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Research Article

Picture Fuzzy Rough Set and Rough Picture Fuzzy Set on Two Different Universes and Their Applications

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The major concern of this article is to propose the notion of picture fuzzy rough sets (PFRSs) over two different universes which depend on $(\delta, \zeta, \vartheta)$-cut of picture fuzzy relation $\mathscr{R}$ on two different universes (i.e., by combining picture fuzzy sets (PFSs) with rough sets (RSs)). Then, we discuss several interesting properties and related results on the PFRSs. Furthermore, we define some notions related to PFRSs such as (Type-I/Type-II) graded PFRSs, the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ on PFRSs, and (Type-I/Type-II) generalized PFRSs based on the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ and investigate the basic properties of above notions. Finally, an approach based on the rough picture fuzzy approximation operators on two different universes in decision-making problem is introduced, and we give an example to show the validity of this approach.

1. Introduction

In the past few years, Pawlak [1] proposed the notion of RS as a mathematical tool to handle with ambiguity and incomplete information systems. The lower/upper approximations (i.e., rough sets) are firstly described through the equivalence classes. That is to say, many datasets cannot be treated properly by way of classical rough sets. In mild of this, the graded rough sets [2], similarity or tolerance relations [3–5], arbitrary binary relation [6, 7], and variable precision rough sets [8, 9] are a few extensions of the classical rough sets. So, several researchers, for example, Dubois and Prade [10], presented the concept of fuzzy rough set (FRS) (i.e., the fuzzy set (FS) [11] and the RS). Many researchers have worked on fuzzy rough models (see [12–16]).

Wong et al. [17] presented the notion of the RS model over two universes and its application. Several applications and the fundamental properties of the FRS model on two universes are studied [18–32]. Yao and Lin [33, 34] proposed the notion of graded rough sets (GRSs) on one universe. Zhang et al. [35] gave a comparative between the variable precision rough set (VPRS) and the GRS. In addition to the previous studies, Liu et al. [36] introduced the notion of GRSs on two universes. Yu et al. [37] presented the notion of a variable precision-graded rough set (VPGRSs) over two universes and Yu and Wang [38] presented a novel type of GRS with VP over two inconsistent universes.

In this paper, we propose the notions of PFRSs and RPFSs over two different universes. The basic properties of PFRSs based on $(\delta, \zeta, \vartheta)$-cut of picture fuzzy relation $\mathscr{R}$ over two different universes are discussed. Meanwhile, we propose two types of graded picture fuzzy rough sets (GPFRSs) based on $(\delta, \zeta, \vartheta)$-cut of $\mathscr{R}$ on two different universes: type-I PFRS is according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ and type-II PFRS is according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$. The interesting properties of Type-I/Type-II PFRSs are investigated in detail. Furthermore, we define the notions of PFRS according to the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ and Type-I/Type-II generalized PFRs according to the degree α and β with respect to $\mathscr{R}_{i[(\delta,\zeta,\vartheta)]}$. The main results of the above notions are studied and explored. Finally, an application of RPFs model over two different universes is presented to solve the decision-making problem.

Sections of this article are arranged as follows. In Section 2, we gave the concepts of PFSs and picture fuzzy relations. In Section 3, we give the notion of PFRSs based on

$(\delta, \zeta, \vartheta)$-cut of picture fuzzy relation $\mathscr{R}$ over two different universes and study some interesting properties on PFRSs. In Section 4, an algorithm is constructed and an application on RPFSs over two different universes in decision-making problem is explored. Lastly, conclusion is discussed in Section 5.

2. Preliminaries

2.1. Picture Fuzzy Sets and Picture Fuzzy Relations. Cuong [39–41] introduced the notion of PFS is an extension of fuzzy FS [42] and intuitionistic fuzzy set IFS [43]. Later on, many researchers defined some notions related to PFSs (e.g., [44–49]) and solved some problems related to PFSs (e.g., [50–57]).

Definition 1 (cf. see [39–41]). Let $U = \{u_1, u_2, \ldots, u_n\}$ be an n-element set (n is a natural number), and a PFS $\mathscr{A} \in \mathbb{I}^U$ is

$$\mathscr{A} = \left\{\frac{(p_1 \circ \mathscr{A}(u_1), p_2 \circ \mathscr{A}(u_1), p_3 \circ \mathscr{A}(u_1))}{u_1}, \frac{(p_1 \circ \mathscr{A}(u_2), p_2 \circ \mathscr{A}(u_2), p_3 \circ \mathscr{A}(u_2))}{u_2}, \ldots, \frac{(p_1 \circ \mathscr{A}(u_n), p_2 \circ \mathscr{A}(u_n), p_3 \circ \mathscr{A}(u_n))}{u_n}\right\}. \tag{1}$$

Moreover, $1 - p_1 \circ \mathscr{A}(u) - p_2 \circ \mathscr{A}(u) - p_3 \circ \mathscr{A}(u)$ is called the refusal degree of u ($u \in U$). A PFS $\mathscr{A} \in \mathbb{I}^U$ with refusal degree 0 at each point $u \in U$ can be identified with an IFS on U and $\mathbb{I}^U$ with the pointwise order $\leq$ is the set of all mappings from a set U (or an universe) to $\mathbb{I} = \{(a_1, a_2, a_3) \in [0,1]^3 | a_1 + a_2 + a_3 \leq 1\}$. Then, each element $\mathscr{A}$ of $\mathbb{I}^U$ is called an $\mathbb{I}$-set or a PFS on U, $p_1 \circ \mathscr{A}(u)$ (i.e., the degree of positive), $p_2 \circ \mathscr{A}(u)$ (i.e., the degree of neutral), and $p_3 \circ \mathscr{A}(u)$ (i.e., the degree of negative) of the element $u \in U$, where p_i: $[0,1]^3 \longrightarrow [0,1]$ (i.e., the ith projection from $[0,1]^3$ to $[0,1]$ ($i = 1,2,3$)).

Definition 2 (cf. see [39–41]). Let $\mathscr{A}, \mathscr{B} \in \mathbb{I}^U$. Then,

(1) The complement $\mathscr{A}^c$ of $\mathscr{A} \in \mathbb{I}^U$ is defined by

$$\mathscr{A}^c = \left\{\frac{(p_3 \circ \mathscr{A}(u_1), p_2 \circ \mathscr{A}(u_1), p_1 \circ \mathscr{A}(u_1))}{u_1}, \frac{(p_3 \circ \mathscr{A}(u_2), p_2 \circ \mathscr{A}(u_2), p_1 \circ \mathscr{A}(u_2))}{u_2}, \ldots, \frac{(p_3 \circ \mathscr{A}(u_n), p_2 \circ \mathscr{A}(u_n), p_1 \circ \mathscr{A}(u_n))}{u_n}\right\}. \tag{2}$$

(2) The union $\cup_{k\in K}\mathscr{A}_k$ (called also supremum $\vee_{k\in K}\mathscr{A}_k$) and the intersection $\cap_{k\in K}\mathscr{A}_k$ (called also infimum $\wedge_{k\in K}\mathscr{A}_k$) of a family $\{\mathscr{A}_k\}_{k\in K} \subseteq \mathbb{I}^U$ can be defined by the following formulae:

$$\begin{aligned}
\bigcup_{k\in K}\mathscr{A}_k(u) &= \left(\bigvee_{k\in K} p_1 \circ \mathscr{A}_k(u), \bigwedge_{k\in K} p_2 \circ \mathscr{A}_k(u), \bigwedge_{k\in K} p_3 \circ \mathscr{A}_k(u)\right),\\
\bigcap_{k\in K}\mathscr{A}_k(u) &= \left(\bigwedge_{k\in K} p_1 \circ \mathscr{A}_k(u), \bigwedge_{k\in K} p_2 \circ \mathscr{A}_k(u), \bigvee_{k\in K} p_3 \circ \mathscr{A}_k(u)\right).
\end{aligned} \tag{3}$$

(3) $\mathscr{A}$ is a subset of $\mathscr{B}$ if $\mathscr{A} \leq \mathscr{B}$ (i.e., $p_1 \circ \mathscr{A}(u) \leq p_1 \circ \mathscr{B}(u)$, $p_2 \circ \mathscr{A}(u) \leq p_2 \circ \mathscr{B}(u)$, and $p_3 \circ \mathscr{A}(u) \geq p_3 \circ \mathscr{B}(u)$, for each $x \in X$).

(4) $\mathscr{A}$ is an equal of $\mathscr{B}$ if $\mathscr{A} \leq \mathscr{B}$ (i.e., $p_1 \circ \mathscr{A}(u) \leq p_1 \circ \mathscr{B}(u)$, $p_2 \circ \mathscr{A}(u) \leq p_2 \circ \mathscr{B}(u)$, and $p_3 \circ \mathscr{A}(u) \geq p_3 \circ \mathscr{B}(u)$) and $\mathscr{A} \geq \mathscr{B}$ (i.e., $p_1 \circ \mathscr{A}(u) \geq p_1 \circ \mathscr{B}(u)$, $p_2 \circ \mathscr{A}(u) \geq p_2 \circ \mathscr{B}(u)$, and $p_3 \circ \mathscr{A}(u) \geq p_3 \circ \mathscr{B}(u)$).

Definition 3 (cf. see [39–41, 57]). Let $\mathscr{R} = (p_1 \circ \mathscr{R}, p_2 \circ \mathscr{R}, p_3 \circ \mathscr{R})$ be a picture fuzzy relation, denoted by $\mathbb{I}^{U\times V}$, where $p_1 \circ \mathscr{R} \in [0,1]^{U\times V}$, $p_2 \circ \mathscr{R} \in [0,1]^{U\times V}$, and $p_3 \circ \mathscr{R} \in [0,1]^{U\times V}$ satisfy $0 \leq p_1 \circ \mathscr{R}(u,v) + p_2 \circ \mathscr{R}(u,v) + p_3 \circ \mathscr{R}(u,v) \leq 1$ for all $(u,v) \in U \times V$ and p_1: $\mathscr{R}^2 \longrightarrow \mathscr{R}$, p_2: $\mathscr{R}^2 \longrightarrow \mathscr{R}$, and $\mathscr{R}^3 \longrightarrow \mathscr{R}$ are first projection, second projection, and third projection, respectively.

Definition 4 (cf. see [39–41, 57]). For two picture fuzzy relations $\mathscr{R} \in \mathbb{I}^{U\times V}$ and $\mathscr{H} \in \mathbb{I}^{V\times W}$, the picture fuzzy relation $\mathscr{H} \circ \mathscr{R} \in \mathbb{I}^{U\times W}$, defined by 1

$$(\mathscr{H} \circ \mathscr{R})(u,w) = \bigvee_{v\in V}[\mathscr{R}(u,v) \wedge \mathscr{H}(v,w)], \quad \forall (u,w) \in U \times W, \tag{4}$$

is said to be a composition of $\mathscr{R}$ and $\mathscr{H}$.

3. Picture Fuzzy Rough Sets over Two Different Universes

3.1. Picture Fuzzy Rough Sets Based on $(\delta, \zeta, \vartheta)$-Cut of $\mathscr{R}$ on Two Different Universes. We will begin by defining the $(\delta, \zeta, \vartheta)$-cut of $\mathscr{R}$ and will subsequently define a picture fuzzy rough set based on $(\delta, \zeta, \vartheta)$-cut.

Definition 5. Let $\mathscr{R} \in \mathbb{I}^{U\times V}$ and $(\delta, \zeta, \vartheta) \in \mathbb{I}$. Then,

(i) $\mathscr{R}_{[(\delta,\zeta,\vartheta)]} = \{(u,v) \in U \times V | \mathscr{R}(u,v) \geq (\delta, \zeta, \vartheta)\} = \{(u,v) \in U \times V | p_1 \circ \mathscr{R}(u,v) \geq \delta, p_2 \circ \mathscr{R}(u,v) \leq \zeta, p_3 \circ \mathscr{R}(u,v) \leq \vartheta\}$ is called the $(\delta, \zeta, \vartheta)$-cut of $\mathscr{R}$, and

$$\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) = \{v \in V | \mathscr{R}(u,v) \geq (\delta, \zeta, \vartheta)\}. \tag{5}$$

(ii) Let $\mathscr{A} \in 2^V$. Then,

$$\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) = \left\{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \wedge \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing\right\}, \tag{6}$$

(i.e., the lower approximation of $\mathscr{A}$) and

$$\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) = \left\{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing \vee \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) = \varnothing\right\}, \tag{7}$$

(i.e., the upper approximation of $\mathscr{A}$).

The pair $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}), \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}))$ is a PFR approximation of $\mathscr{A}$ with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$.

Now, we present some properties based on PFRS as follows.

