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Research Article

On Characterizations of $*$-Hyperconnected Ideal Topological Spaces

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In the present paper, some characterizations of $*$-hyperconnected ideal topological spaces are investigated. Moreover, we introduce the notion of S^*-$\mathcal{I}$-hyperconnected ideal topological spaces. Several characterizations of S^*-$\mathcal{I}$-hyperconnected ideal topological spaces are discussed. Furthermore, we introduce and study θ-$\mathcal{I}$-irreducible ideal topological spaces.

1. Introduction

The concept of hyperconnected spaces was introduced by Steen and Seebach [1]. Several concepts which are equivalent to hyperconnectedness were defined and investigated in the literature. Levine [2] called a topological space X a D-space if every nonempty open set of X is dense in X and showed that X is a D-space if and only if it is hyperconnected. Pipitone and Russo [3] defined a topological space X to be semi-connected if X is not the union of two disjoint nonempty semiopen sets of X and showed that X is semiconnected if and only if it is a D-space. Sharma [4] indicated that a space is a D-space if it is a hyperconnected space. Ajmal and Kohli [5] have investigated further properties of hyperconnected spaces. In [6], the author obtained several characterizations of hyperconnected spaces by using semi-preopen sets and almost feebly continuous functions. Hyperconnected spaces are also called irreducible in [7]. D-spaces, s-connected spaces [8], semiconnected spaces, and irreducible spaces are equivalent to hyperconnected spaces. Janković and Long [9] have introduced and investigated the notion of θ-irreducible spaces. It is pointed out in [9] that hyperconnectedness implies θ-irreducibility and θ-irreducibility implies connectedness. Noiri and Umehara [10] investigated some characterizations and some preservation theorems concerning θ-irreducible spaces. The concept of ideal topological spaces was introduced and studied by Kuratowski [11] and Vaidyanathaswamy [12]. Janković and Hamlett [13] investigated further properties of ideal topological spaces. Hatir and Noiri [14] introduced the notions of semi-$\mathcal{I}$-open sets, α-$\mathcal{I}$-open sets, and β-$\mathcal{I}$-open sets in topological spaces via ideals and used these sets to obtain certain decomposition of continuity. Later, in [15], the same authors investigated further properties of semi-$\mathcal{I}$-open sets and semi-$\mathcal{I}$-continuous functions introduced in [14]. Recently, Ekici and Noiri [16] have introduced the notion of $*$-hyperconnected ideal topological spaces and investigated some characterizations of $*$-hyperconnected ideal topological spaces. The purpose of the present paper is to investigate some characterizations of $*$-hyperconnected ideal topological spaces. Moreover, we introduce and study the notion of S^*-$\mathcal{I}$-hyperconnected ideal topological spaces. In the last section, we introduce the notion of θ-$\mathcal{I}$-irreducible ideal topological spaces. Especially, several characterizations of θ-$\mathcal{I}$-irreducible ideal topological spaces are established.

2. Preliminaries

Throughout the present paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. In a topological space (X, τ), the closure and the interior of any subset A of X will be denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X satisfying the following properties: (1) $A \in \mathcal{J}$ and $B \subseteq A$ imply $B \in \mathcal{J}$; (2) $A \in \mathcal{J}$ and $B \in \mathcal{J}$ imply $A \cup B \in \mathcal{J}$. A topological space (X, τ) with an ideal $\mathcal{J}$ on X is called an ideal topological space and is denoted by $(X, \tau, \mathcal{J})$. For an ideal topological space $(X, \tau, \mathcal{J})$ and a subset A of X, $A^*(\mathcal{J})$ is defined as follows: $A^*(\mathcal{J}) = \{x \in X\colon U \cap A \notin \mathcal{J}$ for every open neighbourhood U of $x\}$. In case there is no chance for confusion, $A^*(\mathcal{J})$ is simply written as A^*. In [11], A^* is called the local function of A with respect to $\mathcal{J}$ and τ and $\mathrm{Cl}^*(A) = A^* \cup A$ defines a Kuratowski closure operator for a topology $\tau^*(\mathcal{J})$. For every ideal topological space $(X, \tau, \mathcal{J})$, there exists a topology $\tau^*(\mathcal{J})$ finer than τ, generated by the base $\mathcal{B}(\mathcal{J}, \tau) = \{U - I \mid U \in \tau \text{ and } I \in \mathcal{J}\}$. However, $\mathcal{B}(\mathcal{J}, \tau)$ is not always a topology [12]. A subset A is said to be $*$-closed [13] if $A^* \subseteq A$. The interior of a subset A in $(X, \tau^*(\mathcal{J}))$ is denoted by $\mathrm{Int}^*(A)$.

