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Research Article

On Generalized Strongly p-Convex Functions of Higher Order

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The aim of this paper is to introduce the definition of a generalized strongly p-convex function for higher order. We will develop some basic results related to generalized strongly p-convex function of higher order. Moreover, we will develop Hermite–Hadamard-, Fejér-, and Schur-type inequalities for this generalization.

1. Introduction

Some geometric properties of convex sets and functions have been studied before 1960 by great mathematicians Hermann, Minkowski, and Werner Fenchel. The classical convexity is defined as follows: a function $f: I \longrightarrow R$ is said to be convex function if

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) \tag{1}$$

for all $x, y \in I$ and $t \in [0,1]$.

The notion of convexity is crucial to the solution of many real-world problems. Fortunately, many problems encountered in constrained control and estimation are convex. Since the convexity of sets and functions has been the main object of studies of recent years, in many new problems encountered in applied mathematics, the notion of classical convexity is not enough to reach favorite results [1–4]. Recently, several extensions have been considered for classical convexity such that some of these new concepts are based on extension of the domain of convex function or set to a generalized form [5–7]. Some new generalized concepts in this point of view are pseudoconvex function, quasi-convex functions, invex functions, preinvex functions, B-convex functions, strongly convex functions, and generalized strongly convex functions. There are several fundamental books devoted to different aspects of convex analysis optimization [8–13]. Among them, we mention convex analysis by Rickafaller [11], convex analysis and minimization algorithm by Hiriart-Urruty and Lemerechel [8], convex analysis and nonlinear optimization by Browan and Lewin [9], and introducing lectures on convex optimization by Nesterov [10].

Several inequalities, including some famous inequalities are of Schur-type, Hermite–Hadamard-type, and Fejér-type inequalities, are satisfied by convex functions. The following inequality is known as Hermite–Hadamard inequality [14–20].

Let $f: I \subset R \longrightarrow R$ be a convex function and let $c, d \in I$ with $c < d$, then the following holds:

$$f\left(\frac{c+d}{2}\right) \leq \frac{1}{d-c}\int_c^d f(x)\mathrm{d}x \leq \frac{f(c)+f(d)}{2}. \tag{2}$$

The present paper is organized as follows: in the first section, we will give some basic definitions and some basic properties related to our work. Next, we will derive Hermite–Hadamard-type, Fejér-type and Schur-type inequalities for our definition.

2. Basic Definitions

Let us recall few definitions related to our work [21–25].

A convex function is called strictly convex if the above inequality holds strictly whenever x and y are distinct points and $t \in (0,1)$.

Definition 1 (see [21]). A function $f: R^n \longrightarrow R$ is said to be quasiconvex if for any $x, y \in R^n$ and $t \in [0,1]$, we have

$$f(x(1-t)+tf(y)) \leq \max(f(x), f(y)). \tag{3}$$

Definition 2. The interval I is said to be p-convex set if

$$\left[tx^p + (1-t)y^p\right]^{1/p} \in I \tag{4}$$

for all $x, y \in I$ and $t \in [0,1]$.

In [26], Zhang et al. introduced p-convexity as follows.

Definition 3. Let I be convex set. A function f is said to be p-convex function if the following inequality holds:

$$f\left[\left(tx^p + \left((1-t)y^p\right)\right)\right]^{1/p} \leq tf(x) + (1-t)f(y), \tag{5}$$

wherever $x, y \in [a,b]$ and $t \in [0,1]$, $p > 0$.

It can be easily seen that, for $p = 1$, p-convexity reduced to classical convexity of functions defined on $I \subset (0,\infty)$.

In [27], Polyak introduced the strongly convex function as follows.

Definition 4. Suppose I is a convex set. A function $f: I \longrightarrow R$ is said to be strongly convex function with modulus $\mu > 0$, if

$$f(tx+(1-t)y) \leq tf(x) + (1-t)f(y) - \mu t(1-t)(y-x)^2 \tag{6}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 5 (see [28]). A function $f: I \longrightarrow R$ is said to be strongly p-convex function, if

$$\begin{aligned} f\left[\left(tx^p + (1-t)y^p\right)\right]^{1/p} \leq{}& tf(x) + (1-t)f(y) \\ &- \mu t(1-t)(y-x)^2 \end{aligned} \tag{7}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 6 (see [29]). A function $f: I \longrightarrow R$ is said to be generalized convex function with respect to $\eta: A \times A \longrightarrow B$ for appropriate $A, B \subseteq R$, if

$$f(tx+(1-t)y) \leq f(y) + t\eta(f(x), f(y)) \tag{8}$$

for all $x, y \in I$ and $t \in [0,1]$.