Theorem 1. *Let $\mathscr{R} \in \mathbb{I}^{U\times V}$, $(\delta, \zeta, \vartheta) \in \mathbb{I}$, and $\mathscr{A} \in 2^V$. Then, the following holds:*

(1) $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\varnothing) = \varnothing$; $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(V) = U$

(2) $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$

(3) If $\mathscr{A} \subseteq \mathscr{B}$, then $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \subseteq \mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{B})$; $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{B})$

(4) $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A} \cap \mathscr{B}) = \mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \cap \mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{B})$; $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A} \cup \mathscr{B}) = \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \cup \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{B})$

(5) $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A} \cup \mathscr{B}) \supseteq \mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \cup \mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{B})$; $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A} \cap \mathscr{B}) \subseteq \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \cap \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{B})$

(6) $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) = (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}^c))^c$; $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) = (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}^c))^c$

Proof. We only prove (6).

$(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}^c))^c = (\{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}^c \neq \varnothing \vee \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) = \varnothing\})^c = \{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}^c = \varnothing \wedge \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing\} = \{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \wedge \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing\} = \mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$, and $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}^c))^c = (\{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A}^c \wedge \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing\})^c = (\{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} = \varnothing \wedge \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing\})^c = \{u \in U | \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing \vee \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) = \varnothing\} = \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$.

The equality of Theorem 1 (5) does not hold as the following example.

Example 1. Suppose $U = \{x_i | i = 1, 2, 3\}$ and $V = \{y_i | i = 1, 2, 3\}$ be two three-element set, and $\mathscr{R} \in \mathbb{I}^{U\times V}$ is defined by

$$\mathscr{R} = \left\{\frac{(0.2,0.5,0.3)}{(x_1,y_1)}, \frac{(0.6,0.3,0)}{(x_1,y_2)}, \frac{(0.4,0.3,0.1)}{(x_1,y_3)}, \frac{(0.4,0.2,0.3)}{(x_2,y_1)}, \frac{(0.3,0.3,0.2)}{(x_2,y_2)}, \frac{(0.1,0.6,0.2)}{(x_2,y_3)}, \frac{(0.5,0.2,0.1)}{(x_3,y_1)}, \frac{(0.5,0.5,0)}{(x_3,y_2)}, \frac{(0.6,0.1,0.1)}{(x_3,y_3)}\right\}. \tag{8}$$

Take $(\delta, \zeta, \vartheta) = (0.5, 0.3, 0.1)$. Then, $\mathscr{R}_{[(0.5,0.3,0.1)]}(x_1) = \{y_2\}$, $\mathscr{R}_{[(0.5,0.3,0.1)]}(x_2) = \varnothing$, and $\mathscr{R}_{[(0.5,0.3,0.1)]}(x_3) = \{y_1, y_3\}$.

Let $\mathscr{A} = \{y_1\}$ and $\mathscr{B} = \{y_2, y_3\}$. Thus, $\mathscr{R}^-_{[(0.5,0.3,0.1)]}(\mathscr{A}) = \varnothing$, $\mathscr{R}^-_{[(0.5,0.3,0.1)]}(\mathscr{B}) = \{x_1\}$, $\mathscr{R}^-_{[(0.5,0.3,0.1)]}(\mathscr{A} \cup \mathscr{B}) = \{x_1, x_3\} \neq \{x_1\} = \mathscr{R}^-_{[(0.5,0.3,0.1)]}(\mathscr{A}) \cup \mathscr{R}^-_{[(0.5,0.3,0.1)]}(\mathscr{B})$ and $\mathscr{R}^+_{[(0.5,0.3,0.1)]}(\mathscr{A}) = \{x_2, x_3\}$, $\mathscr{R}^+_{[(0.5,0.3,0.1)]}(\mathscr{B}) = U$, $\mathscr{R}^+_{[(0.5,0.3,0.1)]}(\mathscr{A} \cap \mathscr{B}) = \{x_2\} \neq \{x_2, x_3\} = \mathscr{R}^+_{[(0.5,0.3,0.1)]}(\mathscr{A}) \cap \mathscr{R}^+_{[(0.5,0.3,0.1)]}(\mathscr{B})$.

Remark 1. In general, $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\varnothing) = \varnothing$ and $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(V) = U$ do not hold. For example, let $U = \{x_i | i = 1, 2\}$ and $V = \{y_i | i = 1, 2\}$ be two two-element sets, and $\mathscr{R} \in \mathbb{I}^{U\times V}$ is defined by

$$\mathscr{R} = \left\{\frac{(0.2,0.4,0.4)}{(x_1,y_1)}, \frac{(0.4,0.2,0.3)}{(x_1,y_2)}, \frac{(0.5,0.3,0.2)}{(x_2,y_1)}, \frac{(0.7,0.1,0.2)}{(x_2,y_2)}\right\}. \tag{9}$$

Take $(\delta, \zeta, \vartheta) = (0.5, 0.1, 0.2)$. Then, $\mathscr{R}_{[(0.5,0.1,0.2)]}(x_1) = \varnothing$ and $\mathscr{R}_{[(0.5,0.1,0.2)]}(x_2) = \{y_2\}$. Thus, $\mathscr{R}^+_{[(0.5,0.1,0.2)]}(\varnothing) = \{x_1\} \neq \varnothing$ and $\mathscr{R}^-_{[(0.5,0.1,0.2)]}(V) = \{x_2\} \neq U$.

Remark 2. It should be pointed that some conclusions in [58–60] which are similar to those in Theorem 1 are wrong. Firstly, the equality $\mathscr{R}^-_{[(\delta,\zeta)]}(\varnothing) = \varnothing$ and $\mathscr{R}^+_{[(\delta,\zeta)]}(V) = U$ in Theorem 3.1(2) in [60] are incorrect. Let $U = \{x_i | i = 1, 2\}$, $V = \{y_i | i = 1, 2\}$, and $\mathscr{R} \in \mathbb{J}^{U\times V}$ be defined by

$$\mathscr{R} = \left\{\frac{(0.5,0.2)}{(x_1,y_1)}, \frac{(0.7,0.1)}{(x_1,y_2)}, \frac{(0.5,0.3)}{(x_2,y_1)}, \frac{(0.3,0.4)}{(x_2,y_2)}\right\}. \tag{10}$$

Take $(\delta, \zeta) = (0.5, 0.1)$. Then, $\mathscr{R}^-_{[(0.5,0.1)]}(\varnothing)$ and $\mathscr{R}^+_{[(0.5,0.1)]}(V)$ in the sense of [60] are not $\varnothing$ and U, but $\{x_2\}$ and $\{x_1\}$, respectively. Secondly, the inclusion $\mathscr{R}^-_{[\delta]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[\delta]}(\mathscr{A})$ in Theorem 3.1(1) in [58] is incorrect. Let $U =$

$\{x_i|i = 1,2,3\}$ and $V = \{y_i|i = 1,2,3\}$ be two three-element sets, and $\mathscr{R} \in [0,1]^{U\times V}$, is defined by

$$\mathscr{R} = \left\{\frac{0.5}{(x_1,y_1)},\frac{0.7}{(x_1,y_2)},\frac{0.4}{(x_1,y_3)},\frac{0.5}{(x_2,y_1)},\frac{0.3}{(x_2,y_2)},\frac{0.4}{(x_2,y_3)},\frac{0.8}{(x_3,y_1)},\frac{0.1}{(x_3,y_2)},\frac{0.6}{(x_3,y_3)}\right\}. \tag{11}$$

Take $\delta = 0.6$ and $\mathscr{A} = \{y_1,y_3\}$. Then, $\mathscr{R}_{[0.6]}(x_1) = \{y_2\}$, $\mathscr{R}_{[0.6]}(x_2) = \varnothing$, and $\mathscr{R}_{[0.6]}(x_3) = \{y_1,y_3\}$, and thus $\mathscr{R}^-_{[0.6]}(\mathscr{A})$ (resp., $\mathscr{R}^+_{[0.6]}(\mathscr{A})$) in the sense of [58] is $\{x_2,x_3\}$ (resp., $\{x_3\}$). Therefore, $\mathscr{R}^-_{[\delta]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[\delta]}(\mathscr{A})$. Analogously, assertions $\mathscr{R}^-_{[(\delta,\zeta)]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[(\delta,\zeta)]}(\mathscr{A})$ in Theorem 3.1(1) [59] and $\mathscr{R}^-_{[(\delta,\zeta)]}(\mathscr{A}) \subseteq \mathscr{A} \subseteq \mathscr{R}^+_{[(\delta,\zeta)]}(\mathscr{A})$ in Theorem 3.1(1) [60] are incorrect. Let $U = \{x_i|i = 1,2,3\}$ and $V = \{y_i|i = 1,2,3\}$ be two three-element sets, and $\mathscr{R} \in ([0,1]^2)^{U\times V}$ and also $\mathscr{R} \in \mathbb{J}^{U\times V}$ is defined by

$$\mathscr{R} = \left\{\frac{(0.2,0.5)}{(x_1,y_1)},\frac{(0.6,0.3)}{(x_1,y_2)},\frac{(0.4,0.3)}{(x_1,y_3)},\frac{(0.4,0.2)}{(x_2,y_1)},\frac{(0.3,0.3)}{(x_2,y_2)},\frac{(0.1,0.6)}{(x_2,y_3)},\frac{(0.5,0.2)}{(x_3,y_1)},\frac{(0.5,0.5)}{(x_3,y_2)},\frac{(0.6,0.1)}{(x_3,y_3)}\right\}. \tag{12}$$

Take $(\delta,\zeta) = (0.5,0.3)$, and $\mathscr{A} = \{y_1,y_3\}$. Then, $\mathscr{R}_{[(0.5,0.3)]}(x_1) = \{y_2\}$, $\mathscr{R}_{[(0.5,0.3)]}(x_2) = \varnothing$, and $\mathscr{R}_{[(0.5,0.3)]}(x_3) = \{y_1,y_3\}$, and thus $\mathscr{R}^-_{[(0.5,0.3)]}(\mathscr{A}) = \{x_2,x_3\} \subseteq \{x_3\} = \mathscr{R}^+_{[(0.5,0.3)]}(\mathscr{A})$ and $\mathscr{R}^-_{[(0.5,0.3)]}(\mathscr{A}) = \{x_2,x_3\} \subseteq \mathscr{A} \subseteq \{x_3\} = \mathscr{R}^+_{[(0.5,0.3)]}(\mathscr{A})$.

To correct some results in [58–60] in above Remark 2, we will give new notations of lower and upper approximations as follow:

Definition 6. Let $\mathscr{A} \in 2^V$. Then,

(1) For $\mathscr{R} \in [0,1]^{U\times V}$ and $\delta \in [0,1]$, we have

$$\mathscr{R}^-_{[\delta]}(\mathscr{A}) = \left\{u \in U|\mathscr{R}_{[\delta]}(u) \subseteq \mathscr{A} \wedge \mathscr{R}_{[\delta]}(u) \neq \varnothing\right\},$$
$$\mathscr{R}^+_{[\delta]}(\mathscr{A}) = \left\{u \in U|\mathscr{R}_{[\delta]}(u) \cap \mathscr{A} \neq \varnothing \vee \mathscr{R}_{[\delta]}(u) = \varnothing\right\}. \tag{13}$$

(2) For $\mathscr{R} \in ([0,1]^2)^{U\times V}$ and $(\delta,\zeta) \in [0,1]^2$, we have

$$\mathscr{R}^-_{[(\delta,\zeta)]}(\mathscr{A}) = \left\{u \in U|\mathscr{R}_{[(\delta,\zeta)]}(u) \subseteq \mathscr{A} \wedge \mathscr{R}_{[(\delta,\zeta)]}(u) \neq \varnothing\right\},$$
$$\mathscr{R}^+_{[(\delta,\zeta)]}(\mathscr{A}) = \left\{u \in U|\mathscr{R}_{[(\delta,\zeta)]}(u) \cap \mathscr{A} \neq \varnothing \vee \mathscr{R}_{[(\delta,\zeta)]}(u) = \varnothing\right\}. \tag{14}$$

(3) For $\mathscr{R} \in \mathbb{J}^{U\times V}$ and $(\delta,\zeta) \in \mathbb{J}$, we have (cf. [61])

$$\mathscr{R}^-_{[(\delta,\zeta)]}(\mathscr{A}) = \left\{u \in U|\mathscr{R}_{[(\delta,\zeta)]}(u) \subseteq \mathscr{A} \wedge \mathscr{R}_{[(\delta,\zeta)]}(u) \neq \varnothing\right\},$$
$$\mathscr{R}^+_{[(\delta,\zeta)]}(\mathscr{A}) = \left\{u \in U|\mathscr{R}_{[(\delta,\zeta)]}(u) \cap \mathscr{A} \neq \varnothing \vee \mathscr{R}_{[(\delta,\zeta)]}(u) = \varnothing\right\}. \tag{15}$$

Theorem 2. *Let $\mathscr{R},\mathscr{S} \in \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, and $\mathscr{A} \in 2^V$. If $\mathscr{R} \leq \mathscr{S}$, then $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \subseteq \mathscr{S}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$ and $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \subseteq \mathscr{S}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$.*

Proof. If $\mathscr{R} \leq \mathscr{S}$, then $p_1 \circ \mathscr{R}(u,v) \leq p_1 \circ \mathscr{S}(u,v)$, $p_2 \circ \mathscr{R}(u,v) \leq p_2 \circ \mathscr{S}(u,v)$ and $p_3 \circ \mathscr{R}(u,v) \geq p_3 \circ \mathscr{S}(u,v)$ for all $(u,v) \in U \times V$. By Definition 5, we have $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) = \{v \in V|\mathscr{R}(u,v) \geq (\delta,\zeta,\vartheta)\}$ $\subseteq \{v \in V|\mathscr{S}(u,v) \geq (\delta,\zeta,\vartheta)\} = \mathscr{S}_{[(\delta,\zeta,\vartheta)]}(u)$. Therefore, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \wedge \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing \subseteq \mathscr{S}_{[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \wedge \mathscr{S}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$ for all $u \in U$. Thus, $\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \subseteq \mathscr{S}^-_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$. Similarly, $\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$.