Definition 1. A subset A of an ideal topological space $(X, \tau, \mathcal{J})$ is said to be

(1) Pre-$\mathcal{J}$-open [17] if $A \subseteq \mathrm{Int}(\mathrm{Cl}^*(A))$
(2) Semi-$\mathcal{J}$-open [14] if $A \subseteq \mathrm{Cl}^*(\mathrm{Int}(A))$
(3) Semi*-$\mathcal{J}$-open [16] if $A \subseteq \mathrm{Cl}(\mathrm{Int}^*(A))$
(4) Strong β-$\mathcal{J}$-open [18] if $A \subseteq \mathrm{Cl}^*(\mathrm{Int}(\mathrm{Cl}^*(A)))$
(5) R-$\mathcal{J}$-open [19] if $A = \mathrm{Int}(\mathrm{Cl}^*(A))$
(6) b-$\mathcal{J}$-open [20] if $A = \mathrm{Int}(\mathrm{Cl}^*(A)) \cup \mathrm{Cl}^*(\mathrm{Int}(A))$
(7) $*$-Dense [21] if $\mathrm{Cl}^*(A) = X$

By $p\mathcal{J}O(X, \tau)$ (resp. $s\mathcal{J}O(X, \tau)$, $s^*\mathcal{J}O(X, \tau)$, $s\beta\mathcal{J}O(X, \tau)$, $r\mathcal{J}O(X, \tau)$, and $b\mathcal{J}O(X, \tau)$), we denote the family of all pre-$\mathcal{J}$-open (resp. semi-$\mathcal{J}$-open, semi*-$\mathcal{J}$-open, strong β-$\mathcal{J}$-open, R-$\mathcal{J}$-open, and b-$\mathcal{J}$-open) sets of an ideal topological space $(X, \tau, \mathcal{J})$.

The complement of a pre-$\mathcal{J}$-open (resp. semi-$\mathcal{J}$-open, semi*-$\mathcal{J}$-open, strong β-$\mathcal{J}$-open, and R-$\mathcal{J}$-open) set is called pre-$\mathcal{J}$-closed [17] (resp. semi-$\mathcal{J}$-closed [14], semi*-$\mathcal{J}$-closed [16, 22], strong β-$\mathcal{J}$-closed [18], and R-$\mathcal{J}$-closed [19]).

Definition 2 (see [16]). The pre-$\mathcal{J}$-closure (resp. semi-$\mathcal{J}$-closure, semi*-$\mathcal{J}$-closure, and strong β-$\mathcal{J}$-closure) of a subset A of an ideal topological space $(X, \tau, \mathcal{J})$, denoted by $p\mathrm{Cl}_{\mathcal{J}}(A)$ (resp. $s\mathrm{Cl}_{\mathcal{J}}(A)$, $s^*\mathrm{Cl}_{\mathcal{J}}(A)$, and $s\beta\mathrm{Cl}_{\mathcal{J}}(A)$), is defined by the intersection of all pre-$\mathcal{J}$-closed (resp. semi-$\mathcal{J}$-closed, semi*-$\mathcal{J}$-closed, and strong β-$\mathcal{J}$-closed) sets of X containing A.

Lemma 1 (see [16]). *For a subset A of an ideal topological space $(X, \tau, \mathcal{J})$, the following properties hold:*

(1) $pCl_{\mathcal{J}}(A) = A \cup Cl(Int^*(A))$
(2) $sCl_{\mathcal{J}}(A) = A \cup Int^*(Cl(A))$
(3) $s^*Cl_{\mathcal{J}}(A) = A \cup Int(Cl^*(A))$
(4) $s\beta Cl_{\mathcal{J}}(A) = A \cup Int^*(Cl(Int^*(A)))$

Lemma 2 (see [23]). *Let A be a subset of an ideal topological space $(X, \tau, \mathcal{J})$ and U be an open set. Then, $U \cap Cl^*(A) \subseteq Cl^*(U \cap A)$.*

3. On Characterizations of $*$-Hyperconnected Ideal Topological Spaces

First, we investigate some characterizations of $*$-hyperconnected ideal topological spaces defined by Ekici and Noiri [16].

Definition 3 (see [16]). An ideal topological space $(X, \tau, \mathcal{J})$ is said to be $*$-hyperconnected if V is $*$-dense for every nonempty open set V of X.

The following theorems give several characterizations of $*$-hyperconnected ideal topological spaces.

Theorem 1. *For an ideal topological space $(X, \tau, \mathcal{J})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{J})$ is $*$-hyperconnected
(2) $Cl^*(U) = X$ for every nonempty set $U \in p\mathcal{J}O(X, \tau)$
(3) $s^*Cl_{\mathcal{J}}(U) = X$ for every nonempty set $U \in p\mathcal{J}O(X, \tau)$

Proof

(1)$\Longrightarrow$(2): suppose that $(X, \tau, \mathcal{J})$ is $*$-hyperconnected. Let U be a nonempty pre-$\mathcal{J}$-open set. Then, U is strong β-$\mathcal{J}$-open; by Theorem 14 of [16], $\mathrm{Cl}^*(U) = X$.

(2)$\Longrightarrow$(3): let U be a nonempty pre-$\mathcal{J}$-open set. By (2), $\mathrm{Cl}^*(U) = X$, and hence $X = \mathrm{Int}(\mathrm{Cl}^*(U)) = U \cup \mathrm{Int}(\mathrm{Cl}^*(U)) = s^*\mathrm{Cl}_{\mathcal{J}}(U)$.

