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Research Article

Icosahedral Group and Classification of PSL(2, Z)-Orbits of Real Quadratic Fields

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Reduced numbers play an important role in the study of modular group action on the $\mathrm{PSL}(2,\mathbb{Z})$-subset of $Q(\sqrt{m})/Q$. For this purpose, we define new notions of equivalent, cyclically equivalent, and similar G-circuits in $\mathrm{PSL}(2,\mathbb{Z})$-orbits of real quadratic fields. In particular, we classify $\mathrm{PSL}(2,\mathbb{Z})$-orbits of $Q(\sqrt{m})/Q = \cup_{k\in N} Q^*(\sqrt{k^2m})$ containing G-circuits of length 6 and determine that the number of equivalence classes of G-circuits of length 6 is ten. We also employ the icosahedral group to explore cyclically equivalence classes of circuits and similar G-circuits of length 6 corresponding to each of these ten circuits. This study also helps us in classifying reduced numbers lying in the $\mathrm{PSL}(2,\mathbb{Z})$-orbits.

1. Introduction

Let $n = k^2m$, where $k \in N$ and m is a square-free positive integer. Now, we define the following set $Q^*(\sqrt{n}) = \{(a+\sqrt{n})/c\colon a, b = (a^2-n)/c, c \in Z, c \neq 0 \text{ and } (a,b,c) = 1\}$ and its subset as $Q^*_{\mathrm{red}}(\sqrt{n}) = \{\gamma \in Q^*(\sqrt{n})\colon \gamma > 1 \text{ and } -1 < \overline{\gamma} < 0\}$. Then, $Q(\sqrt{m})/Q = \cup_{k\in N} Q^*(\sqrt{k^2m})$.

For $\gamma = (a+\sqrt{n})/c \in Q^*(\sqrt{n})$, its algebraic conjugate $\overline{\gamma} = (a-\sqrt{n})/c$ has different signs; then, γ is said to be an ambiguous number, that is, γ is an ambiguous number if and only if $a^2 < n$[1]. A quadratic irrational number γ is said to be reduced if $\gamma > 1$ and $-1 < \overline{\gamma} < 0$. It is obvious from the definition that every reduced number is ambiguous. Note that if γ is a reduced number, then $\overline{\gamma}$, $-\overline{\gamma}$, and $-\gamma$ are the ambiguous numbers but not reduced [1]. The modular group $\mathrm{PSL}(2,\mathbb{Z})$ is the group of all linear fractional transformations $z \longrightarrow (sz+r)/(uz+v)$ with $sv - ru = 1$, where s, r, u, and v are integers.

This group can be presented as $G = \langle x, y\colon x^2 = y^3 = 1\rangle$, where $x\colon z \longrightarrow -1/z$, $y\colon z \longrightarrow (z-1)/z$.

Throughout this paper, $p(q)$ denotes the number of partitions of q, whereas D_n stands for the dihedral group of order $2n$ and S_n stands for the symmetric group of order $n!$. It is easy to see that $|S_n| = ((n-1)!/2) \times 2n = |A_{n-1}| \times |D_n|$.

Coset diagrams for the G-orbit acting on the real quadratic field give some interesting information. A coset diagram is a graph consisting of vertices and edges. It depicts a permutation representation of the modular group G, the 3-cycles of y are denoted by three vertices of a triangle permuted anticlockwise by y, and the two vertices which are interchanged by x are joined by an edge.

In [2], it was proved that the ambiguous numbers in the orbit $\gamma^G, \gamma \in Q^*(\sqrt{n})$, form a single closed path (called G-circuit or simply circuit), and it is the only circuit contained in the coset diagram for the orbit γ^G. The number of disjoint orbits is γ^G, where $\gamma \in Q^*(\sqrt{n})$ is equal to the number of circuits in the coset diagram under the action of G on $Q^*(\sqrt{n})$. Thus, it becomes interesting to classify the circuit.

If $l'_1, l'_3, l'_5, \ldots, l'_{2t-1}$ and $l'_2, l'_4, l'_6, \ldots, l'_{2t}$ are two sequences of positive integers, then by a circuit $[l'_1, l'_2, l'_3, l'_4, \ldots, l'_{2t-1}, l'_{2t}]$, we shall mean the circuit in which l'_{2i-1} triangles have one vertex outside the circuit and l'_{2i} triangles have one vertex inside the circuit, where $1 \le i \le t$.

This circuit induces an element $g = (xy^2)^{l'_1}(xy)^{l'_2}(xy^2)^{l'_3}\ldots(xy^2)^{l'_{2t-1}}(xy)^{l'_{2t}}$ and fixes a particular vertex of a triangle lying on the circuit. Throughout this paper, G-circuit (resp. G-orbit) will be simply denoted by circuit (resp. orbit). The concept of the circuit grew out of the study of group action on $Q(\sqrt{m}) \cup \{\infty\}$ and the study of G-subsets $Q^*(\sqrt{n}) = \{(a+\sqrt{n})/c\colon a, b = (a^2-n)/c, c \in Z, c \neq 0 \text{ and } (a, b, c) = 1\}$.

In this paper, we define what a circuit of specific length is and we classify nonequivalent circuits of length 6 so as to classify orbits containing these circuits. We also consider some of the elementary concepts associated with equivalent circuits, cyclically equivalent circuits, and similar circuits which are introduced to explore transitive G-subsets (called orbits) of $Q^*(\sqrt{n})$.

2. Preliminaries

In this section, necessary definition of an equivalent circuits will be given. The definition of an equivalent circuit, that is now standard and was in formulation since long ago. It was required of course, a definition that is as broad as possible, so that it would include all special cases of the various examples that are useful in group action, G-subset and orbit. However, the definition was also required to be narrow enough that the standard theorems corresponding to these concepts like partition of positive integer q would help in general. The definition finally settled on may seem a bit abstract, but as we work through various ways of determining orbits, we will get a better feeling for what the concept means.

Definition 1. Two circuits $[l'_1, l'_2, l'_3, l'_4, \ldots, l'_{2t-1}, l'_{2t}]$ and $[l_1, l_2, l_3, l_4, l_5, l_6, \ldots, l_{2t-1}, l_{2t}]$ are said to be equivalent iff $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, \ldots, l'_{(2t-1)\theta}, l'_{(2t)\theta}] = [l_1, l_2, l_3, l_4, l_5, l_6, \ldots, l_{2t-1}, l_{2t}]$, where $\theta \in S_{2t}$. That is, the circuits are equivalent to $[l'_1, l'_2, l'_3, l'_4, \ldots, l'_{2t-1}, l'_{2t}]$ if and only if they are obtained just by permuting the entries l'_1, l'_2, l'_3, …, l'_{2t}. Notation for equivalent is " $\sim$." It is easy to see that being equivalent of circuits is an equivalence relation. Thus, a property is possessed by one circuit that is also possessed by all equivalent circuits. Such properties which are preserved under equivalent are called equivalent properties or circuit invariant.