Corollary 1. *(1) Let $\mathscr{R},\mathscr{S} \in [0,1]^{U\times V}$, $\delta \in [0,1]$, and $\mathscr{A} \in 2^V$. If $\mathscr{R} \leq \mathscr{S}$, then $\mathscr{R}^-_{[\delta]}(\mathscr{A}) \subseteq \mathscr{S}^-_{[\delta]}(\mathscr{A})$ and $\mathscr{R}^+_{[\delta]}(\mathscr{A}) \subseteq \mathscr{S}^+_{[\delta]}(\mathscr{A})$.*

(2) Let $\mathscr{R},\mathscr{S} \in ([0,1]^2)^{U\times V}$, $(\delta,\zeta) \in [0,1]^2$, and $\mathscr{A} \in 2^V$. If $\mathscr{R} \leq \mathscr{S}$, then $\mathscr{R}^-_{[(\delta,\zeta)]}(\mathscr{A}) \subseteq \mathscr{S}^-_{[(\delta,\zeta)]}(\mathscr{A})$ and $\mathscr{R}^+_{[(\delta,\zeta)]}(\mathscr{A}) \subseteq \mathscr{S}^+_{[(\delta,\zeta)]}(\mathscr{A})$.

(3) Let $\mathscr{R},\mathscr{S} \in \mathbb{J}^{U\times V}$, $(\delta,\zeta) \in \mathbb{J}$, and $\mathscr{A} \in 2^V$. If $\mathscr{R} \leq \mathscr{S}$, then $\mathscr{R}^-_{[(\delta,\zeta)]}(\mathscr{A}) \subseteq \mathscr{S}^-_{[(\delta,\zeta)]}(\mathscr{A})$ and $\mathscr{R}^+_{[(\delta,\zeta)]}(\mathscr{A}) \subseteq \mathscr{S}^+_{[(\delta,\zeta)]}(\mathscr{A})$.

Proof. It follows from Definition 6.

Next, we give comparison between some properties by Definition 3.3 in [58], Definition 3.2 in [59], Definition 3.3 in [60] and 6 as shown in Tables 1–3:

Remark 3. Assertions $\mathscr{R}^-_{[\delta_1]}(\mathscr{A}) \subseteq \mathscr{R}^-_{[\delta_2]}(\mathscr{A})$ (if $\delta_1 \leq \delta_2$) and $\mathscr{R}^+_{[\delta_2]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[\delta_1]}(\mathscr{A})$ (if $\delta_1 \leq \delta_2$) do not hold by Definition 6 (1), let $U = \{x_i|i = 1,2\}$ and $V = \{y_i|i = 1,2\}$ be two two-element sets, and $\mathscr{R} \in [0,1]^{U\times V}$ is defined by

$$\mathscr{R} = \left\{\frac{0.6}{(x_1,y_1)},\frac{0.4}{(x_1,y_2)},\frac{1}{(x_2,y_1)},\frac{0.6}{(x_2,y_2)}\right\}. \tag{16}$$

Take $\delta_1 = 0.55 \leq 0.7 = \delta_2$ and $\mathscr{A} = \{y_1\}$. Then, $\mathscr{R}_{[0.55]}(x_1) = \{y_1\}$, $\mathscr{R}_{[0.55]}(x_2) = V$, $\mathscr{R}_{[0.7]}(x_1) = \varnothing$, and $\mathscr{R}_{[0.7]}(x_2) = \{y_1\}$, and thus $\mathscr{R}^-_{[0.55]}(\mathscr{A}) = \{x_1\} \neq \{x_2\} = \mathscr{R}^-_{[0.7]}(\mathscr{A})$. Also, if we take $\mathscr{A} = \{y_2\}$, $\mathscr{R}^+_{[0.55]}(\mathscr{A}) = \{x_2\} \neq \{x_1\} = \mathscr{R}^+_{[0.7]}(\mathscr{A})$.

Remark 4. Assertions $\mathscr{R}^-_{[(\delta_1,\zeta_1)]}(\mathscr{A}) \subseteq \mathscr{R}^-_{[(\delta_2,\zeta_2)]}(\mathscr{A})$ (if $\delta_1 \leq \delta_2$, $\zeta_1 \leq \zeta_2$) and $\mathscr{R}^+_{[(\delta_2,\zeta_2)]}(\mathscr{A}) \subseteq \mathscr{R}^+_{[(\delta_1,\zeta_1)]}(\mathscr{A})$ (if $\delta_1 \leq \delta_2$, $\zeta_1 \leq \zeta_2$) in Tables 2 and 3 do not hold by Definition 6 (2) and (3), and let $U = \{x_i|i = 1,2\}$ and $V = \{y_i|i = 1,2\}$ be two two-element sets, and $\mathscr{R} \in ([0,1]^2)^{U\times V}$ and also $\mathscr{R} \in \mathbb{J}^{U\times V}$ is defined by

Table 1: Comparison between some properties by Definition 3.3 in [58] and 6 (1).

	Definition 3.3 in [58]	Definition 6 (1)
$\mathscr{R}^{-}_{[\delta]}(\varnothing)=\varnothing$	×	√
$\mathscr{R}^{+}_{[\delta]}(\varnothing)=\varnothing$	√	×
$\mathscr{R}^{-}_{[\delta]}(V)=U$	√	×
$\mathscr{R}^{+}_{[\delta]}(V)=U$	×	√
$\mathscr{R}^{-}_{[\delta]}(\mathscr{A})\subseteq\mathscr{R}^{+}_{[\delta]}(\mathscr{A})$	×	√
$\mathscr{R}^{-}_{[\delta]}(\mathscr{A})\subseteq\mathscr{S}^{-}_{[\delta]}(\mathscr{A})$ (if $\mathscr{R}\leq\mathscr{S}$)	×	√
$\mathscr{R}^{+}_{[\delta]}(\mathscr{A})\subseteq\mathscr{S}^{+}_{[\delta]}(\mathscr{A})$ (if $\mathscr{R}\leq\mathscr{S}$)	×	√
$\mathscr{R}^{-}_{[\delta_1]}(\mathscr{A})\subseteq\mathscr{R}^{-}_{[\delta_2]}(\mathscr{A})$ (if $\delta_1\leq\delta_2$)	√	×
$\mathscr{R}^{+}_{[\delta_2]}(\mathscr{A})\subseteq\mathscr{R}^{+}_{[\delta_1]}(\mathscr{A})$ (if $\delta_1\leq\delta_2$)	√	×

(√) indicates that the property is satisfied.

Table 2: Comparison between some properties by Definition 3.2 in [59] and 6 (2).

	Definition 3.2 in [59]	Definition 6 (2)
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(\varnothing)=\varnothing$	×	√
$\mathscr{R}^{+}_{[(\delta,\zeta)]}(\varnothing)=\varnothing$	√	×
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(V)=U$	√	×
$\mathscr{R}^{+}_{[(\delta,\zeta)]}(V)=U$	×	√
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(\mathscr{A})\subseteq\mathscr{R}^{+}_{[(\delta,\zeta)]}(\mathscr{A})$	×	√
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(\mathscr{A})\subseteq\mathscr{S}^{-}_{[(\delta,\zeta)]}(\mathscr{A})$ (if $\mathscr{R}\leq\mathscr{S}$)	×	√
$\mathscr{R}^{+}_{[(\delta,\zeta)]}(\mathscr{A})\subseteq\mathscr{S}^{+}_{[(\delta,\zeta)]}(\mathscr{A})$ (if $\mathscr{R}\leq\mathscr{S}$)	×	√
$\mathscr{R}^{-}_{[(\delta_1,\zeta_1)]}(\mathscr{A})\subseteq\mathscr{R}^{-}_{[(\delta_2,\zeta_2)]}(\mathscr{A})$ (if $\delta_1\leq\delta_2,\zeta_1\leq\zeta_2$)	√	×
$\mathscr{R}^{+}_{[(\delta_2,\zeta_2)]}(\mathscr{A})\subseteq\mathscr{R}^{+}_{[(\delta_1,\zeta_1)]}(\mathscr{A})$ (if $\delta_1\leq\delta_2,\zeta_1\leq\zeta_2$)	√	×

(√) indicates that the property is satisfied.

Table 3: Comparison between some properties by Definition 3.3 in [60] and 6 (3).

	Definition 3.3 in [60]	Definition 6 (2)
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(\varnothing)=\varnothing$	×	√
$\mathscr{R}^{+}_{[(\delta,\zeta)]}(\varnothing)=\varnothing$	√	×
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(V)=U$	√	×
$\mathscr{R}^{+}_{[(\delta,\zeta)]}(V)=U$	×	√
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(\mathscr{A})\subseteq\mathscr{R}^{+}_{[(\delta,\zeta)]}(\mathscr{A})$	×	√
$\mathscr{R}^{-}_{[(\delta,\zeta)]}(\mathscr{A})\subseteq\mathscr{S}^{-}_{[(\delta,\zeta)]}(\mathscr{A})$ (if $\mathscr{R}\leq\mathscr{S}$)	×	√
$\mathscr{R}^{+}_{[(\delta,\zeta)]}(\mathscr{A})\subseteq\mathscr{S}^{+}_{[(\delta,\zeta)]}(\mathscr{A})$ (if $\mathscr{R}\leq\mathscr{S}$)	×	√
$\mathscr{R}^{-}_{[(\delta_1,\zeta_1)]}(\mathscr{A})\subseteq\mathscr{R}^{-}_{[(\delta_2,\zeta_2)]}(\mathscr{A})$ (if $\delta_1\leq\delta_2,\zeta_1\leq\zeta_2$)	√	×
$\mathscr{R}^{+}_{[(\delta_2,\zeta_2)]}(\mathscr{A})\subseteq\mathscr{R}^{+}_{[(\delta_1,\zeta_1)]}(\mathscr{A})$ (if $\delta_1\leq\delta_2,\zeta_1\leq\zeta_2$)	√	×

(√) indicates that the property is satisfied.

$$\mathscr{R}=\left\{\frac{0.6}{(x_1,y_1)},\frac{0.4}{(x_1,y_2)},\frac{1}{(x_2,y_1)},\frac{0.6}{(x_2,y_2)}\right\}. \tag{17}$$

Take $\delta_1=0.55\leq 0.7=\delta_2$, and $\mathscr{A}=\{y_1\}$. Then, $\mathscr{R}_{[0.55]}(x_1)=\{y_1\}$, $\mathscr{R}_{[0.55]}(x_2)=V$, $\mathscr{R}_{[0.7]}(x_1)=\varnothing$, and $\mathscr{R}_{[0.7]}(x_2)=\{y_1\}$, and thus $\mathscr{R}^{-}_{[0.55]}(\mathscr{A})=\{x_1\}\neq\{x_2\}=\mathscr{R}^{-}_{[0.7]}(\mathscr{A})$ Also, if we take $\mathscr{A}=\{y_2\}$, thus $\mathscr{R}^{+}_{[0.55]}(\mathscr{A})=\{x_2\}\neq\{x_1\}=\mathscr{R}^{+}_{[0.7]}(\mathscr{A})$

3.2. Graded Picture Fuzzy Rough Sets Based on $\mathscr{R}$ on Two Different Universes

Definition 7. Let $\mathscr{R}\in\mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta)\in\mathbb{I}$, $n\in N$, and $\mathscr{A}\in 2^V$. Then,

$$\begin{aligned}\left(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]}\right)^n(\mathscr{A})&=\left\{u\in U\,\middle|\,\left|\mathscr{R}_{(\delta,\zeta,\vartheta)}(u)-\mathscr{A}\right|\leq n\wedge\mathscr{R}_{(\delta,\zeta,\vartheta)}(u)\neq\varnothing\right\},\\ \left(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]}\right)^n(\mathscr{A})&=\left\{u\in U\,\middle|\,\left|\mathscr{R}_{(\delta,\zeta,\vartheta)}(u)\cap\mathscr{A}\right|>n\vee\mathscr{R}_{(\delta,\zeta,\vartheta)}(u)=\varnothing\right\},\end{aligned} \tag{18}$$

are called the Type-I lower approximation and the Type-I upper approximation of $\mathscr{A}$ according to the graded n with respect to $\mathscr{R}_{(\delta,\zeta,\vartheta)}$ on U and V, respectively, and $((\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}),(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}))$ is called the Type-I picture fuzzy rough approximation of $\mathscr{A}$ according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ (briefly, a Type-I picture fuzzy rough set according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$).