Definition 7 (see [26]). A function $f: I \longrightarrow R$ is said to be generalized p-convex function, if

$$f\left[\left(tx^p + (1-t)y^p\right)\right]^{1/p} \leq f(y) + t\eta(f(x), f(y)) \tag{9}$$

for all $x, y \in I$ and $t \in [0,1]$.

The generalized strongly p-convex function [30] is defined as follows.

Definition 8. A function f is said to be a generalized strongly p-convex function, if

$$\begin{aligned} f\left(tx^p + (1-t)y^p\right)^{1/p} \leq{}& f(y) + t\eta(f(x), f(y)) \\ &- \mu t(1-t)\left(y^p - x^p\right)^2 \end{aligned} \tag{10}$$

holds for $t \in [0,1]$.

In [31], Lin and Fukushima gave the concept of higher order strongly convex functions and also used it many mathematical programs. Mishra and Sharma [32] derived the Hermite–Hadamard-type inequalities of higher order strongly convex function.

Definition 9 (see [31]). A function f on the convex and closed set I is said to be strongly convex function of higher order if

$$f((1-t)x+ty) \leq (1-t)f(y) + tf(x) - \mu\phi(t)\|y-x\|^q \tag{11}$$

for all $x, y \in I, t \in [0,1]$ with $\mu \geq 0$, where $\phi(t) = t(1-t)$ and q is any positive real number. If $q = 2$, then higher order strongly convex function becomes strongly convex functions with $\phi(t) = t(1-t)$.

Now, in the view of above definitions, we are in the position to introduce new generalization of convexity as follows:

Definition 10. A function f is said to be generalized strongly convex of higher order if

$$f(tx+(1-t)y) \leq f(y) + t\eta(f(x), f(y)) - \mu\phi(t)\|y-x\|^q \tag{12}$$

for $x, y \in I$, with $\mu \geq 0$ and q is any positive real number

Some generalizations of strongly p-convex function of higher order are given in [11] for bifunctions.

Definition 11. A function f is said to be a generalized strongly p-convex of higher order if

$$\begin{aligned} f\left(tx^p + (1-t)y^p\right)^{1/p} \leq{}& f(y) + t\eta(f(x), f(y)) \\ &- \mu\phi(t)\left\|y^p - x^p\right\|^q \end{aligned} \tag{13}$$

for $x, y \in I$, with $\mu \geq 0$.

Remark 1

(1) If we take $q = 2$ and $\phi(t) = t(1-t)$ in (13), then we obtain (10).

(2) If we take $p = 1$, then we get (12).

(3) Inserting $p = 1$ and $\eta(x,y) = x - y$, we obtain (11).

3. Basic Results

This section is to introduce some basic results related to a generalized strongly p-convex function of higher order.

Definition 12. A function η is said to be nonnegatively homogeneous if $n(\lambda x, \lambda y) = \lambda\eta(x, y)$ for all $x, y \in R$ and $\lambda \geq 0$.

Definition 13 (see [33]). A function η is said to be additive if $\eta(a, b) + \eta(c, d) = \eta(a + b, c + d)$ for all $a, b, c, d \in R$.

Proposition 1. *Let f, g: $I \longrightarrow R$ be two generalized strongly p-convex functions of higher order, then the following statements hold:*

(i) If η is additive, then $f + g$: $I \longrightarrow R$ is a generalized strongly p-convex function of higher order.

(ii) If η is nonnegatively homogeneous, then for any $\lambda \geq 0$, λf: $I \longrightarrow R$ is a generalized strongly p-convex function of higher order.

(iii) If $f_1, f_2, f_3, \ldots f_n$ be generalized strongly p-convex function of higher order with $\mu \geq 0$ on I and $\alpha_i \geq 0$, then $f = \alpha_1 f_1 + \alpha_2 f_2 + \cdots + \alpha_n f_n = \sum_{i=1}^{n} \alpha_i f_i$. is also generalized strongly p-convex functions of higher order.

