	1	3	5	7	2	4	6	8
y	1	1	2	1	2	1	3	2	1	3	2	1	4	3	2	1	4	3	2	1
z	3	4	5	5	6	6	7	7	7	8	8	8	9	9	9	9	10	10	10	10

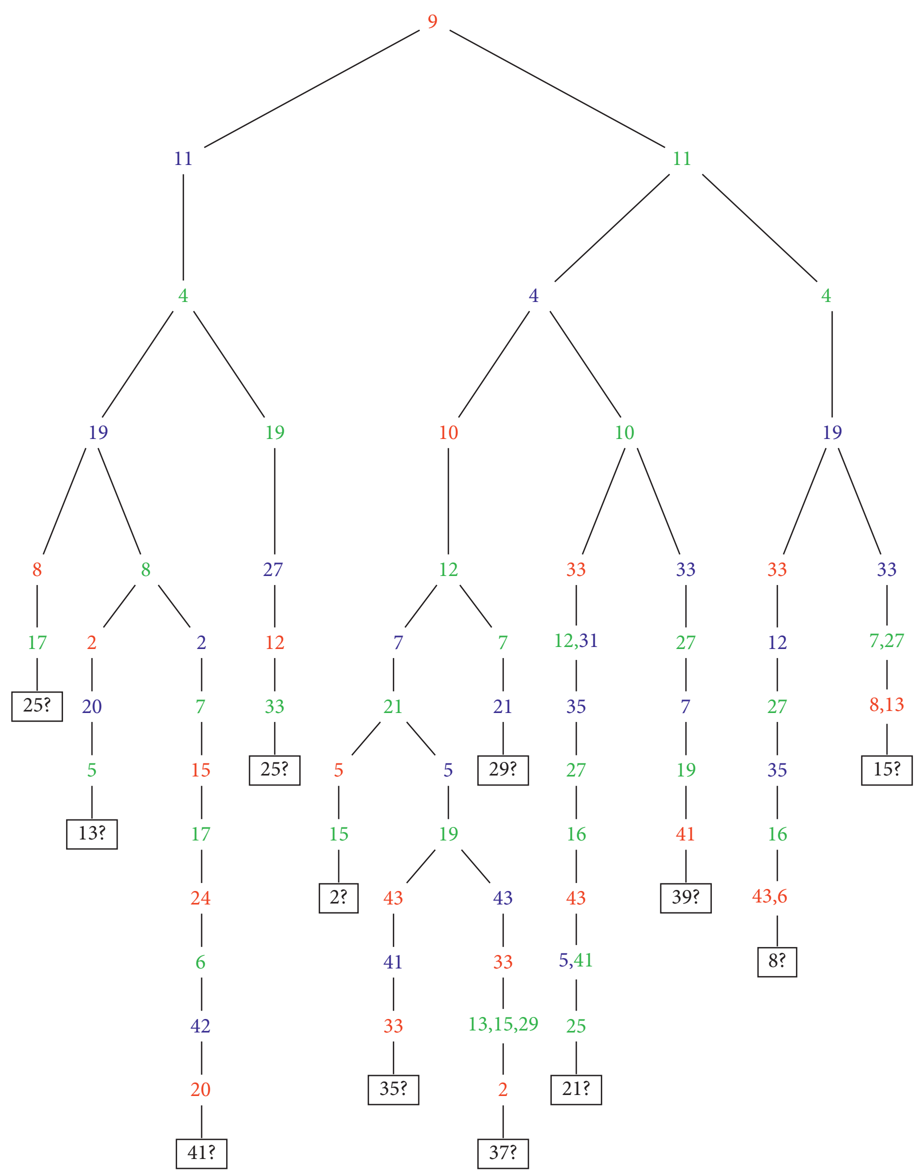

Figure 1: Tree representation for $\chi(9) = \text{red}$.

solution to $x + ay = z$. Using red and blue as the colors, assume, for a contradiction, that there exists a 2-coloring χ of $[1, a^2 + 3a + 1]$ with no monochromatic solution to $x + ay = z$. Since $(1, 1, a + 1)$ and $(a + 1, a + 1, (a + 1)^2)$ cannot be monochromatic, without loss of generality, we can assume that $\chi(1) = \chi((a + 1)^2) = \text{red}$ and $\chi(a + 1) = \text{blue}$. Since neither of the triples $(1, a + 2, (a + 1)^2)$, $((a + 1)^2 - a, 1, (a + 1)^2)$, and $((a + 1)^2, 1, (a + 1)^2 + a)$ can be red, $a + 2$, $(a + 1)^2 - a$, and $(a + 1)^2 + a$ must be blue. From this, we have that $(a + 1, a + 2, (a + 1)^2 + a)$ is blue, contradicting our assumption. □

By Theorems 5 and 6, we have the following corollary.

Corollary 1. *For all* $a \geq 1$, *we have* $s_a = a^2 + 3a + 1$.