(3)$\Longrightarrow$(1): let U be a nonempty open set. Then, U is pre-$\mathcal{J}$-open, and by (3), $\mathrm{Cl}^*(U) \supseteq s^*\mathrm{Cl}_{\mathcal{J}}(U) = X$. Thus, $(X, \tau, \mathcal{J})$ is $*$-hyperconnected. □

Lemma 3. *Let $(X, \tau, \mathcal{J})$ be an ideal topological space. Then, $s^*Cl_{\mathcal{J}}(V) \in s\mathcal{J}O(X, \tau)$ for every $V \in s\mathcal{J}O(X, \tau)$.*

Proof. It follows from Lemma 1(3) and Theorem 3.4(1) of [15]. □

Theorem 2. *For an ideal topological space $(X, \tau, \mathcal{J})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{J})$ is $*$-hyperconnected
(2) $U \cap V \neq \varnothing$ for every nonempty set $U \in s\mathcal{J}O(X, \tau)$ and every nonempty set $V \in s^*\mathcal{J}O(X, \tau)$
(3) The only subsets of X which are both semi-$\mathcal{J}$-open and semi*-$\mathcal{J}$-closed in $(X, \tau, \mathcal{J})$ are empty set and X itself
(4) X cannot be expressed by the disjoint union of a nonempty semi-$\mathcal{J}$-open set and a nonempty semi*-$\mathcal{J}$-open set

(5) $s^*Cl_{\mathcal{J}}(V) = X$ for every nonempty set $V \in s\mathcal{J}O(X,\tau)$

Proof (1)$\Longrightarrow$(2): it follows from Theorem 11 of [16].

(2)$\Longrightarrow$(3): let V be a subset of X, which is both semi-$\mathcal{J}$-open and semi*-$\mathcal{J}$-closed. Then, V is semi-$\mathcal{J}$-open and $X - V$ is semi*-$\mathcal{J}$-open such that $V \cap (X - V) = \varnothing$, which is a contradiction of (2).

(3)$\Longrightarrow$(4): suppose that $X = U \cup V$, where U is a nonempty semi-$\mathcal{J}$-open set and V a nonempty semi*-$\mathcal{J}$-open set such that $U \cap V = \varnothing$. Since $X - U = V$ and V is semi*-$\mathcal{J}$-open, U is semi*-$\mathcal{J}$-closed. This shows that U is both semi-$\mathcal{J}$-open and semi*-$\mathcal{J}$-closed, which is a contradiction of (3).

(4)$\Longrightarrow$(5): suppose that $s^*\mathrm{Cl}_{\mathcal{J}}(V) \neq X$ for some nonempty set $V \in s\mathcal{J}O(X,\tau)$. Then, we have $X - s^*\mathrm{Cl}_{\mathcal{J}}(V) \neq \varnothing$, $s^*\mathrm{Cl}_{\mathcal{J}}(V) \neq \varnothing$ and

$$X = \left(X - s^*\mathrm{Cl}_{\mathcal{J}}(V)\right) \cup s^*\mathrm{Cl}_{\mathcal{J}}(V). \tag{1}$$

Since $s^*\mathrm{Cl}_{\mathcal{J}}(V)$ is semi*-$\mathcal{J}$-closed, $X - s^*\mathrm{Cl}_{\mathcal{J}}(V)$ is semi*-$\mathcal{J}$-open, and by Lemma 3, $s^*\mathrm{Cl}_{\mathcal{J}}(V)$ is semi-$\mathcal{J}$-open, which is a contradiction of (4).

(5)$\Longrightarrow$(1): let V be a nonempty open set. Then, V is semi-$\mathcal{J}$-open, and by (5), $X = s^*\mathrm{Cl}_{\mathcal{J}}(V) \subseteq \mathrm{Cl}^*(V)$. Thus, $\mathrm{Cl}^*(V) = X$. This shows that $(X,\tau,\mathcal{J})$ is $*$-hyperconnected.

Next, we will introduce and study the concept of S^*-$\mathcal{J}$-hyperconnected ideal topological spaces. □

Definition 4. An ideal topological space $(X,\tau,\mathcal{J})$ is called *S^*-$\mathcal{J}$-hyperconnected* if $s^*\mathrm{Cl}_{\mathcal{J}}(U) = X$ for every nonempty open set U.

Remark 1. For an ideal topological space $(X,\tau,\mathcal{J})$, we have the following diagram:

$$\begin{aligned}&(X,\tau,\mathcal{J}) \text{ is } S^*\text{-}\mathcal{J}\text{-hyperconnected}\\ &\quad\Longrightarrow (X,\tau,\mathcal{J}) \text{ is } *\text{-hyperconnected.}\end{aligned} \tag{2}$$

The implication in the diagram is not reversible as shown in the following example.

Example 1 (see [16]). Let $X = \{a,b,c,d\}$ with a topology

$$\tau = \{\varnothing, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}, \tag{3}$$

and an ideal $\mathcal{J} = \{\varnothing, \{b\}\}$. Then $(X,\tau,\mathcal{J})$ is $*$-hyperconnected, but $(X,\tau,\mathcal{J})$ is not S^*-$\mathcal{J}$-hyperconnected.

Lemma 4 (see [15]). *A subset A of an ideal topological space $(X,\tau,\mathcal{J})$ is semi-$\mathcal{J}$-open if and only if there exists $U \in \tau$ such that $U \subseteq A \subseteq Cl^*(U)$.*

Theorem 3. *For an ideal topological space $(X,\tau,\mathcal{J})$, the following properties are equivalent:*

(1) $(X,\tau,\mathcal{J})$ is S^*-$\mathcal{J}$-hyperconnected

(2) $s^*Cl_{\mathcal{J}}(A) = X$ or $Int(s^*Cl_{\mathcal{J}}(A)) = \varnothing$ for every subset A of X

(3) $U \cap V \neq \varnothing$ for every nonempty open set U and every nonempty semi*-$\mathcal{J}$-open set V

(4) $U \cap V \neq \varnothing$ for every nonempty semi-$\mathcal{J}$-open set U and every nonempty semi*-$\mathcal{J}$-open set V

Proof

(1)$\Longrightarrow$(2): let A be a subset of X. Suppose that $\mathrm{Int}(s^*\mathrm{Cl}_{\mathcal{J}}(A)) \neq \varnothing$. Since $(X,\tau,\mathcal{J})$ is S^*-$\mathcal{J}$-hyperconnected, $X = s^*\mathrm{Cl}_{\mathcal{J}}(\mathrm{Int}(s^*\mathrm{Cl}_{\mathcal{J}}(A))) \subseteq s^*\mathrm{Cl}_{\mathcal{J}}(A)$. Thus, $s^*\mathrm{Cl}_{\mathcal{J}}(A) = X$.