Theorem 3. *Let $\mathscr{R}\in\mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta)\in\mathbb{I}$, $n\in N$, and $\mathscr{A}\in 2^V$. Then,* *(1) If $\mathscr{A} \subseteq \mathscr{B}$, then $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$ and $(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$*

(2) $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \cap (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$

(3) $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A} \cup \mathscr{B}) \supseteq (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \cup (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$; $(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \cap (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$

(4) $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) = ((\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}^c))^c$; $(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) = ((\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}^c))^c$

(5) If $m \leq n$, then $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^m(\mathscr{A}) \subseteq (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ and $(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^m(\mathscr{A})$

Proof. (1) By Definition 7, we have $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) = \{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}| \leq n \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\}$. Since $\mathscr{A} \subseteq \mathscr{B}$, then we have $\{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}| \leq n \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\} \subseteq \{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{B}| \leq n \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\} = (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$. Therefore, $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^k(\mathscr{B})$. Similarly, $(\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$.

(2) and (3) Clear.

(4) $((\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}^c))^c = (\{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \cap \mathscr{A}^c| > n \vee \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) = \varnothing\})^c = (\{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}| > n \vee \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) = \varnothing\})^c = \{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}| \leq n \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\} = (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ and $((\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}^c))^c = (\{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}^c| \leq n \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\})^c = (\{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \cap \mathscr{A}| \leq n \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\})^c = \{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \cap \mathscr{A}| > n \vee \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) = \varnothing\} = (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$.

(5) By Definition 7, we have $(\mathscr{R}_{[(\delta,\zeta,\ \vartheta)]^{-I}})^m(\mathscr{A}) = \{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}| \leq m \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\}$. If $m \leq n$, then we have $\{u \in U|\ |\mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}| \leq m \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\} \subseteq \{u \in U \| \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) - \mathscr{A}| \leq n \wedge \mathscr{R}_{(\delta,\zeta,\vartheta)}(u) \neq \varnothing\} = (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$. Hence, $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^m(\mathscr{A}) \subseteq (\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$. Similarly, $(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^m(\mathscr{A})$.

Remark 5. The equality $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\varnothing) = (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\varnothing)\backslash\backslash = \varnothing$ and $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(V) = (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(V) = U$ does not hold. For example, let $U = \{x_i | i = 1,2\}$ and $V = \{y_i | i = 1,2\}$ be two two-element sets, and $\mathscr{R} \in \mathbb{I}^{U \times V}$ is defined by

$$\mathscr{R} = \left\{ \frac{(0.2,0.6,0.2)}{(x_1,y_1)}, \frac{(0.4,0.2,0.3)}{(x_1,y_2)}, \frac{(0.5,0.2,0.3)}{(x_2,y_1)}, \frac{(0.6,0.1,0.2)}{(x_2,y_2)} \right\}. \tag{19}$$

Take $(\delta,\zeta,\vartheta) = (0.5,0.1,0.2)$ and $n = 2$. Then, $\mathscr{R}_{[(0.5,0.1,0.2)]}(x_1) = \varnothing$, $\mathscr{R}_{[(0.5,0.1,0.2)]}(x_2) = \{y_2\}$. Thus, $(\mathscr{R}^{-I}_{[(0.5,0.1,0.2)]})^2(\varnothing) = \{x_2\} \neq \varnothing$, $(\mathscr{R}^{+I}_{[(0.5,0.1,0.2)]})^2(\varnothing) = \{x_1\} \neq \varnothing$, $(\mathscr{R}^{-I}_{[(0.5,0.1,0.2)]})^2(V) = \{x_2\} \neq U$, and $(\mathscr{R}^{+I}_{[(0.5,0.1,0.2)]})^2(V) = \{x_1\} \neq U$.

Remark 6. Let $\mathscr{R} \in \mathbb{I}^{U \times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $n \in N$, and $\mathscr{A} \in 2^V$. Then, there does not exist inclusion relation between $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ and $(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$. Consider $U, V, (\delta,\zeta,\vartheta)$ and $\mathscr{R} \in \mathbb{I}^{U \times V}$ are given in Remark 1. Let $\mathscr{A} = \{y_1\}$ and $n = 1$. Then, $(\mathscr{R}^{-I}_{[(0.5,0.1,0.2)]})^1(\mathscr{A}) = \{x_2\}$ and $(\mathscr{R}^{+I}_{[(0.5,0.1,0.2)]})^1(\mathscr{A}) = \{x_1\}$. This show that there does not exist inclusion relation between $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ and $(\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$.

We study the notion of Type-II PFRS according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ satisfy $(\mathscr{R}^{-I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+I}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$.

Definition 8. Let $\mathscr{R} \in \mathbb{I}^{U \times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $n \in N$ s.t. $n \in [0, \min((|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2), |\mathscr{A}|))$ and $\mathscr{A} \in 2^V$. Then,

$$\begin{aligned} \left(\mathscr{R}^{-\mathrm{II}}_{[(\delta,\zeta,\vartheta)]}\right)^n(\mathscr{A}) &= \left\{u \in U \| \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}| \leq n\right\}, \\ \left(\mathscr{R}^{+\mathrm{II}}_{[(\delta,\zeta,\vartheta)]}\right)^n(\mathscr{A}) &= \left\{u \in U \| \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| > n\right\}, \end{aligned} \tag{20}$$

are called the Type-II lower approximation and the Type-II upper approximation of $\mathscr{A}$ according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ on U and V, respectively, and $((\mathscr{R}^{-\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}), (\mathscr{R}^{+\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}))$ is called the Type-II PFR approximation of $\mathscr{A}$ according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ (briefly, a Type-II PFRS according to the graded n with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$). The subsets $(\mathscr{R}^{-\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$, $U - (\mathscr{R}^{+\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$, and $(\mathscr{R}^{+\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ - $(\mathscr{R}^{-\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$-$(\mathscr{R}^{-\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ are called the $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$-positive region ($\mathscr{R}_{[(\delta,\zeta,\vartheta)]} -$ POS), the $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$-negative region ($\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ −NEG), and the $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$-boundary region ($\mathscr{R}_{[(\delta,\zeta,\vartheta)]} -$ BND) of $\mathscr{A}$ with respect to graded n, respectively.

Lemma 1. *Let $\mathscr{R} \in \mathbb{I}^{U \times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $n \in N$, and $\mathscr{A} \in 2^V$. Then, the following holds:*

(1) $n \in [0, \min((|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2), |\mathscr{A}|))$

(2) The limits of the grade $n = \max[0, \min((|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2), |\mathscr{A}|))$

Proof. (1) For any $u \in U$, $(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}) \cup (\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}) = \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)$ Then, $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}| + |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| = |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|$ implies that $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| = |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| - |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}|$. If $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| > n$, then $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| - |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}| > n$, and if $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}| \leq n$, then $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| - n > n$. Consequently, $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| > 2n$ and thus $n < (|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2)$. Since $n < |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| \leq \min(|\mathscr{A}|, |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)$ and $n < (|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2)$, then $n < \min((|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2), |\mathscr{A}|)$. Thus, $n \in (0, \min((|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2), |\mathscr{A}|))$.

(2) Follows from (1) above.

Theorem 4. *Let $\mathscr{R} \in \mathbb{I}^{U \times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $n \in N$ s.t. $n \in [0, \min((|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2), |\mathscr{A}|))$ and $\mathscr{A} \in 2^V$. Then, the following holds:*

(1) $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(V) = (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(V) = U$; $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\varnothing) = (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\varnothing) = 0$

(2) $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$

(3) If $\mathscr{A} \subseteq \mathscr{B}$, then $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$ and $(\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$

(4) $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \cap (\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$ and $(\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A} \cup \mathscr{B}) \subseteq (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \cap (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$

(5) $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A} \cup \mathscr{B}) \supseteq (\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \cup (\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$ and $(\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \cap (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{B})$

(6) $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) = ((\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}^c))^c$ *and* $(\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) = ((\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}^c))^c$

(7) *If* $m \le n$, *then* $(\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^m(\mathscr{A}) \subseteq (\mathscr{R}^{-II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ *and* $(\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+II}_{[(\delta,\zeta,\vartheta)]})^m(\mathscr{A})$.

Proof. We only prove (2). Let $\exists u \in U$ and $u \in (\mathscr{R}^{-\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$, but $u \notin (\mathscr{R}^{+\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$. From Definition 8, $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}| \le n$, $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| > n$, and for any $u \in U$, $(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}) \cup (\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}) = \mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)$ Since $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}| + |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| = |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|$ and $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) - \mathscr{A}| = |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| - |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}|$, then $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| - |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| \le n$ and thus $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| - n \le n$. Consequently, $|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)| > 2n$ and thus $n < (|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|/2)$. It is contradiction with the limits of n, so the element which only belongs to $(\mathscr{R}^{-\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ but does not belong to $(\mathscr{R}^{+\mathrm{II}}_{[(\delta,\zeta,\vartheta)]})^n(\mathscr{A})$ does not exist.

Remark 7. (1) The equality $(\mathscr{R}^{-}_{[(\delta,\zeta)]})^n(\varnothing) = \varnothing$ and $(\mathscr{R}^{+}_{[(\delta,\zeta)]})^n(V) = U$ in Theorem 3.10 (2) in [60] are incorrect. (2) For example, let $U = \{x_i | i = 1, 2\}$ and $V = \{y_i | i = 1, 2\}$ be two two-element sets, and $\mathscr{R} \in \mathbb{J}^{U\times V}$ is defined by

$$\mathscr{R} = \left\{\frac{(0.2, 0.6)}{(x_1, y_1)}, \frac{(0.4, 0.2)}{(x_1, y_2)}, \frac{(0.5, 0.2)}{(x_2, y_1)}, \frac{(0.8, 0.1)}{(x_2, y_2)}\right\}. \tag{21}$$

Take $(\delta, \zeta) = (0.5, 0.1)$ and $n = 2$. Then, $\mathscr{R}_{[(0.5,0.1)]}(x_1) = \varnothing$, $\mathscr{R}_{[(0.5,0.1)]}(x_2) = \{y_2\}$. Thus, $(\mathscr{R}^{-}_{[(0.5,0.1)]})^2(\varnothing) = U \neq \varnothing$ and $(\mathscr{R}^{+}_{[(0.5,0.1)]})^2(V) = \varnothing \neq U$.

(2) The inclusions in $(\mathscr{R}^{-}_{[(\delta,\zeta)]})^n(\mathscr{A}) \subseteq (\mathscr{R}^{+}_{[(\delta,\zeta)]})^n(\mathscr{A})$ and $(\mathscr{R}^{-}_{[(\delta,\zeta)]})^n(\mathscr{A}) \subseteq \mathscr{A} \subseteq (\mathscr{R}^{-}_{[(\delta,\zeta)]})^n(\mathscr{A})$ in Theorem 3.10(1) in [60] are incorrect. For example, let $U = \{x_i | i = 1, 2, 3\}$ and $V = \{y_i | i = 1, 2, 3\}$ be two three-element sets, and $\mathscr{R} \in \mathbb{J}^{U\times V}$ is defined by

$$\mathscr{R} = \left\{\frac{(0.5, 0.2)}{(x_1, y_1)}, \frac{(0.7, 0.1)}{(x_1, y_2)}, \frac{(0.4, 0.3)}{(x_1, y_3)}, \frac{(0.5, 0.3)}{(x_2, y_1)}, \frac{(0.3, 0.4)}{(x_2, y_2)}, \frac{(0.4, 0.5)}{(x_2, y_3)}, \frac{(0.8, 0.1)}{(x_3, y_1)}, \frac{(0.1, 0.7)}{(x_3, y_2)}, \frac{(0.6, 0.2)}{(x_3, y_3)}\right\}. \tag{22}$$

Take $(\delta, \zeta) = (0.6, 0.2)$, $n = 1$ and $\mathscr{A} = \{y_1, y_3\}$. Then, $\mathscr{R}_{[(0.6,0.2)]}(x_1) = \{y_2\}$, $\mathscr{R}_{[(0.6,0.2)]}(x_2) = \varnothing$, and $\mathscr{R}_{[(0.6,0.2)]}(x_3) = \{y_1, y_3\}$. Thus, $(\mathscr{R}^{-}_{[(0.6,0.2)]})^1(\mathscr{A}) = U \subseteq \varnothing = (\mathscr{R}^{+}_{[(0.6,0.2)]})^1(\mathscr{A})$ and $(\mathscr{R}^{-}_{[(0.6,0.2)]})^1(\mathscr{A}) = U \subseteq \mathscr{A} \subseteq \varnothing = (\mathscr{R}^{+}_{[(0.6,0.2)]})^1(\mathscr{A})$.