Two circuits $[l'_1, l'_2, l'_3, l'_4, \ldots, l'_{2t-1}, l'_{2t}]$ and $[l_1, l_2, l_3, l_4, l_5, l_6, \ldots, l_{2t-1}, l_{2t}]$ are said to be cyclically equivalent if and only if the circuit $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, \ldots, l'_{(2t-1)\theta}, l'_{(2t)\theta}] = [l_1, l_2, l_3, l_4, l_5, l_6, \ldots, l_{2t-1}, l_{2t}]$, where $\theta \in D_t$. Notation for cyclically equivalent is " $\sim_c$." It is easy to see that being cyclically equivalent of circuits is an equivalence relation.

Two circuits are said to be similar if they represent the same orbit. That is, two circuits $[l'_1, l'_2, l'_3, l'_4, \ldots, l'_{2t-1}, l'_{2t}]$ and $[l_1, l_2, l_3, l_4, l_5, l_6, \ldots, l_{2t-1}, l_{2t}]$ are said to be similar if and only if $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, \ldots, l'_{(2t-1)\theta}, l'_{(2t)\theta}] = [l_1, l_2, l_3, l_4, l_5, l_6, \ldots, l_{2t-1}, l_{2t}]$, where $\theta \in C_t = \langle(1\,3\,5\ldots 2t-1), (2\,4\,6\ldots 2t)\rangle$. Notation for similarly equivalent is " $\sim_s$." It is easy to see that being similar circuits is an equivalence relation.

It is interesting to note that the orbit containing circuit $[l'_1, l'_2, l'_3, l'_4, \ldots, l'_{2t-1}, l'_{2t}]$ has exactly $2(l'_1 + l'_2 + l'_3 + \cdots + l'_{2t-1} + l'_{2t})$ ambiguous numbers, while this circuit consists of only t number of reduced numbers. Thus, studying orbits with the help of reduced numbers is fruitful and economical. Throughout this paper, l'_1, l'_2, l'_3, l'_4, l'_5, and l'_6 denote distinct positive integers, and the expression $(d + \sqrt{d^2 + 4ef})/2f$ is replaced by $((d/h) + \sqrt{(d^2 + 4ef)/h^2})/2f/h$ when $(d, (2e, 2f)) = h > 1$. Classification of nonequivalent circuits and cyclically equivalent circuits plays a significant role in determining the orbits of $Q(\sqrt{m})/Q$ because with this, the task of finding orbits is simplified.

In [3], Aslam and Asim found G-subsets of $Q^*(\sqrt{k^2m}) \subseteq Q(\sqrt{m})/Q = \cup_{k\in N} Q^*(\sqrt{k^2m})$. Since G-subsets may or may not be transitive, it becomes interesting to explore transitive G-subsets called orbits. Reduced quadratic irrational numbers and types of G-circuits with length four by modular group and the orbits of $Q^*(\sqrt{p})$, $p = 3 \pmod 4$ under the action of modular group have been studied in [4, 5].

3. Materials and Methods

The following results of [6–8] are used in the sequel.

Lemma 1 (see [7]). *If γ^G has a circuit $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$, then $[l'_2, l'_3, l'_4, l'_5, l'_6, l'_1]$, $[l'_1, l'_6, l'_5, l'_4, l'_3, l'_2]$, and $[l'_6, l'_5, l'_4, l'_3, l'_2, l'_1]$ are the circuits of $(-\gamma)^G$, $(-\overline{\gamma})^G$, and $(\overline{\gamma})^G$, respectively.*

Lemma 2 (see [7]). *If $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$ is the circuit contained in the orbit γ^G, then γ^G, $(-\gamma)^G$, $(-\overline{\gamma})^G$, and $(\overline{\gamma})^G$ are all distinct.*

Lemma 3 (see [8]). *For a given sequence of positive integers $l'_1, l'_2, l'_3, \ldots, l'_{2t}$, there does not exist a circuit which has a period of length $2t'$, where t' divides t.*

Lemma 4 (see [6]). *The number of different arrangements of q objects of which q_1 is alike, q_2 is alike, q_3 is alike, …, q_r is alike is $q!/q_1!q_2!q_3!\ldots q_r!$, where $q_1 + q_2 + q_3 + \cdots + q_r = q$.*

3.1. Classification of Circuits of Length Six. Circuits play an important role in the study of modular group acting on the quadratic field.

We start this section with a consideration of finding the nonequivalent relation of circuits in a specific length q. Given the positive integer q, we say that the sequence of positive integers $q_1, q_2, q_3, q_4, \ldots, q_{r-1}, q_r$ with $q_1 \geq q_2 \geq q_3 \geq q_4 \geq \cdots \geq q_{r-1} \geq q_r$ constitutes a partition of q if $q_1 + q_2 + q_3 + \cdots + q_r = q$.

Theorem 1. *The number of equivalence classes of length $q > 2$ is exactly $p(q) - 1$, where $p(q)$ denotes the partition of q.*

Proof. Let $p(q)$ denote the partition of q, and we are looking in determining all equivalence classes of equivalent circuits of length q. For a given circuit of length q, to find all equivalence classes of equivalent circuits of length q, we adopt partitions of $q_1 + q_2 + q_3 + \cdots + q_r = q$ in the sense that q_1 entries are alike, q_2 entries are alike, q_3 entries are

alike, …, q_r entries are alike, where $r \geq 2$. We get nonequivalent circuits $[l'_1, l'_1, l'_1, \ldots, l'_1, l'_2, l'_2, \ldots, l'_2, \ldots, l'_r, l'_r, \ldots, l'_r]$ of length q corresponding to each partition of q, where l'_i repeats q_i times, $1 \leq i \leq r$.

Here, $r \neq 1$ because if $q = q_1$, then the circuit corresponding to this partition is $[l'_1, l'_1, l'_1, \ldots, l'_1]$, where l'_1 repeats q times. This circuit of length $q > 2$ is not possible by Lemma 3. So, distinct classes of the equivalent circuit of length $q > 2$ are exactly $p(q) - 1$. □

Remark 1. For $q = 2$, there are precisely two partitions, namely, 2 and $1 + 1$. Circuits of length 2 corresponding to these partitions are $[l'_1, l'_1]$ and $[l'_1, l'_2]$.

Note 1. We can get all other circuits which are equivalent to $[l'_1, l'_2, l'_3, \ldots, l'_r]$ by just permuting $l'_1, l'_2, l'_3, \ldots, l'_r$.