(2)$\Longrightarrow$(3): suppose that $U \cap V = \varnothing$ for some nonempty open set U and nonempty semi*-$\mathcal{J}$-open set V. Then, $s^*\mathrm{Cl}_{\mathcal{J}}(U) \cap V = \varnothing$, and by (2), $s^*\mathrm{Cl}_{\mathcal{J}}(U) \neq X$. Moreover, since U is open, $\varnothing \neq U \subseteq \mathrm{Int}(s^*\mathrm{Cl}_{\mathcal{J}}(U))$.

(3)$\Longrightarrow$(4): suppose that $U \cap V = \varnothing$ for some nonempty semi-$\mathcal{J}$-open set U and nonempty semi*-$\mathcal{J}$-open set V. Since U is a nonempty semi-$\mathcal{J}$-open set, by Lemma 4, there exists a nonempty open set G such that $G \subseteq U \subseteq \mathrm{Cl}^*(G)$. Then, G is a nonempty semi-$\mathcal{J}$-open set such that $G \cap V = \varnothing$, which is a contradiction of (3).

(4)$\Longrightarrow$(1): suppose that $(X,\tau,\mathcal{J})$ is not S^*-$\mathcal{J}$-hyperconnected. Therefore, there exists a nonempty open set G of X such that $s^*\mathrm{Cl}_{\mathcal{J}}(G) \neq X$, and hence $X - s^*\mathrm{Cl}_{\mathcal{J}}(G) \neq \varnothing$. Thus, $X - s^*\mathrm{Cl}_{\mathcal{J}}(G)$ is a nonempty semi*-$\mathcal{J}$-open set and G is a nonempty semi-$\mathcal{J}$-open set such that $(X - s^*\mathrm{Cl}_{\mathcal{J}}(G)) \cap G = \varnothing$. This is a contradiction. Consequently, we obtain $(X,\tau,\mathcal{J})$ is S^*-$\mathcal{J}$-hyperconnected. □

Theorem 4. *For an ideal topological space $(X,\tau,\mathcal{J})$, the following properties are equivalent:*

(1) $(X,\tau,\mathcal{J})$ is $*$-hyperconnected

(2) $Cl^*(U) = X$ for every nonempty set $U \in p\mathcal{J}O(X,\tau)$

(3) $s^*Cl_{\mathcal{J}}(U) = X$ for every nonempty set $U \in p\mathcal{J}O(X,\tau)$

(4) $(X,\tau,\mathcal{J})$ is S^*-$\mathcal{J}$-hyperconnected

Proof

(1)$\Longrightarrow$(2) and (2)$\Longrightarrow$(3) follow from Theorem 1.

(3)$\Longrightarrow$(4) and (4)$\Longrightarrow$(1) are obvious.

For a subset A of an ideal topological space $(X,\tau,\mathcal{J})$, we denote by $\tau_{|A}$ the relative topology on A and $\mathcal{J}_{|A} = \{A \cap I \mid I \in \mathcal{J}\}$ is an ideal on A [24]. □

Lemma 5 (see [24]). *Let $(X,\tau,\mathcal{J})$ be an ideal topological space and $B \subseteq A \subseteq X$. Then, $B^*(\tau_{|A},\mathcal{J}_{|A}) = B^*(\tau,\mathcal{J}) \cap A$.*

Lemma 6 (see [23]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $B \subseteq A \subseteq X$. Then, $Cl_A^*(B) = Cl^*(B) \cap A$.*

Lemma 7. *Let U be an open set of an ideal topological space $(X, \tau, \mathcal{I})$. If $V \in p\mathcal{I}O(U, \tau_{|U}, \mathcal{I}_{|U})$, then $V \in p\mathcal{I}O(X, \tau)$.*

Proof. Suppose that U is an open set and $V \in p\mathcal{I}O(U, \tau_{|U}, \mathcal{I}_{|U})$. Since $V \subseteq \mathrm{Int}_U(\mathrm{Cl}_U^*(V))$ and $\mathrm{Int}_U(\mathrm{Cl}_U^*(V))$ is open in $(U, \tau_{|U})$, there exists an open set W such that $U \cap W = \mathrm{Int}_U(\mathrm{Cl}_U^*(V))$. Since $U \subseteq \mathrm{Int}(\mathrm{Cl}^*(U))$ and by Lemma 2,

$$\begin{aligned} V &\subseteq \mathrm{Int}(\mathrm{Cl}^*(U)) \cap W \\ &= \mathrm{Int}(\mathrm{Cl}^*(U) \cap W) \\ &\subseteq \mathrm{Int}(\mathrm{Cl}^*(U \cap W)) \\ &= \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Int}_U(\mathrm{Cl}_U^*(V)))) \\ &\subseteq \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Cl}_U^*(V))) \\ &\subseteq \mathrm{Int}(\mathrm{Cl}^*(\mathrm{Cl}^*(V))) \\ &= \mathrm{Int}(\mathrm{Cl}^*(V)). \end{aligned} \tag{4}$$