(3) The equalities $(\mathscr{R}^{-}_{[(\delta,\zeta)]})^n(\mathscr{A} \cap \mathscr{B}) = (\mathscr{R}^{-}_{[(\delta,\zeta)]})^n(\mathscr{A}) \cap (\mathscr{R}^{-}_{[(\delta,\zeta)]})^n(\mathscr{B})$ and $(\mathscr{R}^{+}_{[(\delta,\zeta)]})^n(\mathscr{A} \cup \mathscr{B}) = (\mathscr{R}^{+}_{[(\delta,\zeta)]})^n(\mathscr{A}) \cup (\mathscr{R}^{+}_{[(\delta,\zeta)]})^n(\mathscr{B})$ in Theorem 3.10(3) in [60] are incorrect. Consider above example in (2), let $\mathscr{A} = \{y_1\}$ and $\mathscr{B} = \{y_2, y_3\}$. Then, $(\mathscr{R}^{-}_{[(0.6,0.2)]})^1(\mathscr{A}) = (\mathscr{R}^{-}_{[(0.6,0.2)]})^1(\mathscr{B}) = U$ and $(\mathscr{R}^{+}_{[(0.6,0.2)]})^1(\mathscr{A}) = (\mathscr{R}^{+}_{[(0.6,0.2)]})^1(\mathscr{B}) = \varnothing$. Thus, $(\mathscr{R}^{-}_{[(0.6,0.2)]})^1(\mathscr{A} \cap \mathscr{B}) = \{x_1, x_2\} \subsetneq U = (\mathscr{R}^{-}_{[(0.6,0.2)]})^1(\mathscr{A}) \cap (\mathscr{R}^{-}_{[(0.6,0.2)]})^1(\mathscr{B})$ and $(\mathscr{R}^{+}_{[(0.6,0.2)]})^1(\mathscr{A} \cup \mathscr{B}) = \{x_3\} \supsetneq \varnothing = (\mathscr{R}^{+}_{[(0.6,0.2)]})^1(\mathscr{A}) \cup (\mathscr{R}^{+}_{[(0.6,0.2)]})^1(\mathscr{B})$.

Definition 9. Let $\mathscr{R} \in \mathbb{I}^{U\times V}$, $(\delta, \zeta, \vartheta) \in \mathbb{I}$, $0 \le \beta < \alpha \le 1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$, and $\mathscr{A} \in 2^V$. Then,

$$\begin{aligned}\left(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]}\right)^{\alpha}(\mathscr{A}) &= \left\{u \in U \,|\, \frac{|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}|}{|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|} \ge \alpha\right\},\\ \left(\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]}\right)^{\beta}(\mathscr{A}) &= \left\{u \in U \,|\, \frac{|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}|}{|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|} > \beta\right\},\end{aligned} \tag{23}$$

are called the lower approximation and the upper approximation of $\mathscr{A}$ according to the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ on U and V, respectively, and $((\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A}), (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A}))$ is called the picture fuzzy rough approximation of $\mathscr{A}$ according to the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ (briefly, a picture fuzzy rough set according to the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$), where $|\mathscr{A}|$ denotes the cardinality of the set $\mathscr{A}$. The subset $bn^{\alpha\beta}_{\mathscr{R}_{[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | \beta < (|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|) < \alpha\}$ is called the boundary region of $\mathscr{A}$ according to the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$.

The main results are as follows.

Theorem 5. *Let* $\mathscr{R} \in \mathbb{I}^{U\times V}$, $(\delta, \zeta, \vartheta) \in \mathbb{I}$, $0 \le \beta < \alpha \le 1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$, *and* $\mathscr{A}, \mathscr{B} \in 2^V$. *Then, the following holds:*

(1) $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\varnothing) = (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\varnothing) = \varnothing$; $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(V) = (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(V) = U$

(2) $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A}) \subseteq (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A})$

(3) $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A}) = ((\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{1-\alpha}(\mathscr{A})^c)^c$; $(\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A}) = ((\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{1-\beta}(\mathscr{A})^c)^c$,

(4) *If* $\mathscr{A} \subseteq \mathscr{B}$, *then* $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A}) \subseteq (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{B})$ *and* $(\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A}) \subseteq (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{B})$

(5) $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A}) \cap (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{B})$ *and* $(\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A}) \cap (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{B})$

(6) $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A} \cup \mathscr{B}) \supseteq (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A}) \cup (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{B})$; $(\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A} \cup \mathscr{B}) \supseteq (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A}) \cup (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{B})$

(7) $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha_1}(\mathscr{A}) \subseteq (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A})$ (*if* $0.5 < \alpha < \alpha_1 \le 1$); $(\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta_1}(\mathscr{A}) \subseteq (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\mathscr{A})$ (*if* $0 \le \beta < \beta_1 < 0.5$)

Proof. (1) For any $u \in U, 0 \le \beta < \alpha \le 1$, we have $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \varnothing| / |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|) = 0 \le \beta < \alpha$, $u \notin (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\varnothing)$, and $u \notin (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\varnothing)$. Therefore, $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\varnothing) = (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(\varnothing) = \varnothing$. Also, $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap V| / |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|) = 1 \ge \beta > \alpha$, $u \in (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(V)$, and $u \in (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(V)$. Thus, $(\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(V) = (\mathscr{R}^{+}_{[(\delta,\zeta,\vartheta)]})^{\beta}(V) = U$.

(2)–(4) It follows from Definition 9.

(5) Let $u \in (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A} \cap \mathscr{B})$. Then, $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap (\mathscr{A} \cap \mathscr{B})| / |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|) \ge \alpha \Longrightarrow (|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|) \ge \alpha$, $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{B}| / |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|) \ge \alpha \Longrightarrow u \in (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{A})$, and $u \in (\mathscr{R}^{-}_{[(\delta,\zeta,\vartheta)]})^{\alpha}(\mathscr{B}) \Longrightarrow u \in$

$(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})\cap(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})$. Similarily, we can obtain $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}\cap\mathscr{B})\subseteq(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})\cap(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})$.

(6) It is analogous to (5).

(7) For $0.5<\alpha<\alpha_1\le 1$, let $u\in(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha_1}(\mathscr{A})$. Then, $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)\ge\alpha_1$. Since $\alpha<\alpha_1$, then $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)\ge\alpha_1>\alpha\Longrightarrow(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)\ge\alpha$. Therefore, $u\in(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})$ Thus $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha_1}(\mathscr{A})\subseteq(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})$ Similarly, $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta_1}(\mathscr{A})\subseteq(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})$ (if $0\le\beta<\beta_1<0.5$).

Equality (5) and (6) of Theorem 5 does not hold as the following example.

Example 2 (continuation of Example 1). Let $\alpha=0.4,\beta=0.3$, $\mathscr{A}=\{y_1\}$, and $\mathscr{B}=\{y_2,y_3\}$. Then,

(1) $(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{A}\cap\mathscr{B})=(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{A}\cap\mathscr{B})=\varnothing$, $(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{A})=(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{A})=\{x_3\}$, $(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{B})=(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{B})=\{x_1,x_3\}$ and thus $(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{A}\cap\mathscr{B})=\varnothing\neq\{x_3\}=(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{A})\cap(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{B})$, $(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{A}\cap\mathscr{B})=\varnothing\neq\{x_3\}=(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{A})\cap(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{B})$

(2) $(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{A}\cup\mathscr{B})=(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{A}\cup\mathscr{B})=U$, and thus $(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{A}\cup\mathscr{B})=U\neq\{x_1,x_3\}=(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{A})\cup(\mathscr{R}^-_{[(0.5,0.3,0.1)]})^{0.4}(\mathscr{B})$, $(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{A}\cup\mathscr{B})=U\neq\{x_1,x_3\}=(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{A})\cup(\mathscr{R}^+_{[(0.5,0.3,0.1)]})^{0.3}(\mathscr{B})$

The relationship among the regulation for lower approximation, upper approximation, and boundary region with error α and β is discussed as the following theorems.

Theorem 6. *Let $\mathscr{R}\in\mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta)\in\mathbb{I}$, $0<r<1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\neq\varnothing$, and $\mathscr{A}\in 2^V$. Then, the following holds:*

(1) $\lim_{\alpha\longrightarrow r^+}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=\cup_{\alpha>r}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$

(2) $\lim_{\beta\longrightarrow r^-}(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})=\cap_{\beta<r}(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})=(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$

(3) $\lim_{\beta\longrightarrow r^-;\alpha\longrightarrow r^+}bn^{\alpha\beta}_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})=\cap_{\beta<r<\alpha}((\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})\setminus(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}))=(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})\setminus(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})=\{u\in U|(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)=r\}$

Proof. (1) From Definition 9, when $\alpha>r$, we have $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=\{u\in U|\ (|\mathscr{R}_{[(\delta,\ \zeta,\vartheta)]}(u)\cap\mathscr{A}|/\ |\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)\ge\alpha\}\subseteq\{u\in U|\ (|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)>r\}=(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$ is hold. In addition, by Theorem 5 (7), we have $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})$ increase with error α decrease. Thus, $\lim_{\alpha\longrightarrow r^+}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=\cap_{\alpha>r}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})\subseteq(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$. Conversely, if there exists $u_0\in(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})\setminus\cup_{\alpha>r}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})$, for any $\alpha>r$, $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)>r$ and $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)<\alpha$. That is, $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)<r$. It contradicts with $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)>r$, i.e., $u_0\in\cup_{\alpha>r}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})$. Consequently, $\cup_{\alpha>r}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$. Thus, $\lim_{\alpha\longrightarrow r^+}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=\cup_{\alpha>r}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$ hold.

(2) It is analogous to (1) above.

(3) We know $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})\setminus(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})=\{u\in U|(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)=r\}\subseteq bn^{\alpha\beta}_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$ is held. In addition, we know that when α decreased to r and β decreased to r, the boundary region $bn^{\alpha\beta}_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})$ will decrease. Then, we have $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})\setminus(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})=\{u\in U|\ (|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)=r\}\subseteq\lim_{\beta\longrightarrow r^-;\alpha\longrightarrow r^+}bn^{\alpha\beta}_{[(\delta,\zeta,\vartheta)]}(\mathscr{A})=\cap_{\beta<r<\alpha}((\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})\setminus(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}))$. Conversely, if there exists $u_0\in\cap_{\beta<r<\alpha}((\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})\setminus(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}))\setminus\{u\in U|\ (|\mathscr{R}_{(\delta,\zeta,\vartheta)}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)=r\}$, then for any $\beta<r<\alpha$, we have $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)>\beta$, $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)<\alpha$ and $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)\neq r$. Furthermore, there is $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)\ge r$ by $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)>\beta$ and $\beta<r$. Similarily, there is $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)\le r$ by $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)<\alpha$ and $\alpha>r$. This is a contradiction. Consequently, there is $(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u_0)|)=r$. That is, $\{u\in U|(|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)|)=r\}=\cap_{\beta<r<\alpha}((\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})\setminus(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}))$.

Theorem 7. *Let $\mathscr{R}\in\mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta)\in\mathbb{I}$, $0<r<1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\neq\varnothing$, and $\mathscr{A}\in 2^V$. Then, the following holds:*

(1) $\lim_{\alpha\longrightarrow r^-}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=\cap_{\alpha<r}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$

(2) $\lim_{\beta\longrightarrow r^+}(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})=\cup_{\beta>r}(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})=(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$

Proof. Obvious.

Remark 8. Let $\mathscr{R}\in\mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta)\in\mathbb{I}$, $0<r<1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\neq\varnothing$, and $\mathscr{A}\in 2^V$. Then, the following does not hold:

(1) $\lim_{\alpha\longrightarrow r^+}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$ (i.e., the lower approximation $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})$ is not right continuous with error α)

(2) $\lim_{\beta\longrightarrow r^-}(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})=(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^r(\mathscr{A})$ (i.e., the upper approximation $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})$ is not left continuous with error β)

Example 3 (continuation of Example 1). Let $\mathscr{A}=\{y_2,y_3\}$. Then, we can obtain the following relations:

(1) $\lim_{\alpha\longrightarrow 0.5^+}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})=\{x_1\}$, $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{0.5}(\mathscr{A})=\{x_1,x_3\}$. Consequently, $\lim_{\alpha\longrightarrow 0.5^-}(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})\neq(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{0.5}(\mathscr{A})$.