Proof

(i) Take $u = [tx^p + (1-t)y^p]^{1/p}$, then by definition of f and g, we obtain

$$\begin{aligned}
&(f+g)\left(tx^p + (1-t)y^p\right)^{1/p} \\
&\quad = f\left(tx^p + (1-t)y^p\right)^{1/p} + g\left(tx^p + (1-t)y^p\right)^{1/p} \\
&\quad \leq f(y) + t\eta(f(x), f(y)) - \mu\phi(t)\left\|y^p - x^p\right\|^q \\
&\qquad + g(y) + t\eta(g(x), g(y)) - \mu\phi(t)\left\|y^p - x^p\right\|^q \\
&\quad = (f+g)(y) + t\eta((f+g)(x), (f+g)(y)) \\
&\qquad - \mu^*\phi(t)\left\|y^p - x^p\right\|^q,
\end{aligned} \tag{14}$$

where $\mu^* = 2\mu$ and $\mu \geq 0$.

(ii) Let $\lambda \geq 0$, then by definition, we obtain

$$\begin{aligned}
&\lambda f\left[tx^p + (1-t)y^p\right]^{1/p} \\
&\quad \leq \lambda\left[f(y) + t\eta(f(x), f(y)) - \mu\phi(t)\left\|y^p - x^p\right\|^q\right]\lambda f(y) \\
&\qquad + t\eta(\lambda f(x), \lambda f(y)) - \mu^*\phi(t)\left\|y^p - x^p\right\|^q,
\end{aligned} \tag{15}$$

where $\mu^* = \lambda\mu$ and $\mu \geq 0$.

(iii) It directly follows from (i) and (ii). □

Proposition 2. *Let f_i: $R^n \longrightarrow \overline{R}$ for $i \in I$ be collection of generalized strongly p-convex functions of higher order. Then, supremum function*

$$f(x) = \sup f_i(x), \quad i \in I, \tag{16}$$

is also a generalized strongly p-convex function of higher order.

Proof. Fix $x, y \in R^n$ and $t \in (0, 1)$. For every $i \in I$, we have

$$\begin{aligned}
&f_i\left(tx^p + (1-t)y^p\right)^{1/p} \\
&\quad \leq f_i(y) + t\eta\left(f_i(x), f_i(y)\right) - \mu\phi(t)\left\|y^p - x^p\right\|^q \\
&\quad \leq f(y) + t\eta(f(x), f(y)) - \mu\phi(t)\left\|y^p - x^p\right\|^q,
\end{aligned} \tag{17}$$

which implies in turn that

$$\begin{aligned}
&f\left(tx^p + (1-t)y^p\right)^{1/p} \\
&\quad = \sup_{i\in I}\left(f_i\left(tx^p + (1-t)y^p\right)^{1/p}\right) \\
&\quad \leq f(y) + t\eta(f(x), f(y)) - \mu\phi(t)\left\|y^p - x^p\right\|^q.
\end{aligned} \tag{18}$$

This justifies the convexity of supremum function. □

Proposition 3. *Let f_i: $R^n \longrightarrow R$ be a generalized strongly p-convex function of higher order, then*

$$f = \max\{f_i, i = 1, 2, 3, \ldots, n\}, \tag{19}$$

is also a generalized strongly p-convex function.

Proof. Take any $x, y \in R^n$ and $t \in (0, 1)$. Denote $f = \max f_i$, where $i = 1, 2, 3, \ldots, n$, then

$$\begin{aligned}
&f\left[\left(tx^p + (1-t)y^p\right)^{1/p}\right] \\
&\quad = \max\left\{f_i\left[\left(tx^p + (1-t)y^p\right)^{1/p}\right], \quad i = 1, 2, 3, \ldots, n\right\} \\
&\quad = f_w\left[\left(tx^p + (1-t)y^p\right)^{1/p}\right] \\
&\quad \leq f_w(y) + t\eta\left(f_w(x), f_w(y)\right) - \mu\phi(t)\left\|y^p - x^p\right\|^q \\
&\quad = \max\{f_i(y)\} + t\eta\left(\max f_i(x), \max f_i(y)\right) \\
&\qquad - \mu\phi(t)\left\|y^p - x^p\right\|^q \\
&\quad = f(y) + t\eta(f(x), f(y)) - \mu\phi(t)\left\|y^p - x^p\right\|^q,
\end{aligned} \tag{20}$$

implying that $f = \max\{f_i, i = 1, 2, 3, \ldots, n\}$ is also a generalized strongly p-convex function of higher order. □

4. Main Results

In this section, we will develop the Hermite–Hadamard-, Fejér-, and Schur-type inequalities for generalized strongly p-convex function of higher order.