Thus, $V \in p\mathcal{I}O(X, \tau)$. □

Lemma 8. *Let U be an open set of an ideal topological space $(X, \tau, \mathcal{I})$. Then, $s^*Cl_{\mathcal{I}_{|U}}(V) = s^*Cl_{\mathcal{I}}(V) \cap U$ for every $V \subseteq U$.*

Proof. Let V be a subset of U. Then, we have

$$\begin{aligned} s^*\mathrm{Cl}_{\mathcal{I}_{|U}}(V) &= V \cup \mathrm{Int}_U(\mathrm{Cl}_U^*(V)) \\ &= V \cup \mathrm{Int}_U(\mathrm{Cl}^*(V) \cap U) \\ &= V \cup \mathrm{Int}(\mathrm{Cl}^*(V) \cap U) \\ &= V \cup ((\mathrm{Int}(\mathrm{Cl}^*(V)) \cap U) \\ &= (V \cup \mathrm{Int}(\mathrm{Cl}^*(V))) \cap U \\ &= s^*\mathrm{Cl}_{\mathcal{I}}(V) \cap U. \end{aligned} \tag{5}$$

□

Theorem 5. *Let U be an open set of an ideal topological space $(X, \tau, \mathcal{I})$. If $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected, then $(U, \tau_{|U}, \mathcal{I}_{|U})$ is S^*-$\mathcal{I}_{|U}$-hyperconnected.*

Proof. Suppose that $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected. Let V be a nonempty pre-$\mathcal{I}_{|U}$-open set of $(U, \tau_{|U}, \mathcal{I}_{|U})$. By Lemma 7, V is pre-$\mathcal{I}$-open in $(X, \tau, \mathcal{I})$. Since $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected, $s^*\mathrm{Cl}_{\mathcal{I}}(V) = X$, and by Lemma 8, $s^*\mathrm{Cl}_{\mathcal{I}_{|U}}(V) = s^*\mathrm{Cl}_{\mathcal{I}}(V) \cap U = X \cap U = U$. Therefore, $(U, \tau_{|U}, \mathcal{I}_{|U})$ is S^*-$\mathcal{I}_{|U}$-hyperconnected. □

Theorem 6. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected

(2) $Cl^*(V) = X$ for every nonempty open set V

Proof

(1)⟹(2): it follows from Theorem 14 of [16].

(2)⟹(1): let V be a nonempty open set. By (2), we have

$$X = \mathrm{Cl}^*(V) = \mathrm{Int}(\mathrm{Cl}^*(V)) \subseteq V \cup \mathrm{Int}(\mathrm{Cl}^*(V)) = s^*\mathrm{Cl}_{\mathcal{I}}(V), \tag{6}$$

and hence $s^*\mathrm{Cl}_{\mathcal{I}}(V) = X$. This shows that $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected. □

Definition 5. A function $f\colon (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is said to be θ-$\mathcal{I}$-continuous if for each point $x \in X$ and each open set V containing $f(x)$, there exists an open set U containing x such that $f(\mathrm{Cl}^*(U)) \subseteq \mathrm{Cl}^*(V)$.

Theorem 7. *If $f\colon (X, \tau, \mathcal{I}) \longrightarrow (Y, \sigma, \mathcal{J})$ is a θ-$\mathcal{I}$-continuous surjection and $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected, then $(Y, \sigma, \mathcal{J})$ is S^*-$\mathcal{J}$-hyperconnected.*

Proof. Suppose that $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected. Let V be a nonempty open set of Y. Since f is surjective, there exists a point x of X such that $f(x) \in V$. Since f is θ-$\mathcal{I}$-continuous, there exists an open set U containing x such that $f(\mathrm{Cl}^*(U)) \subseteq \mathrm{Cl}^*(V)$. Since $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected, by Theorem 6, we have $\mathrm{Cl}^*(U) = X$, and hence $Y = f(X) = f(\mathrm{Cl}^*(U)) \subseteq \mathrm{Cl}^*(V)$. It follows from Theorem 6 that $(Y, \sigma, \mathcal{J})$ is S^*-$\mathcal{J}$-hyperconnected. □

Theorem 8. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected

(2) V is $*$-dense for every nonempty set $V \in s\mathcal{I}O(X, \tau)$

(3) $s^*Cl_{\mathcal{I}}(V) = X$ for every nonempty set $V \in s\mathcal{I}O(X, \tau)$

Proof. The proof follows from Theorem 14 of [16]. □

Theorem 9. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected

(2) V is $*$-dense for every nonempty set $V \in s\mathcal{I}O(X, \tau)$

(3) $s\beta Cl_{\mathcal{I}}(V) = X$ for every nonempty set $V \in s\mathcal{I}O(X, \tau)$

Proof. The proof follows from Theorem 14 of [16]. □

Corollary 1. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{I})$ is S^*-$\mathcal{I}$-hyperconnected

(2) V is $*$-dense for every nonempty set $V \in s\beta\mathcal{I}O(X, \tau)$

(3) V is $*$-dense for every nonempty set $V \in b\mathcal{I}O(X, \tau)$

(4) V is $*$-dense for every nonempty set $V \in p\mathcal{I}O(X, \tau)$

(5) $sCl_{\mathcal{I}}(V) = X$ for every nonempty set $V \in p\mathcal{I}O(X, \tau)$

(6) $pCl_{\mathcal{I}}(V) = X$ for every nonempty set $V \in s\mathcal{I}O(X, \tau)$

(7) $s\beta Cl_{\mathcal{I}}(V) = X$ for every nonempty set $V \in s\mathcal{I}O(X, \tau)$

Proof. The proof follows from Theorem 14 of [16]. □

4. On θ-$\mathcal{I}$-Irreducible Ideal Topological Spaces

In this section, we introduce the notion of θ-$\mathcal{I}$-irreducible ideal topological spaces. Moreover, some characterizations of θ-$\mathcal{I}$-irreducible ideal topological spaces are investigated.