In the following theorems, we discuss equivalence classes of circuits, cyclically equivalence classes of circuits, equivalent circuits, cyclically equivalent circuits, and similar circuits of length 6.

Corollary 1. *There are precisely ten nonequivalent circuits of length six.*

Proof. Let $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ be different positive integers. Then, by Theorem 1, we have ten nonequivalent circuits, namely, $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$, $[l'_1, l'_1, l'_2, l'_3, l'_4, l'_5]$, $[l'_1, l'_1, l'_1, l'_2, l'_3, l'_4]$, $[l'_1, l'_1, l'_2, l'_2, l'_3, l'_4]$, $[l'_1, l'_1, l'_2, l'_2, l'_3, l'_3]$, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_3]$, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_3]$, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_1, l'_1, l'_1, l'_1, l'_2]$, and $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_2]$, corresponding to the number $p(6) - 1$, namely, $1 + 1 + 1 + 1 + 1 + 1$, $2 + 1 + 1 + 1 + 1$, $3 + 1 + 1 + 1$, $2 + 2 + 1 + 1$, $2 + 2 + 2$, $3 + 2 + 1$, $4 + 1 + 1$, $4 + 2$, $5 + 1$, and $3 + 3$, respectively. The circuit corresponding to summand 6 is $[l'_1, l'_1, l'_1, l'_1, l'_1, l'_1]$. This circuit is not possible by Lemma 3. So, these are the only ten nonequivalent circuits of length 6.

The notation used in this paper for equivalence classes of circuits of length 6 is E_{T_i}, and the number of circuits equivalent to T_i is denoted by $|E_{T_i}|$. Similarly, the notation for cyclically equivalent classes is $E^c_{T_i}$, and $|E^c_{T_i}|$ denotes the number of circuits cyclically equivalent to T_i. The number of distinct orbits corresponding to E_{T_i} is denoted by $|O_{T_i}|$. Furthermore, each cyclically equivalent class $E^c_{T_i}$ is discussed in each corresponding relevant corollary. □

Corollary 2. *There are 720 equivalent circuits of length 6 in which all numbers are different.*

Proof. It is well known that $S_n = \{(D_n)\varphi\colon \varphi \in A_{n-1}\}$. In our case, $S_6 = \{(D_6)\varphi\colon \varphi \in A_5\}$. We know that there are $6! = 720$ arrangements of six different numbers $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ taken all at a time, and so circuits corresponding to these arrangements are $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, l'_{(5)\theta}, l'_{(6)\theta}]$ for each $\theta \in S_6$. Hence the proof. □

Corollary 3. *If $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ are distinct positive integers, then there exist 60 cyclically equivalent classes $E^c_{T_1}$.*

Proof. Let $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ be different positive integers. It is well known that the icosahedral group is isomorphic to alternating group A_5. Now, the cyclically equivalent classes $E^c_{T_1}$ are obtained by $E^c_{[l'_{(1)\varphi}, l'_{(2)\varphi}, l'_{(3)\varphi}, l_{(4)\varphi}', l'_{(5)\varphi}, l'_{(6)\varphi}]}$ for each $\varphi \in A_5$. There are exactly 60 cyclically equivalent classes, namely, $E^c_{[l'_1,l'_2,l'_3,l'_4,l'_5,l'_6]}$, $E^c_{[l'_2,l'_1,l'_4,l'_3,l'_5,l'_6]}$, $E^c_{[l'_2,l'_1,l'_5,l'_4,l'_3,l'_6]}$, $E^c_{[l'_4,l'_3,l'_2,l'_1,l'_5,l'_6]}$, $E^c_{[l'_5,l'_3,l'_2,l'_4,l'_1,l'_6]}$, $E^c_{[l'_3,l'_4,l'_1,l'_2,l'_5,l'_6]}$, $E^c_{[l'_5,l'_4,l'_3,l'_2,l'_1,l'_6]}$, $E^c_{[l'_5,l'_2,l'_4,l'_3,l'_1,l'_6]}$, $E^c_{[l'_4,l'_2,l'_5,l'_1,l'_3,l'_6]}$, $E^c_{[l'_2,l'_1,l'_3,l'_5,l'_4,l'_6]}$, $E^c_{[l'_3,l'_2,l'_1,l'_5,l'_4,l'_6]}$, $E^c_{[l'_3,l'_5,l'_1,l'_4,l'_2,l'_6]}$, $E^c_{[l'_4,l'_5,l'_3,l'_1,l'_2,l'_6]}$, $E^c_{[l'_1,l'_4,l'_5,l'_2,l'_3,l'_6]}$, $E^c_{[l'_1,l'_5,l'_4,l'_3,l'_2,l'_6]}$, $E^c_{[l'_2,l'_4,l'_3,l'_1,l'_5,l'_6]}$, $E^c_{[l'_2,l'_5,l'_3,l'_4,l'_1,l'_6]}$, $E^c_{[l'_3,l'_2,l'_4,l'_1,l'_5,l'_6]}$, $E^c_{[l'_3,l'_2,l'_5,l'_4,l'_1,l'_6]}$, $E^c_{[l'_4,l'_2,l'_3,l'_5,l'_1,l'_6]}$, $E^c_{[l'_4,l'_3,l'_1,l'_5,l'_2,l'_6]}$, $E^c_{[l'_5,l'_3,l'_1,l'_4,l'_2,l'_6]}$, $E^c_{[l'_5,l'_4,l'_2,l'_3,l'_1,l'_6]}$, $E^c_{[l'_1,l'_4,l'_2,l'_5,l'_3,l'_6]}$, $E^c_{[l'_1,l'_5,l'_4,l'_3,l'_2,l'_6]}$, $E^c_{[l'_1,l'_3,l'_5,l'_4,l'_2,l'_6]}$, $E^c_{[l'_1,l'_2,l'_5,l'_3,l'_4,l'_6]}$, $E^c_{[l'_5,l'_2,l'_3,l'_1,l'_4,l'_6]}$, $E^c_{[l'_1,l'_3,l'_4,l'_2,l'_5,l'_6]}$, $E^c_{[l'_1,l'_5,l'_3,l'_2,l'_4,l'_6]}$, $E^c_{[l'_1,l'_4,l'_3,l'_5,l'_2,l'_6]}$, $E^c_{[l'_5,l'_2,l'_1,l'_4,l'_3,l'_6]}$, $E^c_{[l'_1,l'_5,l'_2,l'_4,l'_3,l'_6]}$, $E^c_{[l'_2,l'_3,l'_4,l'_5,l'_1,l'_6]}$, $E^c_{[l'_2,l'_4,l'_5,l'_3,l'_1,l'_6]}$, $E^c_{[l'_3,l'_4,l'_2,l'_5,l'_1,l'_6]}$, $E^c_{[l'_3,l'_5,l'_4,l'_2,l'_1,l'_6]}$, $E^c_{[l'_4,l'_3,l'_5,l'_2,l'_1,l'_6]}$, $E^c_{[l'_4,l'_5,l'_2,l'_3,l'_1,l'_6]}$, $E^c_{[l'_3,l'_1,l'_4,l'_5,l'_2,l'_6]}$, $E^c_{[l'_4,l'_1,l'_5,l'_3,l'_2,l'_6]}$, $E^c_{[l'_4,l'_3,l'_1,l'_5,l'_2,l'_6]}$, $E^c_{[l'_5,l'_3,l'_4,l'_1,l'_2,l'_6]}$, $E^c_{[l'_3,l'_4,l'_5,l'_1,l'_2,l'_6]}$, $E^c_{[l'_4,l'_1,l'_2,l'_5,l'_3,l'_6]}$, $E^c_{[l'_5,l'_4,l'_2,l'_1,l'_3,l'_6]}$, $E^c_{[l'_5,l'_1,l'_4,l'_2,l'_3,l'_6]}$, $E^c_{[l'_2,l'_3,l'_5,l'_1,l'_4,l'_6]}$, $E^c_{[l'_3,l'_5,l'_2,l'_1,l'_4,l'_6]}$, $E^c_{[l'_3,l'_1,l'_5,l'_2,l'_4,l'_6]}$, $E^c_{[l'_5,l'_4,l'_3,l'_1,l'_2,l'_6]}$, $E^c_{[l'_2,l'_5,l'_1,l'_3,l'_4,l'_6]}$, $E^c_{[l'_2,l'_5,l'_4,l'_1,l'_3,l'_6]}$, $E^c_{[l'_5,l'_1,l'_2,l'_3,l'_4,l'_6]}$, $E^c_{[l'_1,l'_3,l'_2,l'_5,l'_4,l'_6]}$, $E^c_{[l'_5,l'_4,l'_1,l'_3,l'_2,l'_6]}$, $E^c_{[l'_2,l'_4,l'_1,l'_5,l'_3,l'_6]}$, and $E^c_{[l'_4,l'_5,l'_1,l'_2,l'_3,l'_6]}$. □