3. A General Lower Bound

First, suppose that $a = 2$. The coloring $1^2 0^6 1^2$ shows that $s_2(2) \geq 11$. One can extend to the coloring $0^2 1^6 0^2 2^{22} 0^2 1^6 0^2$ to obtain that $s_2(3) \geq 43$. Continuing in this way, we have the following.

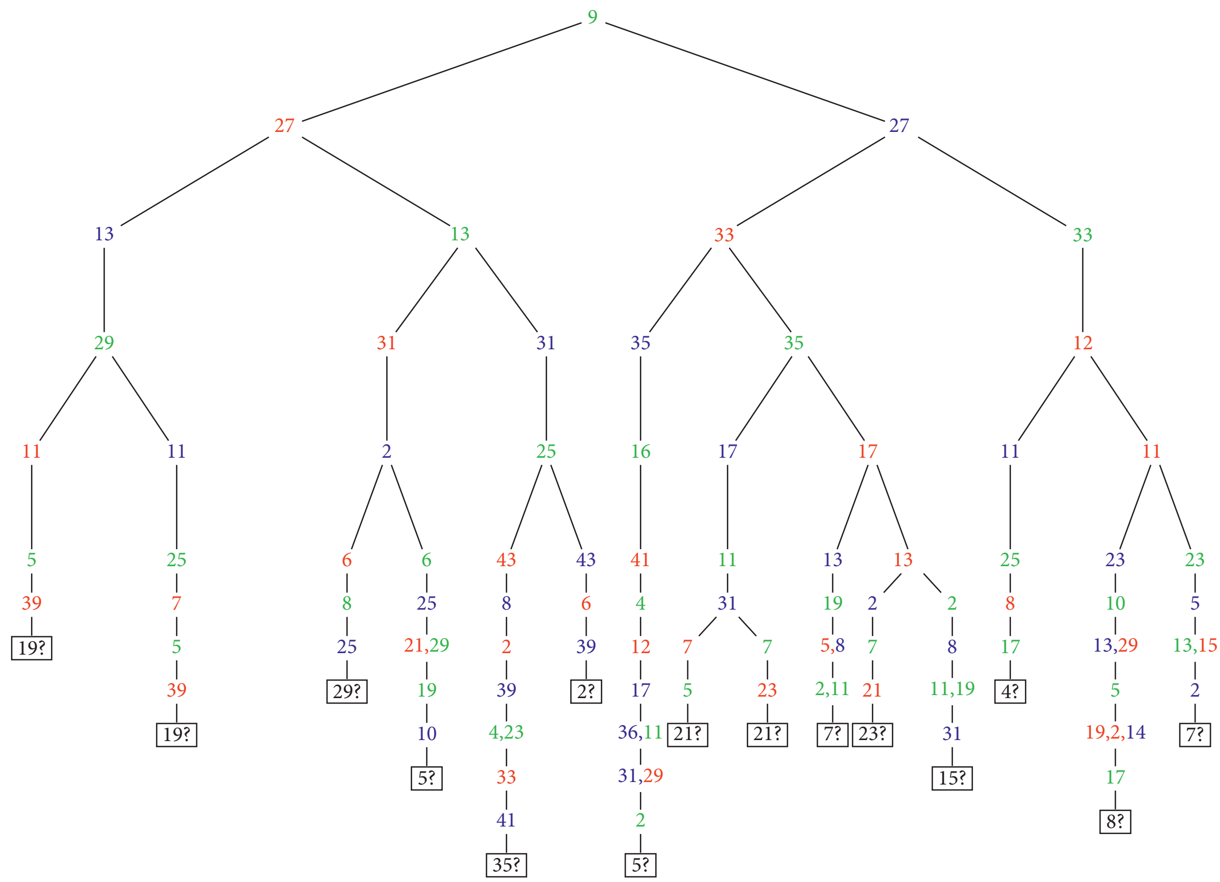

Figure 2: Tree representation for $\chi(9)$ = green.

Theorem 7. *For all* $r \geq 1$, $s_2(r) \geq (2 \times 4^r + 1)/3$.

Proof. Let $n \geq 1$ and assume χ: $[1, n] \longrightarrow \{1, \ldots, r\}$ is an r-coloring of $[1, n]$ with no monochromatic solution to $x + 2y = z$. Define an $(r+1)$-coloring $\hat{\chi}$: $[1, 4n+1] \longrightarrow \{1, \ldots, r+1\}$ of $[1, 4n+2]$ that extends χ as follows. For all $x \in [n+1, 3n+2]$, let $\hat{\chi}(x) = r+1$, and for all $x \in [1, n] \cup [3n+3, 4n+2]$, let $\hat{\chi}(x) = \chi(y)$, where $x \equiv y \pmod{3n+2}$. Then, $\hat{\chi}$ contains no monochromatic solution to $x + 2y = z$. So if $s_2(r) \geq n+1$, then $s_2(r+1) \geq 4n+3$ and thus, $s_2(r+1) \geq 4s_2(r) - 1$. We now prove by induction that $s_2(r) \geq (2 \times 4^r + 1)/3$. It is clearly true for $r = 1$. Now, by induction hypothesis,

$$s_2(r+1) \geq 4s_2(r) - 1 \geq 4\left(\frac{2 \times 4^r + 1}{3}\right) - 1 = \frac{2 \times 4^{r+1} + 1}{3}. \tag{1}$$