(2) $\lim_{\beta\longrightarrow 0.5^-}(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})=\{x_1,x_3\}$, $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{0.5}(\mathscr{A})=\{x_1\}$. Consequently, $\lim_{\beta\longrightarrow 0.5^-}(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})\neq(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{0.5}(\mathscr{A})$.

Definition 10. Let $\mathscr{R}\in\mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta)\in\mathbb{I}$, $0\le\beta<\alpha\le 1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u)\neq\varnothing$, and $\mathscr{A}\in 2^V$. Then, the precision $(\rho_{[(\delta,\zeta,\vartheta)]})^\alpha_\beta(\mathscr{A})$ and rough degree $(\mu_{[(\delta,\zeta,\vartheta)]})^\alpha_\beta(\mathscr{A})$ of $\mathscr{A}$ with error α and β are defined by

$$\left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_\beta^\alpha(\mathscr{A}) = 1 - \frac{\left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{A})\right|}{\left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\right|},$$
$$\left(\mu_{[(\delta,\zeta,\vartheta)]}\right)_\beta^\alpha(\mathscr{A}) = 1 - \left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_\beta^\alpha(\mathscr{A}). \tag{24}$$

Obviously, $0 \le (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) \le 1$ and $0 \le (\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) \le 1$.

Lemma 2. *Let $\mathscr{R} \in \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $0 \le \beta < \alpha \le 1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \ne \emptyset$, and $\mathscr{A} \in 2^V$. Then, the following holds:*

(1) $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})$ does not decrease with error α and does not increase with error β

(2) $(\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})$ does not increase with error α and does not decrease with error β

Proof. We only prove (1) and then the proof of (2) can be obtained using similar techniques.

(1) Since $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha_1}(\mathscr{A}) \subseteq (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})$ $(0.5 < \alpha < \alpha_1 \le 1)$ by Theorem 5 (7), then there is $(|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha_1}(\mathscr{A})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})|) \le (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})|)$ implies that $1 - (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha_1}(\mathscr{A})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})|) \ge 1 - (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})|)$. Therefore, $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) \le (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^{\alpha_1}(\mathscr{A})$. Similarly, $(\rho_{[(\delta,\zeta,\vartheta)]})_{\beta_1}^\alpha(\mathscr{A}) \le (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})$ for $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta_1}(\mathscr{A}) \subseteq (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})$ $(0 \le \beta < \beta_1 < 0.5)$.

Theorem 8. *Let $\mathscr{R} \in \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $0 \le \beta < \alpha \le 1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \ne \emptyset$, $\mathscr{A}, \mathscr{B} \in 2^V$, and $\mathscr{A} \subseteq \mathscr{B}$. Then, the following holds:*

(1) If $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}) = (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})$, there is $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) \le (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B})$ and $(\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B}) \le (\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})$

(2) If $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) = (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})$, there is $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B}) \le (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})$ and $(\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) \le (\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B})$

(3) If $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}) = (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})$ and $(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) = (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})$, there is $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) \le (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B})$ and $(\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) = (\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B})$

Proof. It follows from Theorem 5 (4) and Definition 10.

Theorem 9. *Let $\mathscr{R} \in \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $0 \le \beta < \alpha \le 1$, $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}(u) \ne \emptyset$, and $\mathscr{A}, \mathscr{B} \in 2^V$. Then, the following holds:*

(1) $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}\cup\mathscr{B})\,|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| \le (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})| + (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| - (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}\cap\mathscr{B})\ |(\mathscr{R}_{[(\delta,\zeta,\vartheta)]^+})^\beta(\mathscr{A}) \cap (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})|$

(2) $(\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}\cup\mathscr{B})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| \ge (\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})| + (\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| - (\mu_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}\cap\mathscr{B})|(\mathscr{R}_{[(\delta,\zeta,\vartheta)]^+})^\beta(\mathscr{A}) \cap (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})|$

Proof. (1) From Definition 10, we have $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}\cup\mathscr{B}) = 1 - (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}\cup\mathscr{B})|/\ |(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}\cup\mathscr{B})|) = 1 - (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}\cup\mathscr{B})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})|) \le 1 - (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}) \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})|)$. Consequently, $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}\cup\mathscr{B})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| \le |(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| - |(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}) \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})|$. Similarly, $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}\cap\mathscr{B})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(X) \cap (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| \le |(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A}) \cap (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| - |(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A}) \cap (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})|$. We know $|X \cup Y| = |X| + |Y| - |X \cap Y|$. Then,

$$\begin{aligned}
&\left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_\beta^\alpha\left(\mathscr{A}\bigcup\mathscr{B}\right)\left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\bigcup\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right| \\
&\le \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\bigcup\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right| - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{A})\bigcup\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{B})\right| \\
&= \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\right| + \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right| - \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\cap\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right| \\
&\quad - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{A})\right| + \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{B})\right| - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{A})\cap\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{B})\right| \\
&= \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\right| + \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right| - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{A})\right| - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{B})\right| \\
&\quad - \left(\left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\cap\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right| - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{A})\cap\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{B})\right|\right) \\
&\le \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\right| + \left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right| - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{A})\right| - \left|\left(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]}\right)^\alpha(\mathscr{B})\right| \\
&\quad - \left(\rho_p\right)_\beta^\alpha(\mathscr{A}\cap\mathscr{B})\left|\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{A})\cap\left(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]}\right)^\beta(\mathscr{B})\right|.
\end{aligned} \tag{25}$$

Also, by the relation of $(\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A}) = 1 - (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})|)$ and $(\rho_p)_\beta^\alpha(\mathscr{B}) = 1 - (|(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})|/|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})|)$, then $|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})| - |(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{A})| = (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{A})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{A})|$ and $|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})| - |(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^\alpha(\mathscr{B})| = (\rho_{[(\delta,\zeta,\vartheta)]})_\beta^\alpha(\mathscr{B})|(\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^\beta(\mathscr{B})|$. Thus, we have

$$\begin{aligned}
&\left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_{\beta}^{\alpha}\left(\mathscr{A} \bigcup \mathscr{B}\right)\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{A}) \bigcup\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{B})\right| \\
&\leq\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{A})\right|+\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{B})\right|-\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{-}\right)^{\alpha}(\mathscr{A})\right| \\
&\quad-\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{-}\right)^{\alpha}(\mathscr{B})\right|-\left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_{\beta}^{\alpha}(\mathscr{A} \cap \mathscr{B})\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{A}) \cap\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{B})\right| \\
&=\left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_{\beta}^{\alpha}(\mathscr{A})\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{A})+\left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_{\beta}^{\alpha}(\mathscr{B})\right|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{B}) \\
&\quad-\left(\rho_{[(\delta,\zeta,\vartheta)]}\right)_{\beta}^{\alpha}(\mathscr{A} \cap \mathscr{B})\left|\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{A}) \cap\left(\mathscr{R}_{[(\delta,\zeta,\vartheta)]}^{+}\right)^{\beta}(\mathscr{B})\right| .
\end{aligned} \tag{26}$$

Hence, (1) is holding.

(2) It can be easily proved by the relationship $(\mu_{[(\delta,\zeta,\vartheta)]})_{\beta}^{\alpha}(\mathscr{A}) = 1 - (\rho_{[(\delta,\zeta,\vartheta)]})_{\beta}^{\alpha}(\mathscr{A})$.

3.3. Generalized Picture Fuzzy Rough Sets Based on (δ,ζ,ϑ)-Cut of $\mathscr{R}$ on Two Different Universes

Definition 11. Let $\mathscr{R} = \{\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}, i \in N\} \subseteq \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $0 \leq \beta < \alpha \leq 1$, $\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$, and $\mathscr{A} \in 2^V$. Then,

(1) $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | (|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)|) \geq \alpha \wedge (|\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u)|) \geq \alpha \wedge \cdots \wedge (|\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u)|) \geq \alpha\}$ and $(\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | (|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)|) > \beta \wedge (|\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u)|) > \beta \wedge \cdots \wedge (|\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u)|) > \beta\}$ are called the Type-I lower $\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}$-approximation and the Type-I upper $\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}$-approximation of $\mathscr{A}$ according to the degree α and β with respect to $\mathscr{R}_{i[(\delta,\zeta,\vartheta)]}$ on U and V, respectively, and $((\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}), (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}))$ is called the Type-I generalized picture fuzzy rough approximation of $\mathscr{A}$ according to the degree α and β with respect to $\mathscr{R}_{i[(\delta,\zeta,\vartheta)]}$ (briefly, the Type-I generalized picture fuzzy rough set according to the degree α and β with respect to $\mathscr{R}_{i[(\delta,\zeta,\vartheta)]}$).

(2) $(\mathscr{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | (|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)|) \geq \alpha \vee (|\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u)|) \geq \alpha \vee \cdots \vee (|\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u)|) \geq \alpha\}$ and $(\mathscr{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | (|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)|) > \beta \vee (|\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u)|) > \beta \vee \cdots \vee (|\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A}| / |\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u)|) > \beta\}$ are called the Type-II lower $\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}$-approximation and the Type-II upper $\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}$-approximation of $\mathscr{A}$ according to the degree α and β with respect to $\mathscr{R}_{i[(\delta,\zeta,\vartheta)]}$ on U and V, respectively, and $((\mathscr{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}), (\mathscr{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}))$ is called the Type-II generalized picture fuzzy rough approximation of $\mathscr{A}$ according to the degree α and β with respect to $\mathscr{R}_{i[(\delta,\zeta,\vartheta)]}$ (briefly, the Type-II generalized picture fuzzy rough set according to the degree α and β with respect to $\mathscr{R}_{i[(\delta,\zeta,\vartheta)]}$).

Remark 9. (1) From Definition 11, we have

(i) $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$

(ii) $(\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-)^{II\beta}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$

(2) If we take $\alpha = 1$ and $\beta = 0$ in Definition 11, then we can obtain in the following definitions:

(i) $(\mathscr{R}^-)^{I}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | \widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \wedge \widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \wedge \cdots \wedge \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A}\}$ and $(\mathscr{R}^+)^{I}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | \widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing \wedge \widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing \wedge \cdots \wedge \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing\}$.

(ii) $(\mathscr{R}^-)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | \widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \vee \widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A} \vee \cdots \vee \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \subseteq \mathscr{A}\}$ and $(\mathscr{R}^+)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u \in U | \widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing \vee \widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing \vee \cdots \vee \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \cap \mathscr{A} \neq \varnothing\}$.

The main results are as follows.

Theorem 10. *Let $\mathscr{R} = \{\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}, i \in N\} \subseteq \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $0 \leq \beta < \alpha \leq 1$, $\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$, and $\mathscr{A}, \mathscr{B} \in 2^V$. Then, the following holds:*

(1) $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = \varnothing$; $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = V$

(2) $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(A) \subseteq (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(A)$

(3) If $\mathscr{A} \subseteq \mathscr{B}$, *then* $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$; $(\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$

(4) $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap (\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$; $(\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A} \cap \mathscr{B}) \subseteq (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B}) \cap (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$

(5) $(\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A} \cup \mathscr{B}) \supseteq (\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$; $(\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A} \cup \mathscr{B}) \supseteq (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$

(6) $(\mathscr{R}^-)^{I\alpha_1}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-)^{I\alpha}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$ *(if* $0.5 < \alpha < \alpha_1 \leq 1$*);* $(\mathscr{R}^+)^{I\beta_1}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^+)^{I\beta}_{\wedge_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$ *(if* $0 \leq \beta < \beta_1 < 0.5$*)*

Proof. By Definition 11, the result can be similarly proven as Theorem 5.

Remark 10. Let $\mathcal{A} \in 2^V$ and $0 \leq \beta < \alpha \leq 1$. Then,

(1) $(\mathcal{R}^-)^{I\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) = ((\mathcal{R}^+)^{I(1-\alpha)}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})^c)^c$

(2) $(\mathcal{R}^+)^{I\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) = ((\mathcal{R}^-)^{I(1-\beta)}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})^c)^c$

does not hold by the following example.