Corollary 4. *If $l'_1, l'_2, l'_3, l'_4, l'_5, l'_6$ are distinct positive integers, then each cyclically equivalent class contains 12 cyclically equivalent circuits.*

Proof. Circuits cyclically equivalent to circuit $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$ are $[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, l'_{(5)\theta}, l'_{(6)\theta}]$ for each $\theta \in D_6$ which is shown in Table 1. Similarly, we can find cyclically equivalent classes corresponding to the remaining circuits.

Circuits of length 6 which are given in Table 1 are cyclically equivalent. Moreover, by Lemma 1, $[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6] \sim_s [l'_3, l'_4, l'_5, l'_6, l'_1, l'_2] \sim_s [l'_5, l'_6, l'_1, l'_2, l'_3, l'_4]$ is the circuit contained in γ^G, $[l'_2, l'_3, l'_4, l'_5, l'_6, l'_1] \sim_s [l'_4, l'_5, l'_6, l'_1, l'_2, l'_3] \sim_s [l'_6, l'_1, l'_2, l'_3, l'_4, l'_5]$ is the circuit contained in $(-\gamma)^G$, $[l'_1, l'_6, l'_5, l'_4, l'_3, l'_2] \sim_s [l'_5, l'_4, l'_3, l'_2, l'_1, l'_6] \sim_s [l'_3, l'_2, l'_1, l'_6, l'_5, l'_4]$ is the circuit contained in $(-\overline{\gamma})^G$, and $[l'_6, l'_5, l'_4, l'_3, l'_2, l'_1] \sim_s [l'_4, l'_3, l'_2, l'_1, l'_6, l'_5] \sim_s [l'_2, l'_1, l'_6, l'_5, l'_4, l'_3]$ is the circuit contained in $(\overline{\gamma})^G$.

From Table 1, it is easy to see that the effect of permutation on the circuit is the same as to change the places of the circuit accordingly. So, if the circuit in which at least two entries are the same, we change places of circuits according to permutation. As each circuit of length 6 contains 3 reduced numbers, each cyclically equivalent class contains 12 reduced numbers. Since there are 60 cyclically equivalent classes, each class contains 12 reduced numbers, so reduced numbers used in E_{T_1} are $12 \times 60 = 720$ which equals to $|E_{T_1}|$.

Table 2 is of considerable utility because it provides us with the exact number of circuits of length 6 and hence the number of G-orbits of $Q^*(\sqrt{n})$.

The following corollaries are an immediate consequence of Theorem 1. □

Corollary 5. *There are 6 equivalent circuits of length 6 in which 5 numbers are alike and one number is different.*

Proof. By Lemma 4, the number of equivalent circuits of length 6 in which 5 numbers are alike is $6!/5!1! = 6$. These 6

Table 1: Cyclically equivalent circuits of length 6.

θ	$[l'_{(1)\theta}, l'_{(2)\theta}, l'_{(3)\theta}, l'_{(4)\theta}, l'_{(5)\theta}, l'_{(6)\theta}]$
$e = (1)(2)(3)(4)(5)(6)$	$[l'_1, l'_2, l'_3, l'_4, l'_5, l'_6]$
$\alpha = (123456)$	$[l'_2, l'_3, l'_4, l'_5, l'_6, l'_1]$
$\alpha^2 = (135)(246)$	$[l'_3, l'_4, l'_5, l'_6, l'_1, l'_2]$
$\alpha^3 = (14)(25)(36)$	$[l'_4, l'_5, l'_6, l'_1, l'_2, l'_3]$
$\alpha^4 = (153)(264)$	$[l'_5, l'_6, l'_1, l'_2, l'_3, l'_4]$
$\alpha^5 = (165432)$	$[l'_6, l'_1, l'_2, l'_3, l'_4, l'_5]$
$\beta = (16)(25)(34)$	$[l'_6, l'_5, l'_4, l'_3, l'_2, l'_1]$
$\alpha\beta = (15)(24)$	$[l'_5, l'_4, l'_3, l'_2, l'_1, l'_6]$
$\alpha^2\beta = (14)(23)(56)$	$[l'_4, l'_3, l'_2, l'_1, l'_6, l'_5]$
$\alpha^3\beta = (13)(46)$	$[l'_3, l'_2, l'_1, l'_6, l'_5, l'_4]$
$\alpha^4\beta = (12)(45)(36)$	$[l'_2, l'_1, l'_6, l'_5, l'_4, l'_3]$
$\alpha^5\beta = (35)(26)$	$[l'_1, l'_6, l'_5, l'_4, l'_3, l'_2]$