Definition 6. A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be

(1) R^*-$\mathcal{I}$-open if $A = \operatorname{Int}^*(\operatorname{Cl}(A))$

(2) R^*-$\mathcal{I}$-closed if its complement is R^*-$\mathcal{I}$-open

By $r^*\mathcal{I}O(X, \tau)$ (resp. $r^*\mathcal{I}C(X, \tau)$), we denote the family of all R^*-$\mathcal{I}$-open (resp. R^*-$\mathcal{I}$-closed) sets of an ideal topological space $(X, \tau, \mathcal{I})$.

Definition 7. An ideal topological space $(X, \tau, \mathcal{I})$ is called θ-$\mathcal{I}$*-irreducible* if every pair of nonempty R^*-$\mathcal{I}$-closed sets of X has a nonempty intersection.

Remark 2. For an ideal topological space $(X, \tau, \mathcal{I})$, we have the following diagram:

$$(X, \tau, \mathcal{I}) \text{ is } *\text{-hyperconnected} \Longrightarrow (X, \tau, \mathcal{I}) \text{ is } \theta\text{-}\mathcal{I}\text{-irreducible}. \tag{7}$$

The implication in the diagram is not reversible as shown in the following example.

Example 2. Let $X = \{a, b, c\}$ with a topology $\tau = \{\varnothing, X, \{a\}, \{b\}, \{a, b\}\}$ and an ideal $\mathcal{I} = \{\varnothing\}$. Then, the ideal topological space $(X, \tau, \mathcal{I})$ is θ-$\mathcal{I}$-irreducible, but $(X, \tau, \mathcal{I})$ is not $*$-hyperconnected.

Definition 8 (see [25]). A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be θ-$\mathcal{I}$-closed if $\operatorname{Cl}^*_\theta(A) = A$, where

$$\operatorname{Cl}^*_\theta(A) = \{x \in X \mid A \cap \operatorname{Cl}^*(U) \neq \varnothing \text{ for each open set } U \text{ containing } x\}. \tag{8}$$

Theorem 10. *An ideal topological space $(X, \tau, \mathcal{I})$ is θ-$\mathcal{I}$-irreducible if and only if $Cl^*_\theta(Cl^*(U)) = X$ for each nonempty open set U of X.*

Proof. Let $(X, \tau, \mathcal{I})$ be θ-$\mathcal{I}$-irreducible. Suppose that there exists a nonempty open set U such that $\operatorname{Cl}^*_\theta(\operatorname{Cl}^*(U)) \neq X$. Then, there exists a nonempty open set V such that $\operatorname{Cl}^*(V) \cap \operatorname{Cl}^*(U) = \varnothing$. Since $\operatorname{Cl}^*(\operatorname{Int}(U))$ and $\operatorname{Cl}^*(\operatorname{Int}(V))$ are R^*-$\mathcal{I}$-closed sets, $(X, \tau, \mathcal{I})$ is not θ-$\mathcal{I}$-irreducible.

Conversely, let $\operatorname{Cl}^*_\theta(\operatorname{Cl}^*(U)) = X$ for each nonempty open set U of X. Suppose that $(X, \tau, \mathcal{I})$ is not θ-$\mathcal{I}$-irreducible. There exist nonempty R^*-$\mathcal{I}$-closed sets F_1 and F_2 such that $F_1 \cap F_2 = \varnothing$. Then, $\operatorname{Int}(F_1)$ and $\operatorname{Int}(F_2)$ are nonempty open sets. Since $\operatorname{Int}(F_1) \neq \varnothing$, there exists an open set G such that $x \in G \subseteq F_1$, and hence $\operatorname{Cl}^*(G) \subseteq \operatorname{Cl}^*(F_1)$. This implies that $\operatorname{Cl}^*(G) \cap \operatorname{Cl}^*(F_2) \subseteq \operatorname{Cl}^*(F_1) \cap \operatorname{Cl}^*(F_2) = \operatorname{Cl}^*(\operatorname{Int}(F_1)) \cap \operatorname{Cl}^*(\operatorname{Int}(F_2)) = F_1 \cap F_2 = \varnothing$. Thus, $\operatorname{Cl}^*(G) \cap \operatorname{Cl}^*(F_2) = \varnothing$. Therefore, $x \notin \operatorname{Cl}^*_\theta(\operatorname{Cl}^*(F_2))$ and $\operatorname{Cl}^*_\theta(\operatorname{Cl}^*(\operatorname{Int}(F_2))) = \operatorname{Cl}^*_\theta(\operatorname{Cl}^*(F_2)) \neq X$. This is a contradiction. □

Theorem 11. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{I})$ is θ-$\mathcal{I}$-irreducible

(2) $Cl^*(U) \cap Cl^*(V) \neq \varnothing$ for every nonempty $U, V \in s\beta\mathcal{I}O(X, \tau)$