Theorem 1 (Hermite–Hadamard-type inequality). Let f: $I \longrightarrow R$ be a generalized strongly p-convex function of higher order with $\mu \geq 0$ and $p > 0$ such that $\eta(.,.)$ is bounded above M_η, then the function satisfies the following:

$$f\left(\frac{a^p+b^p}{2}\right)^{1/p}-\frac{M_\eta}{2}+\mu\phi\left(\frac{1}{2}\right)\left|b^p-a^p\right|^q\left(\frac{1+(-1)^{q+1}}{2(q+1)}\right)$$
$$\leq\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x$$
$$\leq\frac{f(a)+f(b)}{2}+\frac{M_\eta}{2}-\mu\left\|b^p-a^p\right\|^q\int_0^1\phi(t)\mathrm{d}t. \tag{21}$$

Proof. We begin the proof by proving the left-hand side of theorem.

Let $f: I\longrightarrow R$ be a generalized strongly p-convex function of higher order for all $x, y\in I$, we set $t=1/2$, and we have

$$f\left(\frac{x^p+y^p}{2}\right)^{1/p}\leq f(y)+\frac{1}{2}\eta(f(x),f(y))-\mu\phi\left(\frac{1}{2}\right)\left\|(y^p-x^p)\right\|^q. \tag{22}$$

Integrating above equation with respect to t over [0, 1], we have

$$\int_0^1 f\left(\frac{a^p+b^p}{2}\right)^{1/p}\mathrm{d}t$$
$$\leq\int_0^1 f(y)\mathrm{d}t+\frac{1}{2}\int_0^1\eta(f(x),f(y))\mathrm{d}t \tag{23}$$
$$-\mu\phi\left(\frac{1}{2}\right)\int_0^1\left\|y^p-x^p\right\|^q.$$

Take $x=(ta^p+(1-t)b^p)^{1/p}$ and $y=(tb^p+(1-t)a,^p)^{1/p}$, we obtain

$$\int_0^1 f\left(\frac{a^p+b^p}{2}\right)\mathrm{d}t$$
$$\leq\int_0^1 f\left(tb^p+(1-t)a^p\right)^{1/p}\mathrm{d}t$$
$$+\frac{1}{2}\int_0^1\eta\left(f\left(ta^p+(1-t)b^p\right)\right)^{1/p},f\left(\left(tb^p+(1-t)a^p\right)^{1/p}\right)$$
$$-\mu\phi\left(\frac{1}{2}\right)\int_0^1|2t-1|^q\left|b^p-a^p\right|^q\mathrm{d}t. \tag{24}$$

We obtain

$$f\left(\frac{a^p+b^p}{2}\right)^{1/p}-\frac{M_\eta}{2}+\mu\phi\frac{1}{2}\left|b^p-a^p\right|^q\left(\frac{1+(-1)^{q+1}}{2(q+1)}\right)$$
$$\leq\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x, \tag{25}$$

which is the left-hand side of theorem.

To prove right-hand side, we take $x=(ta^p+(1-t)b^p)^{1/p}$ for all $t\in[0,1]$. By definition, we have

$$f\left(tx^p+(1-t)y^p\right)^{1/p}\leq f(y)+t\eta(f(x),f(y))-\mu\phi(t)\left\|y^p-x^p\right\|^q. \tag{26}$$

Integrating the above inequality with respect to t over [0, 1], we get

$$\int_0^1 f\left(ta^p+(1-t)b^p\right)^{1/p}\mathrm{d}t$$
$$\leq f(b)+t\eta(f(a),f(b))\int_0^1 t\,\mathrm{d}t-\mu\left\|b^p-a^p\right\|^q\int_0^1\phi(t)\mathrm{d}t. \tag{27}$$

After solving this, we get

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x$$
$$\leq f(b)+\frac{\eta}{2}(f(a),f(b))-\mu\left\|b^p-a^p\right\|^q\int_0^1\phi(t)\mathrm{d}t \tag{28}$$
$$=A.$$

Also in the similar way, we obtain

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x$$
$$\leq f(a)+\frac{\eta}{2}(f(b),f(a))-\mu\left\|b^p-a^p\right\|^q\int_0^1\phi(t)\mathrm{d}t \tag{29}$$
$$=B.$$