circuits are $[l'_1, l'_1, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_1]$, $[l'_1, l'_2, l'_1, l'_1, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_1]$, and $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_1]$. Clearly, all these types are cyclically equivalent as well. Moreover, by Lemma 1, $[l'_1, l'_1, l'_1, l'_1, l'_1, l'_2] \sim_s [l'_1, l'_2, l'_1, l'_1, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_1, l'_2, l'_1, l'_1]$ is the circuit contained in $\gamma^G = (-\overline{\gamma})^G$, and $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_1, l'_1, l'_2, l'_1] \sim_s [l'_1, l'_1, l'_2, l'_1, l'_1, l'_1]$ is the circuit contained in $(\overline{\gamma})^G = (-\gamma)^G$. In this situation, there is only one cyclically equivalent class, namely, $E^c_{[l'_1,l'_1,l'_1,l'_1,l'_1,l'_2]}$. □

Corollary 6. *There are 15 equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are alike.*

Proof. By Lemma 4, the number of equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are alike is 6!/4!2! = 15. These possible 15 circuits are $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_2]$, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_1]$, $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_2, l'_1, l'_2, l'_1]$, $[l'_1, l'_1, l'_2, l'_2, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_2, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_2, l'_1, l'_1, l'_2, l'_1]$, $[l'_1, l'_2, l'_1, l'_2, l'_1, l'_1]$, $[l'_1, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_1]$, $[l'_2, l'_1, l'_2, l'_1, l'_1, l'_1]$, and $[l'_2, l'_1, l'_1, l'_2, l'_1, l'_1]$, which are equivalent circuits.

In the aforementioned circuits, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_1]$, $[l'_1, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_2, l'_2, l'_1, l'_1]$, and $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_2]$ are cyclically equivalent. Moreover, by Lemma 1, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_2] \sim_s [l'_1, l'_1, l'_2, l'_2, l'_1, l'_1] \sim_s [l'_2, l'_2, l'_1, l'_1, l'_1, l'_1]$ is the circuit contained in $\alpha^G = (\overline{\alpha})^G$, and $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_2] \sim_s [l'_1, l'_2, l'_2, l'_1, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_1, l'_2, l'_2, l'_1]$ is the circuit contained in $(-\overline{\alpha})^G = (-\alpha)^G$.

Also, $[l'_2, l'_1, l'_1, l'_2, l'_1, l'_1]$, $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_2]$, and $[l'_1, l'_2, l'_1, l'_1, l'_2, l'_1]$ are cyclically equivalent. Moreover, by Lemma 1, $[l'_2, l'_1, l'_1, l'_2, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_2, l'_1, l'_1, l'_2] \sim_s [l'_1, l'_2, l'_1, l'_1, l'_2, l'_1]$ is the circuit contained in $(\beta)^G = (\overline{\beta})^G = (-\beta)^G = (-\overline{\beta})^G$.

These circuits $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_2]$, $[l'_1, l'_1, l'_2, l'_1, l'_2, l'_1]$, $[l'_1, l'_2, l'_1, l'_2, l'_1, l'_1]$, $[l'_2, l'_1, l'_2, l'_1, l'_1, l'_1]$, $[l'_1, l'_2, l'_1, l'_1, l'_1, l'_2]$, and $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_1]$ are cyclically equivalent. Moreover, by Lemma 1, $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_2] \sim_s [l'_1, l'_2, l'_1, l'_2, l'_1, l'_1] \sim_s [l'_1, l'_2, l'_1, l'_1, l'_1, l'_2]$ is the circuit contained in $(\gamma)^G = (-\overline{\gamma})^G$, and $[l'_1, l'_1, l'_2, l'_1, l'_2, l'_1] \sim_s [l'_2, l'_1, l'_2, l'_1, l'_1, l'_1] \sim_s [l'_2, l'_1, l'_1, l'_1, l'_2, l'_1]$ is the circuit contained in $(-\gamma)^G = (\overline{\gamma})^G$.

Table 3 summarizes all the information. □

Corollary 7. *There are 18 equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are alike.*

Proof. By Lemma 4, the number of possible equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are alike is 6!/3!3! = 20. These possible 20 circuits are given by $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_2]$, $[l'_1, l'_1, l'_2, l'_1, l'_2, l'_2]$, $[l'_1, l'_1, l'_2, l'_2, l'_1, l'_2]$, $[l'_1, l'_1, l'_2, l'_2, l'_2, l'_1]$, $[l'_1, l'_2, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_2, l'_1, l'_2, l'_1, l'_2]$, $[l'_1, l'_2, l'_1, l'_2, l'_2, l'_1]$, $[l'_1, l'_2, l'_2, l'_1, l'_1, l'_2]$, $[l'_1, l'_2, l'_2, l'_1, l'_2, l'_1]$, $[l'_1, l'_2, l'_2, l'_2, l'_1, l'_1]$, $[l'_2, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_2, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_2, l'_1]$, $[l'_2, l'_1, l'_2, l'_1, l'_2, l'_1]$, $[l'_2, l'_1, l'_1, l'_2, l'_2, l'_1]$, $[l'_2, l'_1, l'_1, l'_2, l'_1, l'_2]$, $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_2, l'_1, l'_2, l'_2, l'_1, l'_1]$, and $[l'_2, l'_1, l'_2, l'_1, l'_1, l'_2]$ which are equivalent circuits. In the aforementioned circuits, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_2]$, $[l'_1, l'_1, l'_2, l'_2, l'_2, l'_1]$, $[l'_1, l'_2, l'_2, l'_2, l'_1, l'_1]$, $[l'_2, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_1, l'_2]$, and $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_2]$ are cyclically equivalent. Moreover, by Lemma 1, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_2] \sim_s [l'_1, l'_2, l'_2, l'_2, l'_1, l'_1] \sim_s [l'_2, l'_2, l'_1, l'_1, l'_1, l'_2]$ is the circuit of $\alpha^G = (-\overline{\alpha})^G$, and $[l'_2, l'_1, l'_1, l'_1, l'_2, l'_2] \sim_s [l'_1, l'_1, l'_2, l'_2, l'_2, l'_1] \sim_s [l'_2, l'_2, l'_2, l'_1, l'_1, l'_1]$ is the circuit of $(\overline{\alpha})^G = (-\alpha)^G$. Also, circuits $[l'_1, l'_1, l'_2, l'_2, l'_1, l'_2]$, $[l'_1, l'_2, l'_1, l'_1, l'_2, l'_2]$, $[l'_2, l'_2, l'_1, l'_2, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_2, l'_1, l'_2]$, $[l'_2, l'_2, l'_1, l'_1, l'_2, l'_1]$, $[l'_1, l'_1, l'_2, l'_1, l'_2, l'_2]$, $[l'_1, l'_2, l'_2, l'_1, l'_1, l'_2]$, $[l'_1, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_1, l'_2, l'_1, l'_1, l'_2]$, $[l'_2, l'_1, l'_1, l'_2, l'_2, l'_1]$, and $[l'_1, l'_2, l'_1, l'_2, l'_2, l'_1]$ are cyclically equivalent. Moreover, by Lemma 1, $[l'_1, l'_1, l'_2, l'_2, l'_1, l'_2] \sim_s [l'_1, l'_2, l'_1, l'_1, l'_2, l'_2] \sim_s [l'_2, l'_2, l'_1, l'_2, l'_1, l'_1]$ is the circuit of β^G, $[l'_1, l'_2, l'_2, l'_1, l'_2, l'_1] \sim_s [l'_2, l'_1, l'_2, l'_1, l'_1, l'_2] \sim_s [l'_2, l'_1, l'_1, l'_2, l'_2, l'_1]$ is the circuit contained in $(-\beta)^G$, $[l'_1, l'_2, l'_1, l'_2, l'_2, l'_1] \sim_s [l'_1, l'_2, l'_2, l'_1, l'_1, l'_2] \sim_s [l'_2, l'_1, l'_1, l'_2, l'_1, l'_2]$ is the circuit of $(-\overline{\beta})^G$, and $[l'_2, l'_1, l'_2, l'_2, l'_1, l'_1] \sim_s [l'_2, l'_2, l'_1, l'_1, l'_2, l'_1] \sim_s [l'_1, l'_1, l'_2, l'_1, l'_2, l'_2]$ is the circuit contained in $(\overline{\beta})^G$. Moreover, $[l'_1, l'_2, l'_1, l'_2, l'_1, l'_2]$ and $[l'_2, l'_1, l'_2, l'_1, l'_2, l'_1]$ are not possible by Lemma 3. Table 4 summarizes all the information. □