(3) $Cl^*(U) \cap Cl^*(V) \neq \varnothing$ for every nonempty $U, V \in s\mathcal{I}O(X, \tau)$

(4) $Cl^*(U) \cap Cl^*(V) \neq \varnothing$ for every nonempty $U, V \in \tau$

(5) $Cl^*(U) \cap Cl^*(V) \neq \varnothing$ for every nonempty $U, V \in s^*\mathcal{I}O(X, \tau)$

Proof

(1)$\Longrightarrow$(2): let U and V be nonempty strong β-$\mathcal{I}$-open sets. Then, we have $\operatorname{Cl}^*(U) = \operatorname{Cl}^*(\operatorname{Int}(\operatorname{Cl}^*(U)))$ and $\operatorname{Cl}^*(V) = \operatorname{Cl}^*(\operatorname{Int}(\operatorname{Cl}^*(V)))$. This implies that $\operatorname{Cl}^*(U)$ and $\operatorname{Cl}^*(V)$ are R^*-$\mathcal{I}$-closed sets. Since $(X, \tau, \mathcal{I})$ is θ-$\mathcal{I}$-irreducible, $\operatorname{Cl}^*(U) \cap \operatorname{Cl}^*(V) \neq \varnothing$.

(2)$\Longrightarrow$(3) and (3)$\Longrightarrow$(4) are obvious.

(4)$\Longrightarrow$(5): for a nonempty set $U \in s^*\mathcal{I}O(X, \tau)$, $\operatorname{Cl}^*(U) = \operatorname{Cl}^*(\operatorname{Int}(\operatorname{Cl}^*(U)))$ and $\operatorname{Int}(U) \neq \varnothing$. By (4), we have
$\varnothing \neq \operatorname{Cl}^*(\operatorname{Int}(U)) \cap \operatorname{Cl}^*(\operatorname{Int}(V)) = \operatorname{Cl}^*(U) \cap \operatorname{Cl}^*(V)$ for every nonempty $U, V \in s^*\mathcal{I}O(X, \tau)$.

(5)$\Longrightarrow$(1): this is obvious since $r^*\mathcal{I}C(X, \tau) \subseteq s^*\mathcal{I}O(X, \tau)$. □

Definition 9. Let A be a subset of an ideal topological space $(X, \tau, \mathcal{I})$. A point $x \in X$ is called a θ*-semi-*$\mathcal{I}$*-cluster point* of A if $A \cap \operatorname{Cl}^*(U) \neq \varnothing$ for every semi-$\mathcal{I}$-open set U containing x. The set of all θ-semi-$\mathcal{I}$-cluster points of A is called the θ*-semi-*$\mathcal{I}$*-closure* of A and is denoted by $s\operatorname{Cl}^*_\theta(A)$. A point $x \in X$ is called a θ*-semi-*$\mathcal{I}$*-interior point* of A if there exists a semi-$\mathcal{I}$-open set U containing x such that $\operatorname{Cl}^*(U) \subseteq A$. The set of all θ-semi-$\mathcal{I}$-interior points of A is called the θ*-semi-*$\mathcal{I}$*-interior* of A and is denoted by $s\operatorname{Int}^*_\theta(A)$.

Lemma 9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then, $sCl^*_\theta(A) \subseteq Cl^*_\theta(A)$ for every subset A of X.*

Theorem 12. *For an ideal topological space $(X, \tau, \mathcal{I})$, the following properties are equivalent:*

(1) $(X, \tau, \mathcal{I})$ is θ-$\mathcal{I}$-irreducible

(2) $sCl^*_\theta(Cl^*(U)) = X$ for every nonempty $U \in \tau$

(3) $sCl^*_\theta(Cl^*(U)) = X$ for every nonempty $U \in s\mathcal{I}O(X, \tau)$

(4) $Cl^*_\theta(Cl^*(U)) = X$ for every nonempty $U \in s\mathcal{I}O(X, \tau)$

Proof

(1)$\Longrightarrow$(2): suppose that there exists a nonempty open set U such that $s\mathrm{Cl}^*_\theta(\mathrm{Cl}^*(U)) \neq X$. Then, there exists a nonempty semi-$\mathcal{I}$-open set V such that $\mathrm{Cl}^*(U) \cap \mathrm{Cl}^*(V) = \varnothing$. Since $V \in s\mathcal{I}O(X,\tau)$, $\mathrm{Cl}^*(V) = \mathrm{Cl}^*(\mathrm{Int}(V)) \in r^*\mathcal{I}C(X,\tau)$, and hence $(X,\tau,\mathcal{I})$ is not θ-$\mathcal{I}$-irreducible.

(2)$\Longrightarrow$(3): for a nonempty set $U \in s\mathcal{I}O(X,\tau)$, we have $\mathrm{Int}(U) \neq \varnothing$, and hence $X = s\mathrm{Cl}^*_\theta(\mathrm{Cl}^*(\mathrm{Int}(U))) = s\mathrm{Cl}^*_\theta(\mathrm{Cl}^*(U))$.

(3)$\Longrightarrow$(4): it follows from Lemma 9.

(4)$\Longrightarrow$(1): since $\tau \subseteq s\mathcal{I}O(X,\tau)$, this is obvious by Theorem 10. □

Definition 10. An ideal topological space $(X,\tau,\mathcal{I})$ is said to be *semi-$\mathcal{I}$-Urysohn* if for each distinct points x and y of X, there exist $U, V \in s\mathcal{I}O(X,\tau)$ such that $x \in U$, $y \in V$ and $\mathrm{Cl}^*(U) \cap \mathrm{Cl}^*(V) = \varnothing$.