For $a \geq 1$, the coloring $1^a 0^{a(a+1)} 1^a$ shows that $s_a(2) \geq a(a+3)$. We can extend this coloring to $1^a 0^{a(a+1)} 1^a 2^{a(a(a+3)+1)} 1^a 0^{a(a+1)}$ to obtain $s_a(3) > a((a+2)(a+3)+1) = a((a^2+5a+7)$. Similarly, the coloring $1^a 0^{a(a+1)} 1^a 2^{a(a(a+3)+1)} 1^a 0^{a(a+1)} 3^{a(a(a^2+5a+7)+1)} 1^a 0^{a(a+1)} 1^a 2^{a(a(a+3)+1)} 1^a 0^{a(a+1)}$ shows that $s_a(4) > a(((a+2)(a+3)+1)(a+2)+1)$. Continuing in this way, we can easily check that $s_a(5) > a((((a+2)(a+3)+1)(a+2)+1)(a+2)+1)$. In general, we obtain the following theorem. ☐

Theorem 8. *For all* $r \geq 1$, $s_a(r) \geq (a(a+2)^r + 1)/(a+1)$.

Proof. Let $n \geq 1$ and assume χ: $[1, n] \longrightarrow \{1, \ldots, r\}$ is an r-coloring of $[1, n]$ with no monochromatic solution to $x + ay = z$. Define an $(r+1)$-coloring $\hat{\chi}$: $[1, a + (a+2)n] \longrightarrow \{1, \ldots, r+1\}$ of $[1, a + (a+2)n]$ that extends χ as follows. For all $x \in [n+1, a + (a+1)n]$, let $\hat{\chi}(x) = r+1$, and for all $x \in [1, n] \cup [(a+1)(n+1), a + (a+2)n]$, let $\hat{\chi}(x) = \chi(y)$, where $x \equiv y \pmod{(a+1)(n+1) - 1 = a + (a+1)n}$. Then, $\hat{\chi}$ contains no monochromatic solution to $x + ay = z$. So if $s_a(r) \geq n+1$, then $s_a(r+1) \geq (a+2)n + a + 1$ and thus, $s_a(r+1) \geq (a+2)s_a(r) - 1$. We now prove by induction that $s_a(r) \geq (a(a+2)^r + 1)/(a+1)$. It is clearly true for $r = 1$. Now, by induction hypothesis,

$$s_a(r+1) \geq (a+2)s_a(r) - 1 \geq (a+2)\left(\frac{a(a+2)^r + 1}{a+1}\right) - 1 = \frac{a(a+2)^{r+1} + 1}{a+1}. \tag{2}$$

☐

4. $s_2(3) = 43$ and $s_3(3) = 94$

In this section, we show that $s_2(3) = 43$. The coloring $0^2 1^6 0^2 2^{22} 0^2 1^6 0^2$ shows that $s_2(3) \geq 43$. We now show the other direction.

Theorem 9. *We have* $s_2(3) \leq 43$.

Proof. To show that $s_2(3) \leq 43$, we must show that every 3-coloring of $[1, 43]$ admits a monochromatic solution to $x + 2y = z$. Using red, blue, and green as the colors, assume, for a contradiction, that there exists a 3-coloring χ of $[1, 43]$ with no monochromatic solution to $x + 2y = z$. Since $(1, 1, 3)$ cannot be monochromatic, without loss of generality, we can assume that $\chi(1) =$ red and $\chi(3) =$ blue. Then, $\chi(9)$ is red or green. We suppose $\chi(9) =$ red and leave the other case to the reader. Since $(1, 9, 11)$ cannot be red, then $\chi(11) =$ blue or green. If $\chi(11) =$ blue, then $\chi(4) =$ green, and if $\chi(11) =$ green, then $\chi(4) =$ blue or green. For a complete solution, we use the tree representation (see Figures 1 and 2). □

A computer search showed us that $s_3(3) \leq 94$, and so by the above upper bound, we have the following.

Theorem 10. *We have* $s_3(3) = 94$.

5. Concluding Remark

In this paper, we obtain the lower bound for $s_a(r)$ and find the exact value in the case of two colors. Moreover, we show that $s_2(3)$ and $s_3(3)$ are equal to the obtained lower bounds 43 and 94, respectively. It is interesting to prove the general formula for $s_a(3)$. It seems that $s_a(3) = a((a + 2)(a + 3) + 1) + 1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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