Corollary 8. *There are 30 equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are different.*

Proof. By Lemma 4, the number of equivalent circuits of length 6 in which 4 numbers are alike and 2 numbers are different is 6!/4!1!1! = 30. These 30 circuits are $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_3]$, $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_3]$, $[l'_1, l'_1, l'_1, l'_2, l'_3, l'_1]$, $[l'_1, l'_1, l'_1, l'_3, l'_1, l'_2]$, $[l'_1, l'_1, l'_1, l'_1, l'_3, l'_2]$, $[l'_1, l'_1, l'_3, l'_2, l'_1, l'_1]$, $[l'_1, l'_1, l'_3, l'_1, l'_2, l'_1]$, $[l'_1, l'_1, l'_3, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_1, l'_3, l'_2, l'_1]$, $[l'_1, l'_1, l'_2, l'_3, l'_1, l'_1]$, $[l'_1, l'_1, l'_2, l'_1, l'_3, l'_1]$, $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_3]$, $[l'_1, l'_2, l'_1, l'_1, l'_1, l'_3]$, $[l'_1, l'_2, l'_1, l'_1, l'_3, l'_1]$, $[l'_1, l'_2, l'_1, l'_3, l'_1, l'_1]$, $[l'_1, l'_2, l'_3, l'_1, l'_1, l'_1]$, $[l'_1, l'_3, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_3, l'_1, l'_1, l'_2, l'_1]$, $[l'_1, l'_3, l'_1, l'_2, l'_1, l'_1]$, $[l'_1, l'_3, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_3]$, $[l'_2, l'_1, l'_1, l'_1, l'_3, l'_1]$, $[l'_2, l'_1, l'_1, l'_3, l'_1, l'_1]$, $[l'_2, l'_1, l'_3, l'_1, l'_1, l'_1]$, $[l'_2, l'_3, l'_1, l'_1, l'_1, l'_1]$, $[l'_3, l'_1, l'_1, l'_1, l'_1, l'_2]$, $[l'_3, l'_1, l'_1, l'_1, l'_2, l'_1]$, $[l'_3, l'_1, l'_1, l'_2, l'_1, l'_1]$, $[l'_3, l'_1, l'_2, l'_1, l'_1, l'_1]$, and $[l'_3, l'_2, l'_1, l'_1, l'_1, l'_1]$.

In the aforementioned circuits, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_3]$, $[l'_1, l'_1, l'_2, l'_3, l'_1, l'_1]$, $[l'_2, l'_3, l'_1, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_1, l'_2, l'_3, l'_1]$, $[l'_1, l'_2, l'_3, l'_1, l'_1, l'_1]$, $[l'_3, l'_1, l'_1, l'_1, l'_1, l'_2]$, $[l'_3, l'_2, l'_1, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_3, l'_2, l'_1, l'_1]$, $[l'_1, l'_1, l'_1, l'_1, l'_3, l'_2]$, $[l'_1, l'_3, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_1, l'_3]$, and $[l'_1, l'_1, l'_1, l'_3, l'_2, l'_1]$ are cyclically equivalent. By Lemma 1, $[l'_1, l'_1, l'_1, l'_1, l'_2, l'_3] \sim_s [l'_1, l'_1, l'_2, l'_3, l'_1, l'_1] \sim_s [l'_2, l'_3, l'_1, l'_1, l'_1, l'_1]$ is the circuit of α^G, $[l'_1, l'_1, l'_1, l'_2, l'_3, l'_1] \sim_s [l'_1, l'_2, l'_3, l'_1, l'_1, l'_1] \sim_s [l'_3, l'_1, l'_1, l'_1, l'_1, l'_2]$ is the circuit contained in $(-\alpha)^G$, $[l'_3, l'_2, l'_1, l'_1, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_3, l'_2, l'_1, l'_1] \sim_s [l'_1, l'_1, l'_1, l'_1, l'_3, l'_2]$ is the circuit of $(\overline{\alpha})^G$, and $[l'_1, l'_3, l'_2, l'_1, l'_1, l'_1] \sim_s [l'_2, l'_1, l'_1, l'_1, l'_1, l'_3] \sim_s [l'_1, l'_1, l'_1, l'_3, l'_2, l'_1]$ is the circuit contained in $(-\overline{\alpha})^G$. Also, $[l'_1, l'_1, l'_1, l'_3, l'_1, l'_2]$, $[l'_1, l'_3, l'_1, l'_2, l'_1, l'_1]$, $[l'_1, l'_2, l'_1, l'_1, l'_1, l'_3]$, $[l'_1, l'_1, l'_3, l'_1, l'_2, l'_1]$, $[l'_3, l'_1, l'_2, l'_1, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_3, l'_1]$, $[l'_1, l'_2, l'_1, l'_3, l'_1, l'_1]$, $[l'_1, l'_3, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_1, l'_2, l'_1, l'_3]$, $[l'_2, l'_1, l'_3, l'_1, l'_1, l'_1]$, $[l'_3, l'_1,$

Table 2: Enumeration of orbits.