Example 4. Let $U = \{x_i | i = 1, 2\}$ and $V = \{y_i | i = 1, 2\}$ be two two-element set, and $\widehat{\mathcal{R}}_1, \widehat{\mathcal{R}}_2, \widehat{\mathcal{R}}_3 \in \mathbb{I}^{U\times V}$ is defined by

$$\begin{aligned}
\widehat{\mathcal{R}}_1 &= \left\{\frac{(0.2, 0.4, 0.4)}{(x_1, y_1)}, \frac{(0.4, 0.2, 0.3)}{(x_1, y_2)}, \frac{(0.5, 0.3, 0.2)}{(x_2, y_1)}, \frac{(0.7, 0.1, 0.2)}{(x_2, y_2)}\right\},\\
\widehat{\mathcal{R}}_2 &= \left\{\frac{(0.3, 0.2, 0.4)}{(x_1, y_1)}, \frac{(0.1, 0.2, 0.2)}{(x_1, y_2)}, \frac{(0.5, 0.3, 0.1)}{(x_2, y_1)}, \frac{(0.6, 0.1, 0.3)}{(x_2, y_2)}\right\},\\
\widehat{\mathcal{R}}_3 &= \left\{\frac{(0.2, 0.3, 0.5)}{(x_1, y_1)}, \frac{(0.6, 0.1, 0.2)}{(x_1, y_2)}, \frac{(0.7, 0.1, 0.1)}{(x_2, y_1)}, \frac{(0.4, 0.3, 0.3)}{(x_2, y_2)}\right\}.
\end{aligned} \tag{27}$$

Take $(\delta,\zeta,\vartheta) = (0.5, 0.3, 0.2)$. Then, $\widehat{\mathcal{R}}_{1[(0.5,0.3,0.2)]}(x_1) = \varnothing$, $\widehat{\mathcal{R}}_{1[(0.5,0.3,0.2)]}(x_2) = V$, $\widehat{\mathcal{R}}_{2[(0.5,0.3,0.2)]}(x_1) = \varnothing$, $\widehat{\mathcal{R}}_{2[(0.5,0.3,0.2)]}(x_2) = \{y_1\}$, $\widehat{\mathcal{R}}_{3[(0.5,0.3,0.2)]}(x_1) = \{y_2\}$, $\widehat{\mathcal{R}}_{3[(0.5,0.3,0.2)]}(x_2) = \{y_1\}$. Let $\mathcal{A} = \{y_1\}$, $\alpha = 0.6$, and $\beta = 0.4$. Thus,

(1) $(\mathcal{R}^-)^{I\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) = \varnothing \neq \{x_1\} = ((\mathcal{R}^+)^{I(1-\alpha)}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})^c)^c$

(2) $(\mathcal{R}^+)^{I\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) = \{x_2\} \neq U = ((\mathcal{R}^-)^{I(1-\beta)}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})^c)^c$

From Remark 9 (2), we can conclude the following corollary.

Corollary 2. *Let $\mathcal{R} = \{\widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}, i \in N\} \subseteq \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $\widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$, and $\mathcal{A}, \mathcal{B} \in 2^V$. Then, the following holds:*

(1) $(\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = \varnothing$; $(\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = V$

(2) $(\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})$

(3) If $\mathcal{A} \subseteq \mathcal{B}$, *then* $(\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$; $(\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$

(4) $(\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cap \mathcal{B}) \subseteq (\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cap (\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$; $(\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cap \mathcal{B}) \subseteq (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cap (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$

(5) $(\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cup \mathcal{B}) \supseteq (\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cup (\mathcal{R}^-)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$; $(\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cup \mathcal{B}) \supseteq (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cup (\mathcal{R}^+)^{I}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$

Theorem 11. *Let $\mathcal{R} = \{\widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}, i \in N\} \subseteq \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $0 \leq \beta < \alpha \leq 1$, $\widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$, and $\mathcal{A}, \mathcal{B} \in 2^V$.*

Then, the following holds:

(1) $(\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = \varnothing$; $(\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = V$

(2) $(\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})$

(3) If $\mathcal{A} \subseteq \mathcal{B}$, *then* $(\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^-)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$; $(\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^-)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$

(4) $(\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cap \mathcal{B}) \subseteq (\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cap (\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$; $(\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cap \mathcal{B}) \subseteq (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cap (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$

(5) $(\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cup \mathcal{B}) \supseteq (\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cup (\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$; $(\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A} \cup \mathcal{B}) \supseteq (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \cup (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$

(6) $(\mathcal{R}^-)^{II\alpha_1}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})$ *(if* $0.5 < \alpha < \alpha_1 \leq 1$*);* $(\mathcal{R}^+)^{II\beta_1}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})$ *(if* $0 \leq \beta < \beta_1 < 0.5$*)*

Proof. It follows from Definition 11 (2).

From Remark 9 (2), we can conclude the following corollary.

Corollary 3. *Let $\mathcal{R} = \{\widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}, i \in N\} \subseteq \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta) \in \mathbb{I}$, $\widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}(u) \neq \varnothing$, and $\mathcal{A}, \mathcal{B} \in 2^V$. Then the following holds:*

(1) $(\mathcal{R}^-)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = (\mathcal{R}^+)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\varnothing) = \varnothing$; $(\mathcal{R}^-)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = (\mathcal{R}^+)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(U) = V$

(2) $(\mathcal{R}^-)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A})$

(3) If $\mathcal{A} \subseteq \mathcal{B}$, *then* $(\mathcal{R}^-)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^-)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$; $(\mathcal{R}^+)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{A}) \subseteq (\mathcal{R}^+)^{II}_{\vee_i^n \widehat{\mathcal{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathcal{B})$

(4) $(\mathscr{R}^-)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}\cap\mathscr{B}) \subseteq (\mathscr{R}^-)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap (\mathscr{R}^-)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$;

$(\mathscr{R}^+)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}\cap\mathscr{B}) \subseteq (\mathscr{R}^+)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap (\mathscr{R}^+)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$

(5) $(\mathscr{R}^-)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}\cup\mathscr{B}) \supseteq (\mathscr{R}^-)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^-)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$; $(\mathscr{R}^+)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}\cup\mathscr{B}) \supseteq (\mathscr{R}^+)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^+)^{II}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{B})$

Theorem 12. *Let $\mathscr{R} = \{\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}, i\in N\} \subseteq \mathbb{I}^{U\times V}$, $(\delta,\zeta,\vartheta)\in\mathbb{I}$, $0\le\beta<\alpha\le 1$, $\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}(u)\neq\varnothing$, and $\mathscr{A},\mathscr{B}\in 2^V$. Then, the following holds:*

(1) $(\mathscr{R}^-)^{I\alpha}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap \cdots \cap (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$

(2) $(\mathscr{R}^+)^{I\beta}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cap \cdots \cap (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta}_{\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$

(3) $(\mathscr{R}^-)^{II\alpha}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup \cdots \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$

(4) $(\mathscr{R}^+)^{II\beta}_{\vee_i^n \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup \cdots \cup (\mathscr{R}^+_{[(\delta,\zeta,\vartheta)]})^{\beta}_{\widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$

Proof. We only prove (3), while rest can be proven similarly.

(3) We show that $(\mathscr{R}^-)^{II\alpha}_{\vee_{i=1}^2 \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$. Let $u\in (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A})\,(u\in U)$, where $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) = \{u\in U \mid (|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)|)\ge\alpha\}$, but $u\in (\mathscr{R}^-)^{II\alpha}_{\vee_{i=1}^2 \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$. So, $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-)^{II\alpha}_{\vee_{i=1}^2 \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$. Similarly, $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-)^{II\alpha}_{\vee_{i=1}^2 \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$. Consequently, $(\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-)^{II\alpha}_{\vee_{i=1}^2 \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$. Conversely, for $u\in (\mathscr{R}^-)^{II\alpha}_{\vee_{i=1}^2 \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$ and by Definition 11, we have either $(|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)|)\ge\alpha$, then $u\in (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$ or $(|\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}(u)\cap\mathscr{A}|/|\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}(u)|)\ge\alpha$, and then $u\in (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$. Hence, $u\in (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$, and thus $(\mathscr{R}^-)^{II\alpha}_{\vee_{i=1}^2 \widehat{\mathscr{R}}_{i[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \subseteq (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{1[(\delta,\zeta,\vartheta)]}}(\mathscr{A}) \cup (\mathscr{R}^-_{[(\delta,\zeta,\vartheta)]})^{\alpha}_{\widehat{\mathscr{R}}_{2[(\delta,\zeta,\vartheta)]}}(\mathscr{A})$. Therefore, (3) is holding.

4. An Application of Rough Picture Fuzzy Sets on Two Different Universes

We will present the notion of RPFSs over two different universes and also introduce an application of rough picture fuzzy sets on two different universes.

Definition 12. Let $\mathscr{R}\in\mathbb{I}^{U\times V}$ and $\mathscr{A}\in\mathbb{I}^V$. Then, $\forall u\in U$

$$\mathscr{R}^-(\mathscr{A})(u) = \left(\bigwedge_{v\in V}[p_3\circ\mathscr{R}(u,v)\vee p_1\circ\mathscr{A}(v)], \bigwedge_{v\in V}[p_2\circ\mathscr{R}(u,v)\vee p_2\circ\mathscr{A}(v)], \bigvee_{v\in V}[p_1\circ\mathscr{R}(u,v)\wedge p_3\circ\mathscr{A}(v)]\right) \tag{28}$$

(i.e., the lower approximation of $\mathscr{A}$) and

$$\mathscr{R}^+(\mathscr{A})(u) = \left(\bigwedge_{v\in V}[p_1\circ\mathscr{R}(u,v)\wedge p_1\circ\mathscr{A}(v)], \bigwedge_{v\in V}[p_2\circ\mathscr{R}(u,v)\vee p_2\circ\mathscr{A}(v)], \bigvee_{v\in V}[p_3\circ\mathscr{R}(u,v)\vee p_3\circ\mathscr{A}(v)]\right) \tag{29}$$

(i.e., the upper approximation of $\mathscr{A}$).

A pair $(\mathscr{R}^-(\mathscr{A}), \mathscr{R}^+(\mathscr{A}))$ is said to be the rough picture fuzzy approximation of $\mathscr{A}$ with respect to $\mathscr{R}$.

Example 5. Let $U=\{x_i \mid i=1,2,3\}$ and $V=\{y_i \mid i=1,2,3\}$ be three-element sets, $\mathscr{R}\in\mathbb{I}^{U\times V}$ defined by

$$\mathscr{R} = \left\{\frac{(0.2,0.5,0.2)}{(x_1,y_1)}, \frac{(0.6,0.3,0.1)}{(x_1,y_2)}, \frac{(0.4,0.3,0.2)}{(x_1,y_3)}, \frac{(0.4,0.2,0.3)}{(x_2,y_1)}, \frac{(0.3,0.3,0.4)}{(x_2,y_2)}, \frac{(0.1,0.6,0.2)}{(x_2,y_3)}, \frac{(0.5,0.2,0.3)}{(x_3,y_1)}, \frac{(0.5,0.4,0.1)}{(x_3,y_2)}, \frac{(0.4,0.1,0.2)}{(x_3,y_3)}\right\}, \tag{30}$$

and $\mathscr{A}\in\mathbb{I}^V$ defined by

$$\mathscr{A} = \left\{\frac{(0.4,0.3,0.2)}{y_1}, \frac{(0.2,0.1,0.5)}{y_2}, \frac{(0.3,0.5,0.2)}{y_3}\right\}. \tag{31}$$

By Definition 12, we obtain

$$\begin{aligned}\mathscr{R}^{-}(\mathscr{A})(x_1) &= \left(\bigwedge_{v\in V}[p_3\circ\mathscr{R}(x_1,v)\vee p_1\circ\mathscr{A}(v)], \bigwedge_{v\in V}[p_2\circ\mathscr{R}(x_1,v)\vee p_2\circ\mathscr{A}(v)], \bigvee_{v\in V}[p_1\circ\mathscr{R}(x_1,v)\wedge p_3\circ\mathscr{A}(v)]\right)\\ &= ([(0.2\vee 0.4)\wedge(0.1\vee 0.2)\wedge(0.2\vee 0.3)], [(0.5\vee 0.3)\wedge(0.3\vee 0.1)\wedge(0.3\vee 0.5)], [(0.2\wedge 0.2)\vee(0.6\wedge 0.5)\vee(0.4\wedge 0.2)])\\ &= (0.2, 0.3, 0.5).\end{aligned} \tag{32}$$

Similarly, $\mathscr{R}^{-}(\mathscr{A})(x_2) = (0.3, 0.3, 0.3)$, $\mathscr{R}^{-}(\mathscr{A})(x_3) = (0.2, 0.3, 0.5)$, $\mathscr{R}^{+}(\mathscr{A})(x_1) = (0.3, 0.3, 0.2)$, $\mathscr{R}^{+}(\mathscr{A})(x_2) = (0.4, 0.3, 0.2)$, and $\mathscr{R}^{+}(\mathscr{A})(x_3) = (0.4, 0.3, 0.2)$.

Therefore,

$$\begin{aligned}\mathscr{R}^{-}(\mathscr{A}) &= \left\{\frac{(0.2, 0.3, 0.5)}{x_1}, \frac{(0.3, 0.3, 0.3)}{x_2}, \frac{(0.2, 0.3, 0.5)}{x_3}\right\},\\ \mathscr{R}^{+}(\mathscr{A}) &= \left\{\frac{(0.3, 0.3, 0.2)}{x_1}, \frac{(0.4, 0.3, 0.2)}{x_2}, \frac{(0.4, 0.3, 0.2)}{x_3}\right\}.\end{aligned} \tag{33}$$

Definition 13. Suppose that U be a set and $\mathscr{A}, \mathscr{B} \in \mathbb{I}^U$. The sum $\mathscr{A}\hat{\oplus}\mathscr{B}$, is defined by

$$\mathscr{A}\hat{\oplus}\mathscr{B} = \{\mathscr{A}(u) + \mathscr{B}(u)\backslash u | u \in U\}, \tag{34}$$

where $\mathscr{A}(u) + \mathscr{B}(u)$ is defined in Definition 3 in [50].