Definition 11. A function $f\colon (X,\tau,\mathcal{I}) \longrightarrow (Y,\sigma,\mathcal{J})$ is said to be *θ-$\mathcal{I}$-irresolute* if for each point $x \in X$ and each semi-$\mathcal{J}$-open set V containing $f(x)$, there exists a semi-$\mathcal{I}$-open set U containing x such that $f(\mathrm{Cl}^*(U)) \subseteq \mathrm{Cl}^*(V)$.

Theorem 13. *If $(X,\tau,\mathcal{I})$ is θ-$\mathcal{I}$-irreducible, $(Y,\sigma,\mathcal{J})$ is semi-$\mathcal{J}$-Urysohn, and $f\colon (X,\tau,\mathcal{I}) \longrightarrow (Y,\sigma,\mathcal{J})$ is θ-$\mathcal{I}$-irresolute, then f is constant.*

Proof. Suppose that there exist distinct points x, y of X such that $f(x) \neq f(y)$. Since $(Y,\sigma,\mathcal{J})$ is semi-$\mathcal{J}$-Urysohn, there exist $U, V \in s\mathcal{J}O(Y,\sigma)$ such that $f(x) \in U$, $f(y) \in V$ and $\mathrm{Cl}^*(U) \cap \mathrm{Cl}^*(V) = \varnothing$. By the θ-$\mathcal{I}$-irresoluteness of f, there exist $G, W \in s\mathcal{I}O(X,\tau)$ containing x and y, respectively, such that $f(\mathrm{Cl}^*(G)) \subseteq \mathrm{Cl}^*(U)$ and $f(\mathrm{Cl}^*(W)) \subseteq \mathrm{Cl}^*(V)$. Thus, $\mathrm{Cl}^*(G) \cap \mathrm{Cl}^*(W) = \varnothing$. By Theorem 11, this contradicts the assumption that $(X,\tau,\mathcal{I})$ is θ-$\mathcal{I}$-irreducible. □

Lemma 10. *If $f\colon (X,\tau,\mathcal{I}) \longrightarrow (Y,\sigma,\mathcal{J})$ is θ-$\mathcal{I}$-irresolute, then*

$$f\left(s\mathrm{Cl}^*_\theta(A)\right) \subseteq s\mathrm{Cl}^*_\theta(f(A)), \tag{9}$$

for every subset A of X.

Proof. Let A be a subset of X and $x \notin f^{-1}(s\mathrm{Cl}^*_\theta(f(A)))$. Then, $f(x) \notin s\mathrm{Cl}^*_\theta(f(A))$, and there exists a semi-$\mathcal{J}$-open set V containing $f(x)$ such that $\mathrm{Cl}^*(V) \cap f(A) = \varnothing$. Since f is θ-$\mathcal{I}$-irresolute, there exists a semi-$\mathcal{I}$-open set U containing x such that $f(\mathrm{Cl}^*(U)) \subseteq \mathrm{Cl}^*(V)$. Thus, $f(\mathrm{Cl}^*(U)) \cap f(A) = \varnothing$ and $\mathrm{Cl}^*(U) \cap A = \varnothing$. Therefore, $x \notin s\mathrm{Cl}^*_\theta(A)$. This shows that $s\mathrm{Cl}^*_\theta(A) \subseteq f^{-1}(s\mathrm{Cl}^*_\theta(f(A)))$. Consequently, we obtain $f(s\mathrm{Cl}^*_\theta(A)) \subseteq s\mathrm{Cl}^*_\theta(f(A))$. □

Theorem 14. *If $f\colon (X,\tau,\mathcal{I}) \longrightarrow (Y,\sigma,\mathcal{J})$ is a θ-$\mathcal{I}$-irresolute surjection and $(X,\tau,\mathcal{I})$ is θ-$\mathcal{I}$-irreducible, then $(Y,\sigma,\mathcal{J})$ is θ-$\mathcal{J}$-irreducible.*

Proof. Suppose that $(X,\tau,\mathcal{I})$ is θ-$\mathcal{I}$-irreducible. Let V be a nonempty semi-$\mathcal{J}$-open set of Y. Since f is surjective, there exists a point x of X such that $f(x) \in V$. By the θ-$\mathcal{I}$-irresoluteness of f, there exists a semi-$\mathcal{I}$-open set U containing x such that $f(\mathrm{Cl}^*(U)) \subseteq \mathrm{Cl}^*(V)$. Since $(X,\tau,\mathcal{I})$ is θ-$\mathcal{I}$-irreducible, by Theorem 12, we have $s\mathrm{Cl}_{\theta^*(\mathrm{Cl}^*(U))=X}$. By Lemma 10,

$$\begin{aligned} Y = f(X) &= f\left(s\mathrm{Cl}^*_\theta(\mathrm{Cl}^*(U))\right) \subseteq s\mathrm{Cl}^*_\theta\left(f(\mathrm{Cl}^*(U))\right) \\ &\subseteq s\mathrm{Cl}^*_\theta(\mathrm{Cl}^*(V)). \end{aligned} \tag{10}$$

It follows from Theorem 12 that $(Y,\sigma,\mathcal{J})$ is θ-$\mathcal{J}$-irreducible. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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