Classes of equivalent circuits of length 6	$\lvert E_{T_i}\rvert$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_i}\rvert$	Regarding distinct orbits
E_{T_1}	720	60	240	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
E_{T_2}	360	30	120	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
E_{T_3}	168	14	56	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
	12	2	4	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
E_{T_4}	120	10	40	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
E_{T_5}	48	4	16	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
	18	3	6	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
	18	3	6	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
	6	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
E_{T_6}	48	4	16	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
	12	2	4	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
E_{T_7}	24	2	8	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
	6	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
E_{T_8}	12	1	4	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$
	6	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
E_{T_9}	6	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
	3	1	1	$\gamma^G = (-\overline{\gamma})^G = (-\gamma)^G = (\overline{\gamma})^G$
	6	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
$E_{T_{10}}$	6	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$

Table 3: Cyclically equivalent classes of T_9.

$E^c_{T_9}$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_9}\rvert$	Structure of orbits
$E^c_{[l'_1,l'_1,l'_1,l'_1,l'_2,l'_2]}$	1	2	$\gamma^G = (\overline{\gamma})^G$, $(-\gamma)^G = (-\overline{\gamma})^G$
$E^c_{[l'_1,l'_2,l'_1,l'_2,l'_1,l'_1]}$	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
$E^c_{[l'_1,l'_1,l'_2,l'_1,l'_1,l'_2]}$	1	1	$\gamma^G = (-\overline{\gamma})^G = (-\gamma)^G = (\overline{\gamma})^G$

Table 4: Cyclically equivalent classes of T_8.

$E^c_{T_8}$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_8}\rvert$	Structure of orbits
$E^c_{[l'_1,l'_1,l'_1,l'_2,l'_2,l'_2]}$	1	2	$\gamma^G = (-\overline{\gamma})^G$, $(-\gamma)^G = (\overline{\gamma})^G$
$E^c_{[l'_1,l'_1,l'_2,l'_2,l'_1,l'_2]}$	1	4	γ^G, $(\overline{\gamma})^G$, $(-\gamma)^G$, $(-\overline{\gamma})^G$

$l'_1, l'_1, l'_2, l'_1]$, and $[l'_1, l'_1, l'_2, l'_1, l'_3, l'_1]$ are cyclically equivalent. By Lemma 1, $[l'_1, l'_1, l'_1, l'_3, l'_1, l'_2] \sim_s [l'_1, l'_3, l'_1, l'_2, l'_1, l'_1] \sim_s [l'_1, l'_2, l'_1, l'_1, l'_1, l'_3]$ is the circuit of β^G, $[l'_1, l'_1, l'_3, l'_1, l'_2, l'_1] \sim_s [l'_3, l'_1, l'_2, l'_1, l'_1, l'_1] \sim_s [l'_2, l'_1, l'_1, l'_1, l'_3, l'_1]$ is the circuit of $(-\beta)^G$, $[l'_1, l'_2, l'_1, l'_3, l'_1, l'_1] \sim_s [l'_1, l'_3, l'_1, l'_1, l'_1, l'_2] \sim_s [l'_1, l'_1, l'_1, l'_2, l'_1, l'_3]$ is the circuit contained in $(-\overline{\beta})^G$, and $[l'_2, l'_1, l'_3, l'_1, l'_1, l'_1] \sim_s [l'_3, l'_1, l'_1, l'_1, l'_2, l'_1] \sim_s [l'_1, l'_1, l'_2, l'_1, l'_3, l'_1]$ is the circuit contained in $(\overline{\beta})^G$. Also, $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_3]$, $[l'_2, l'_1, l'_1, l'_3, l'_1, l'_1]$, $[l'_1, l'_3, l'_1, l'_1, l'_2, l'_1]$, $[l'_1, l'_2, l'_1, l'_1, l'_3, l'_1]$, $[l'_1, l'_1, l'_3, l'_1, l'_1, l'_2]$, and $[l'_3, l'_1, l'_1, l'_2, l'_1, l'_1]$ are cyclically equivalent. Moreover, by Lemma 1, $[l'_1, l'_1, l'_2, l'_1, l'_1, l'_3] \sim_s [l'_2, l'_1, l'_1, l'_3, l'_1, l'_1] \sim_s [l'_1, l'_3, l'_1, l'_1, l'_2, l'_1]$ is the circuit of $(\gamma)^G = (-\overline{\gamma})^G$, and $[l'_1, l'_2, l'_1, l'_1, l'_3, l'_1] \sim_s [l'_1, l'_1, l'_3, l'_1, l'_1, l'_2] \sim_s [l'_3, l'_1, l'_1, l'_2, l'_1, l'_1]$ is the circuit of $(-\gamma)^G = (\overline{\gamma})^G$.

Table 5 summarizes all the information. □

Corollary 9. *There are 60 equivalent circuits of length 6 in which 3 numbers are alike, 2 numbers are alike, and 1 number is different.*

Proof. By Lemma 4, the number of equivalent length 6 in which 3 numbers are alike, 2 numbers are alike, and 1 number is different is 6!/3!2!1! = 60. These circuits are $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_3]$, $[l'_1, l'_2, l'_2, l'_3, l'_1, l'_1]$, $[l'_2, l'_3, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_2, l'_2, l'_3, l'_1]$, $[l'_2, l'_2, l'_3, l'_1, l'_1, l'_1]$, $[l'_2, l'_1, l'_1, l'_1, l'_3, l'_2]$, $[l'_3, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_3, l'_2, l'_2, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_1, l'_3]$, $[l'_1, l'_1, l'_1, l'_3, l'_2, l'_2]$, $[l'_3, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_3, l'_2, l'_2, l'_1]$, and so on.