Definition 14. (cf. see [62])Let $U = \{u_1, u_2, \ldots, u_n\}$ be an n-element set and $\mathscr{A}, \mathscr{A}^* \in \mathbb{I}^U$. A cosine similarity measure is defined by

$$C(\mathscr{A}, \mathscr{A}^*) = \frac{1}{n}\sum_{i=1}^{n}\frac{p_1\circ\mathscr{A}(u_i)\cdot p_1\circ\mathscr{A}^*(u_i) + p_2\circ\mathscr{A}(u_i)\cdot p_2\circ\mathscr{A}^*(u_i) + p_3\circ\mathscr{A}(u_i)\cdot p_3\circ\mathscr{A}^*(u_i)}{\sqrt{(p_1\circ\mathscr{A}(u_i))^2 + (p_2\circ\mathscr{A}(u_i))^2 + (p_3\circ\mathscr{A}(u_i))^2}\cdot\sqrt{(p_1\circ\mathscr{A}^*(u_i))^2 + (p_2\circ\mathscr{A}^*(u_i))^2 + (p_3\circ\mathscr{A}^*(u_i))^2}}. \tag{35}$$

Next, we present the RPFS decision-making medical diagnosis problem over two different universes as indicated below.

Suppose that $U = \{u_1, u_2, \ldots, u_p\}$ (a p-element set) be the set of a disease, $V = \{v_1, v_2, \ldots, v_q\}$ the set of symptoms (where $p, q \in N$), and $\mathscr{R} \in \mathbb{I}^{U\times V}$ be picture fuzzy relation from U to V. For any $\mathscr{R}(u_p, v_q)$, it represents the picture fuzzy relation between the disease $u_p (u_p \in U)$ and the symptom $v_q (v_q \in V)$ (i.e., by a doctor in advance). For $\mathscr{A} \in \mathbb{I}^V$ (i.e., any patient set) who has some symptoms in V, where $\mathscr{A}$ is PFS on symptom V, that is,

$$\mathscr{A} = \left\{\frac{(p_1\circ\mathscr{A}(v_1), p_2\circ\mathscr{A}(v_1), p_3\circ\mathscr{A}(v_1))}{v_1}, \frac{(p_1\circ\mathscr{A}(v_2), p_2\circ\mathscr{A}(v_2), p_3\circ\mathscr{A}(v_2))}{v_2}, \ldots, \frac{(p_1\circ\mathscr{A}(v_q), p_2\circ\mathscr{A}(v_q), p_3\circ\mathscr{A}(v_q))}{v_q}\right\}, \tag{36}$$

where $p_1\circ\mathscr{A}(v_q) \in [0,1]$ (i.e., the degree of positive membership) to the symptom $v_q \in V$ of $\mathscr{A}$, $p_2\circ\mathscr{A}(v_q) \in [0,1]$ (i.e., the degree of neutral membership) to the symptom $v_q \in V$ of $\mathscr{A}$, and $p_3\circ\mathscr{A}(v_q)$ (i.e., the degree of negative membership) to the symptom $v_q \in V$ of $\mathscr{A}$ (clearly, $\forall v_q \in V,\ p_1\circ\mathscr{A}(v_q) + p_2\circ\mathscr{A}(v_q) + p_3\circ\mathscr{A}(v_q) \leq 1$). By Definition 12, we compute the lower and the upper RPF approximations $\mathscr{R}^{-}(\mathscr{A})$ and $\mathscr{R}^{+}(\mathscr{A})$, respectively, of $\mathscr{A}$. Then, by Definition 14, we obtain $\mathscr{R}^{-}(\mathscr{A})\oplus\mathscr{R}^{+}(\mathscr{A})$ as follows:

$$\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}) = \{\mathscr{R}^{-}(\mathscr{A})(u) + \mathscr{R}^{+}(\mathscr{A})(u)\backslash u | u \in U\}. \tag{37}$$

From Definition 14, we calculate the cosine similarity measure between the $(\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}))_{u_p}$ corresponding to u_p and the ideal $(\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}))^*$. Finally, we determine $\hat{x} = \arg\max\{C((\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}))_{u_1}, (\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A})^*)), C((\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}))_{u_2}, (\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}))^*), \ldots, C((\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}))_{u_p}, (\mathscr{R}^{-}(\mathscr{A})\hat{\oplus}\mathscr{R}^{+}(\mathscr{A}))^*)\}$. Hence, the best choice is to select u_p, that is, we can conclude that patient $\mathscr{A}$ is suffering from the diseases.

The corresponding algorithm is as follows.

Now, we introduce the following an example (i.e., an application of RPFSs over two different universes) by using an Algorithm 1.

Step 1. Input the picture fuzzy relation $\mathscr{R} \in \mathbb{I}^{U\times V}$, where $\mathscr{R}(u_p, v_q)$ represents the picture fuzzy relation between the disease $u_p(u_p \in U)$ and the symptom $v_q(v_q \in V)$, which is evaluated by a doctor in advance
Step 2. Define $\mathscr{A}$ (patient set) is PFS on symptom V, that is, $\mathscr{A} = \{((p_1 \circ \mathscr{A}(v_1), p_2 \circ \mathscr{A}(v_1), p_3 \circ \mathscr{A}(v_1))/v_1), ((p_1 \circ \mathscr{A}(v_2), p_2 \circ \mathscr{A}(v_2), p_3 \circ \mathscr{A}(v_2))/v_2), \ldots, ((p_1 \circ \mathscr{A}(v_q), p_2 \circ \mathscr{A}(v_q), p_3 \circ \mathscr{A}(v_q))/v_q)\}$
Step 3. Compute and write $\mathscr{R}^-(\mathscr{A})$ and $\mathscr{R}^+(\mathscr{A})$, respectively, of $\mathscr{A}$
Step 4. By Definition 13, we calculate $\mathscr{R}^-(\mathscr{A})\oplus\mathscr{R}^+(\mathscr{A})$ as $\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}) = \{\mathscr{R}^-(\mathscr{A})(u) + \mathscr{R}^+(\mathscr{A})(u)\setminus u | u \in U\}$
Step 5. By Definition 14, we compute the cosine similarity measure between the $(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}))_{u_p}$ corresponding to u_p and the ideal $(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}))^*$
Step 6. Get the best choice to select u_p, that is, we can conclude that patient $\mathscr{A}$ is suffering from the diseases

ALGORITHM 1: Determine the best choice of rough picture fuzzy soft sets over two different universes.

Example 6. Assume that five diseases ($U = \{x_i | i = 1, 2, 3, 4, 5\}$), where x_1 stands for the "viral fever," x_2 stands for the "malaria," x_3 stands for the "typhoid," x_4 stands for the "stomach problem," and x_5 stands for the "chest problem." The five symptoms are in clinic (let $V = \{y_i | i = 1, 2, 3, 4, 5\}$), where y_1 stands for the "temperature," y_2 stands for the "headache," y_3 stands for the "stomach Pain," y_4 stands for the "cough," and y_5 stands for the "Chest-Pain." Let $\mathscr{R} \in \mathbb{I}^{U\times V}$ be a picture fuzzy relation from U to V, where $\mathscr{R}$ is a medical knowledge statistic data of the relationship of the disease $x_p(x_p \in U)$ and the symptom $y_q(y_q \in V)$ (where $p, q = 1, 2, 3, 4, 5$). The statistic data are given in Table 4.

By the Step 2 of Algorithm 1, suppose the symptoms of a patient $\mathscr{A}$ are defined by a PFS on V, and

$$\mathscr{A} = \left\{\frac{(0.4, 0.3, 0.2)}{y_1}, \frac{(0.2, 0.1, 0.5)}{y_2}, \frac{(0.3, 0.5, 0.2)}{y_3}, \frac{(0.6, 0.2, 0.1)}{y_4}, \frac{(0.4, 0.4, 0.2)}{y_5}\right\}. \tag{38}$$

Then, by Definition 12, we obtain on the $\mathscr{R}^-(\mathscr{A})$ and $\mathscr{R}^+(\mathscr{A})$ of patient $\mathscr{A}$ in Step 3 of Algorithm 1, respectively, as follows:

$$\begin{aligned}
\mathscr{R}^-(\mathscr{A}) &= \left\{\frac{(0.4, 0.3, 0.3)}{x_1}, \frac{(0.3, 0.2, 0.5)}{x_2}, \frac{(0.3, 0.1, 0.5)}{x_3}, \frac{(0.2, 0.3, 0.5)}{x_4}, \frac{(0.2, 0.1, 0.4)}{x_5}\right\},\\
\mathscr{R}^+(\mathscr{A}) &= \left\{\frac{(0.5, 0.3, 0.2)}{x_1}, \frac{(0.4, 0.2, 0.3)}{x_2}, \frac{(0.5, 0.1, 0.2)}{x_3}, \frac{(0.4, 0.3, 0.1)}{x_4}, \frac{(0.4, 0.1, 0.1)}{x_5}\right\}.
\end{aligned} \tag{39}$$

By Step 4 of Algorithm 1, we obtain

$$\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}) = \left\{\frac{(0.7, 0.09, 0.06)}{x_1}, \frac{(0.58, 0.04, 0.15)}{x_2}, \frac{(0.65, 0.01, 0.1)}{x_3}, \frac{(0.52, 0.09, 0.05)}{x_4}, \frac{(0.52, 0.01, 0.04)}{x_5}\right\}. \tag{40}$$

Table 4: Picture fuzzy relation $\mathscr{R} \in \mathbb{I}^{U\times V}$ between the diseases and symptoms.

$\mathscr{R}$	Temperature (y_1)	Headache (y_2)	Stomach pain (y_3)	Cough (y_4)	Chest-pain (y_5)
Viral fever (x_1)	(0.1, 0.5, 0.4)	(0.3, 0.3, 0.4)	(0.3, 0.2, 0.4)	(0.5, 0.3, 0.2)	(0.1, 0.4, 0.4)
Malaria (x_2)	(0.2, 0.4, 0.3)	(0.5, 0.2, 0.3)	(0.5, 0.1, 0.3)	(0.2, 0.3, 0.5)	(0.4, 0.2, 0.4)
Typhoid (x_3)	(0.7, 0.2, 0.1)	(0.6, 0.1, 0.3)	(0.3, 0.6, 0.1)	(0.5, 0.1, 0.4)	(0.7, 0.2, 0.1)
Stomach problem (x_4)	(0.4, 0.4, 0.2)	(0.7, 0.3, 0)	(0.8, 0.1, 0.1)	(0.2, 0.3, 0.1)	(0.6, 0.2, 0.2)
Chest problem (x_5)	(0.1, 0.8, 0.1)	(0.4, 0.1, 0.2)	(0.4, 0.2, 0.3)	(0.1, 0.8, 0.1)	(0.9, 0, 0.1)

Then, by Step 4 of Algorithm 1, we can get the cosine similarity measure between the $(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}))_{u_p}$ corresponding to u_p and the ideal $(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}))^*$ as follows:

$$\begin{aligned}
C\left(\left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)_{x_1}, \left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)^*\right) &= 0.141,\\
C\left(\left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)_{x_2}, \left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)^*\right) &= 0.181,\\
C\left(\left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)_{x_3}, \left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)^*\right) &= 0.198,\\
C\left(\left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)_{x_4}, \left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)^*\right) &= 0.196,\\
C\left(\left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)_{x_5}, \left(\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A})\right)^*\right) &= 0.199.
\end{aligned} \tag{41}$$

Thus, according to Step 6, we conclude the maximum value is $C((\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}))_{x_5}, (\mathscr{R}^-(\mathscr{A})\widehat{\oplus}\mathscr{R}^+(\mathscr{A}))^*) = 0.199$. Thus, the patient is suffering from the disease chest problem (x_5).

5. Conclusions

In this paper, we suggest novel notion of picture fuzzy rough sets (PFRSs) over two different universes which depend on $(\delta, \zeta, \vartheta)$-cut. Also, we discussed some interesting properties and related results on the PFRSs. Furthermore, we presented several notions related to PFRSs such as Type-I-/Type-II-graded PFRSs, the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ on PFRSs, and Type-I-/Type-II-generalized PFRSs based on the degree α and β with respect to $\mathscr{R}_{[(\delta,\zeta,\vartheta)]}$ and investigate the basic properties of above notions. Lastly, we gave an approach based on the rough picture fuzzy approximation RPFA operators on two different universes in decision-making problem is introduced, and we present an example to show the validity of this approach.

Data Availability

All data required for this paper are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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