From (A) and (B), we have

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x\leq\min(A,B). \tag{30}$$

This implies that

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x\leq\frac{A+B}{2}, \tag{31}$$

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)\mathrm{d}x\leq\frac{f(a)+f(b)}{2}$$
$$+\frac{\eta(f(a),f(b))+\eta(f(b),f(a))}{4}$$
$$-\mu\left\|b^p-a^p\right\|^q\int_0^1\phi(t)\mathrm{d}t$$
$$\leq\frac{f(a)+f(b)}{2}+\frac{M_\eta}{2}$$
$$-\mu\left\|b^p-a^p\right\|^q\int_0^1\phi(t)\mathrm{d}t, \tag{32}$$

which is the right-hand side of theorem. This completes the proof. □

Remark 2. Imposing some conditions on Theorem 1, we obtain different versions of Hermite–Hadamard inequality [30, 33]:

(1) If we take $\phi(t) = t(1-t)$, $q = 2$ gives the Hermite–Hadamard inequality for generalized strongly p-convex functions.

(2) If we take $\mu = 0$ and $p = 1$, Theorem 1 gives Hermite–Hadamard inequality for η-convex functions.

(3) If we take $\eta(x, y) = x - y$, then we obtain the Hermite–Hadamard-type inequality of strongly convex function of higher order [34].

Theorem 2 (Schur-type inequality). Let $f: I \longrightarrow R$ be a generalized strongly convex function of higher order with $\mu \geq 0$ and $\eta(.,.)$, be a bifunction, then for all $x_1, x_2, x_3 \in I$ such that $x_1 < x_2 < x_3$ and $x_3^p - x_1^p, x_3^p - x_2^p, x_2^p - x_1^p \in (0,1)$, the following inequality holds:

$$f(x_3)\left(x_3^p - x_1^p\right) - f(x_2)\left(x_3^p - x_1^p\right) + \eta(f(x), f(y)) \cdot \left(x_3^p - x_1^p\right) - \mu\phi\left(x_3^p - x_2^p\right)\left\|x_3^p - x_1\right\|^q \geq 0. \tag{33}$$

Proof. Let f be a generalized strongly p-convex function of higher order with $\mu \geq 0$ and $x_1, x_2, x_3 \in I$, then

$$\frac{x_3^p - x_2^p}{x_3^p - x_1^p}, \frac{x_2^p - x_1^p}{x_3^p - x_1^p} \in (0,1).$$
$$\frac{x_3^p - x_2^p}{x_3^p - x_1^p} + \frac{x_2^p - x_1^p}{x_3^p - x_1^p} = 1. \tag{34}$$

Take $t = ((x_3^p - x_2^p)/(x_3^p - x_1^p)), x = x_1$, and $y = x_3$ in inequality (26), we have

$$x_2^p = tx_1^p + (1-t)x_3^p, \tag{35}$$

$$f(x_2) \leq f(x_3) + \frac{x_3^p - x_2^p}{x_3^p - x_1^p}\eta(f(x_1), f(x_3)) - \mu\phi\left(\frac{x_3^p - x_2^p}{x_3^p - x_1^p}\right)\left\|x_3^p - x_1^p\right\|^q. \tag{36}$$

Multiplying (36) by $(x_3^p - x_1^p)$ on both sides, we get

$$f(x_3)\left(x_3^p - x_1^p\right) - f(x_2)\left(x_3^p - x_1^p\right) + \eta(f(x), f(y)) \cdot \left(x_3^p - x_1^p\right) - \mu\phi\left(x_3^p - x_2^p\right)\left\|x_3^p - x_1\right\|^q \geq 0. \tag{37}$$

□

Remark 3. If we take $q = 2, \phi(t) = t(t-1)$, then we obtain the Schur-type inequality of a generalized strongly p-convex function [30].

Now we will derive the weighted version of Hermite–Hadamard inequality via a generalized strongly convex function of higher order.

Theorem 3 (Fejér-type inequality). Let $f: I \longrightarrow R$ be a generalized strongly function of higher order with $\mu \geq 0$ and provided $\eta(.,.)$ is bounded from above on $f(I) \times f(I)$. Also suppose that w: $[a,b] \longrightarrow R$ is nonnegative, integrable, and symmetric with respect to $[(a^p + b^p)/2]^{1/p}$, then

$$\begin{aligned} f\left(\frac{a^p + b^p}{2}\right)^{1/p} & \int_a^b x^{p-1} w(x)\mathrm{d}x + \mu\phi\left(\frac{1}{2}\right)\int_a^b \\ & \cdot \left\|2x^p - a^p - b^p\right\|^q w(x)x^{p-1}\mathrm{d}x \\ & - \frac{1}{2}\int_a^b \eta\left(f\left(a^p + b^p - x^p\right)^{1/p}, f(x)\right)x^{p-1} f(x)w(x)\mathrm{d}x \\ \leq & \int_a^b x^{p-1} f(x)w(x)\mathrm{d}x \\ \leq & \frac{f(a) + f(b)}{2}\int_a^b w(x)x^{p-1}\mathrm{d}x + \frac{M_\eta}{a^p - b^p} \\ & \cdot \int_a^b (x^p - b^p)w(x)x^{p-1}\mathrm{d}x \\ & - \mu\left\|b^p - a^p\right\|^q \int_a^b \phi\left(\frac{x^p - b^p}{a^p - b^p}\right)w(x)x^{p-1}\mathrm{d}x. \end{aligned} \tag{38}$$

Proof. Let us prove the left-hand side of theorem. Take $p > 0$ and $f: I \longrightarrow R$ be a generalized strongly p-convex function of higher order. We take $t = 1/2$ for all $x, y \in [a,b]$ in (26):

$$f\left(\frac{a^p + b^p}{2}\right)^{1/p} \leq f(y) + \frac{1}{2}\eta(f(x), f(y)) - \mu\phi\left(\frac{1}{2}\right)\left\|y^p - x^p\right\|^q. \tag{39}$$

Take

$$x = \left(ta^p + (1-t)b^p\right)^{1/p},$$
$$y = \left(tb^p + (1-t)a^p\right)^{1/p}.$$

$$\begin{aligned} & f\left(\frac{a^p + b^p}{2}\right)^{1/p} \\ & \leq f\left(tb^p + (1-t)a^p\right)^{1/p} + \frac{1}{2}\eta\left(f\left(ta^p + (1-t)b^p\right)^{1/p}\right), \\ & \cdot f\left(\left(tb^p + (1-t)a^p\right)^{1/p}\right) \\ & - \mu\phi\left(\frac{1}{2}\right)\left\|b^p - a^p\right\|^q \|2t - 1\|^q. \end{aligned} \tag{40}$$

Since w: $[a,b] \longrightarrow R$ be a nonnegative, integrable, and p-symmetric with respect $(a^p + b^p)/2$, then

$$w(x) = w\left(a^p + b^p - x^p\right)^{1/p}, \tag{41}$$

$$w\left(ta^p + (1-t)b^p\right)^{1/p} = w\left(tb^p + (1-t)a^p\right)^{1/p}, \tag{42}$$

for all $x, y \in [a, b]$. Multiplying (40) by $w = (tb^p + (1-t)a^p)^{1/p}$,

$$f\left(\frac{a^p+b^p}{2}\right)^{1/p}\int_0^1 w\left(tb^p+(1-t)a^p\right)^{1/p}\mathrm{d}t$$
$$\leq \int_0^1 f\left(tb^p+(1-t)a^p\right)^{1/p} w\left(tb^p+(1-t)a^p\right)^{1/p}\mathrm{d}t$$
$$+\frac{1}{2}\int_0^1 \eta\left(f\left(ta^p+(1-t)b^p\right), f\left(tb^p+(1-t)a^p\right)^{1/p}\right)$$
$$-\mu\phi\left(\frac{1}{2}\right)\left|b^p-a^p\right|^q\int_0^1 \|2t-1\|^q w\left(tb^p+(1-t)a^p\right)^{1/p}\mathrm{d}t. \tag{43}$$

Choose $x = (tb^p + (1-t)a^p)^{1/p}$ and rearranging the above equation yields that

$$f\left(\frac{a^p+b^p}{2}\right)^{1/p}\int_a^b x^{p-1}w(x)\mathrm{d}x - \frac{1}{2}\int_0^1 \eta$$
$$\cdot\left(f\left(a^p+b^p-x^p\right), f(x)\right)x^{p-1}w(x)\mathrm{d}x$$
$$-\mu\phi\left(\frac{1}{2}\right)\int_a^b \left\|2x^p-a^p-b^p\right\|^q w(x)x^{p-1}\mathrm{d}x \tag{44}$$
$$\leq \int_a^b x^{p-1}f(x)w(x)\mathrm{d}x.$$

For the right-hand side of theorem, we choose $x = (ta^p + (1-t)b^p)^{1/p}$:

$$f\left(ta^p+(1-t)b^p\right)^{1/p} \leq f(b) + t\eta(f(x), f(y)) - \mu\phi(t)\left\|y^p-x^p\right\|^q. \tag{45}$$

Multiplying on both sides by $w(tb^p + (1-t)a^p)^{1/p}$ and integrating w.r.t t over [0, 1], we get

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)w(x)\mathrm{d}x$$
$$\leq f(b)\int_a^b w(x)x^{p-1}\mathrm{d}x + \eta(f(a), f(b))\int_a^b \frac{1}{a^p-b^p}$$
$$\cdot\int_a^b \left(x^p-b^p\right)w(x)x^{p-1}\mathrm{d}x$$
$$-\mu\left\|b^p-a^p\right\|^q\int_a^b \phi\left(\frac{x^p-b^p}{a^p-b^p}\right)w(x)x^{p-1}\mathrm{d}x. \tag{46}$$

Similarly, we get

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)w(x)\mathrm{d}x$$
$$\leq f(a)\int_a^b w(x)x^{p-1}\mathrm{d}x + \eta(f(b), f(a))\int_a^b \frac{1}{a^p-b^p}$$
$$\cdot\int_a^b \left(x^p-b^p\right)w(x)x^{p-1}\mathrm{d}x$$
$$-\mu\left|b^p-a^p\right|^q\int_a^b \phi\left(\frac{x^p-b^p}{a^p-b^p}\right)w(x)x^{p-1}\mathrm{d}x. \tag{47}$$

By adding (46) and (47), we get

$$\frac{p}{b^p-a^p}\int_a^b x^{p-1}f(x)w(x)\mathrm{d}x$$
$$\leq \frac{f(a)+f(b)}{2}\int_a^b w(x)x^{p-1}\mathrm{d}x$$
$$+\frac{\eta(f(b), f(a)) + \eta(f(a), f(b))}{2\left(a^p-b^p\right)}\int_a^b \left(x^p-b^p\right)w(x)x^{p-1}\mathrm{d}x$$
$$-\mu\left\|b^p-a^p\right\|^q\int_a^b \phi\left(\frac{x^p-b^p}{a^p-b^p}\right)w(x)x^{p-1}\mathrm{d}x. \tag{48}$$

We get

$$f\left(\frac{a^p+b^p}{2}\right)^{1/p}\int_a^b x^{p-1}w(x)\mathrm{d}x + \mu\phi\left(\frac{1}{2}\right)$$
$$\cdot\int_a^b \left\|2x^p-a^p-b^p\right\|^q w(x)x^{p-1}\mathrm{d}x$$
$$-\frac{1}{2}\int_a^b \eta\left(f\left(a^p+b^p-x^p\right)^{1/p}, f(x)\right)x^{p-1}f(x)w(x)\mathrm{d}x$$
$$\leq \int_a^b x^{p-1}f(x)w(x)\mathrm{d}x$$
$$\leq \frac{f(a)+f(b)}{2}\int_a^b w(x)x^{p-1}\mathrm{d}x + \frac{M_\eta}{a^p-b^p}\int_a^b$$
$$\cdot\left(x^p-b^p\right)w(x)x^{p-1}\mathrm{d}x$$
$$-\mu\left\|b^p-a^p\right\|^q\int_a^b \phi\left(\frac{x^p-b^p}{a^p-b^p}\right)w(x)x^{p-1}\mathrm{d}x. \tag{49}$$

This completes the proof. □

Remark 4. Imposing some conditions on Theorem 3, we get different versions of Hermite–Fejér-type inequality:

(1) If we put $q = 2$ and $\phi(t) = t(1-t)$, then we obtain the Fejér-type inequality of a generalized strongly p-convex function [30].

(2) For $w(x) = 1$ in the Theorem 3, we obtain Theorem 1.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors' Contributions

Chahn Yong Jung improved the presentation of the paper. Muhammad Shoaib Saleem supervised this work. Yu-Ming Chu revised the paper and arranged funding. Nazia Jahangir proved the main results. Huma Akhtar wrote the first draft of the paper.

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