In the aforementioned circuits, $[l'_1, l'_1, l'_1, l'_2, l'_2, l'_3]$, $[l'_1, l'_2, l'_2, l'_3, l'_1, l'_1]$, $[l'_2, l'_3, l'_1, l'_1, l'_1, l'_2]$, $[l'_1, l'_1, l'_2, l'_2, l'_3, l'_1]$, $[l'_3, l'_1, l'_1, l'_1, l'_2, l'_2]$, $[l'_1, l'_3, l'_2, l'_2, l'_1, l'_1]$, $[l'_2, l'_2, l'_1, l'_1, l'_1, l'_3]$, $[l'_1, l'_1, l'_1, l'_3, l'_2, l'_2]$, $[l'_3, l'_2, l'_2, l'_1, l'_1, l'_1]$, $[l'_1, l'_1, l'_3, l'_2, l'_2, l'_1]$ $[l'_2, l'_1, l'_1, l'_1, l'_3, l'_2]$, and $[l'_2, l'_2, l'_3, l'_1, l'_1, l'_1]$ are cyclically equivalent. By Lemma 1, $[l'_2, l'_3, l'_1, l'_1, l'_1, l'_2] \sim_s [l'_1, l'_1, l'_1, l'_2, l'_2, l'_3] \sim_s [l'_1, l'_2, l'_2, l'_3, l'_1, l'_1]$ is the circuit of α^G, $[l'_1, l'_1, l'_2, l'_2, l'_3, l'_1] \sim_s [l'_2, l'_2, l'_3, l'_1, l'_1, l'_1] \sim_s [l'_3, l'_1, l'_1, l'_1, l'_2, l'_2]$ is the circuit contained in $(-\alpha)^G$, $[l'_1, l'_3, l'_2, l'_2, l'_1, l'_1] \sim_s [l'_2, l'_2, l'_1, l'_1, l'_1, l'_3] \sim_s [l'_1, l'_1, l'_1, l'_3, l'_2, l'_2]$ is the circuit of $(-\overline{\alpha})^G$, $[l'_3, l'_2, l'_2, l'_1, l'_1, l'_1] \sim_s [l'_2, l'_1, l'_1, l'_1, l'_3, l'_2] \sim_s [l'_1, l'_1, l'_3, l'_2, l'_2, l'_1]$ is the circuit contained in $(\overline{\alpha})^G$, and so on.

Table 6 summarizes all the information. □

Corollary 10. *There are 90 equivalent circuits of length 6 in which 2 numbers are alike, 2 numbers are alike, and 2 numbers are alike.*

Proof. By Lemma 4, the number of equivalent circuits of length 6 in which 2 numbers are alike, 2 numbers are alike, and 2 numbers are alike is 6!/2!2!2! = 90. Some of these circuits are $[l_1', l_1', l_2', l_2', l_3', l_3']$, $[l_2', l_2', l_3', l_3', l_1', l_1']$, $[l_3', l_3', l_1', l_1', l_2', l_2']$, $[l_1', l_2', l_2', l_3', l_3', l_1']$, $[l_2', l_3', l_3', l_1', l_1', l_2']$, $[l_2', l_2', l_3', l_1', l_1', l_3']$, $[l_3', l_1', l_1', l_2', l_2', l_3']$, $[l_1', l_3', l_3', l_2', l_2', l_1']$, $[l_3', l_2', l_2', l_1', l_1', l_3']$, $[l_2', l_1', l_1', l_3', l_3', l_2']$, $[l_3', l_3', l_2', l_2', l_1', l_1']$, $[l_2', l_3', l_1', l_1', l_3', l_2']$, $[l_2', l_2', l_1', l_1', l_3', l_3']$, $[l_1', l_1', l_3', l_3', l_2', l_2']$, $[l_1', l_1', l_3', l_2', l_2', l_3']$, $[l_1', l_3', l_2', l_2', l_3', l_1']$, $[l_3', l_2', l_2', l_3', l_1', l_1']$, $[l_3', l_1', l_1', l_3', l_2', l_2']$, and so on.

In th aforementioned circuits, $[l_1', l_1', l_2', l_2', l_3', l_3']$, $[l_2', l_2', l_3', l_3', l_1', l_1']$, $[l_3', l_3', l_1', l_1', l_2', l_2']$, $[l_1', l_2', l_2', l_3', l_3', l_1']$, $[l_2', l_3', l_3', l_1', l_1', l_2']$, $[l_3', l_1', l_1', l_2', l_2', l_3']$, $[l_1', l_3', l_3', l_2', l_2', l_1']$, $[l_3', l_2', l_2', l_1', l_1', l_3']$, $[l_2', l_1', l_1', l_3', l_3', l_2']$, $[l_3', l_3', l_2', l_2', l_1', l_1']$, $[l_2', l_2', l_1', l_1', l_3', l_3']$, and $[l_1', l_1', l_3', l_3', l_2', l_2']$ are cyclically equivalent. By Lemma 1, $[l_1', l_1', l_2', l_2', l_3', l_3'] \sim_s [l_2', l_2', l_3', l_3', l_1', l_1'] \sim_s [l_3', l_3', l_1', l_1', l_2', l_2']$ is the circuit of α^G, $[l_1', l_2', l_2', l_3', l_3', l_1'] \sim_s [l_2', l_3', l_3', l_1', l_1', l_2'] \sim_s [l_3', l_1', l_1', l_2', l_2', l_3']$ is the circuit contained in $(-\alpha)^G$, $[l_1', l_3', l_3', l_2', l_2', l_1'] \sim_s [l_3', l_2', l_2', l_1', l_1', l_3'] \sim_s [l_2', l_1', l_1', l_3', l_3', l_2']$ is the circuit of $(-\overline{\alpha})^G$, $[l_3', l_3', l_2', l_2', l_1', l_1'] \sim_s [l_2', l_2', l_1', l_1', l_3', l_3'] \sim_s [l_1', l_1', l_3', l_3', l_2', l_2']$ is the circuit contained in $(\overline{\alpha})^G$, and so on.

Table 7 summarizes all the information. □

Corollary 11. *There are 120 equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are different.*

Proof. By Lemma 4, the number of equivalent circuits of length 6 in which 3 numbers are alike and 3 numbers are different is 6!/3!1!1!1! = 120. Some of these circuits are $[l_1', l_1', l_1', l_2', l_3', l_4']$, $[l_1', l_2', l_3', l_4', l_1', l_1']$, $[l_3', l_4', l_1', l_1', l_1', l_2']$, $[l_1', l_1', l_2', l_3', l_4', l_1']$, $[l_2', l_3', l_4', l_1', l_1', l_1']$, $[l_4', l_2', l_3', l_1', l_1', l_1']$, $[l_3', l_2', l_4', l_1', l_1', l_1']$, $[l_4', l_1', l_1', l_1', l_2', l_3']$, $[l_1', l_4', l_3', l_2', l_1', l_1']$, $[l_3', l_2', l_1', l_1', l_1', l_4']$, $[l_1', l_1', l_1', l_4', l_3', l_2']$, $[l_4', l_3', l_2', l_1', l_1', l_1']$, $[l_2', l_4', l_1', l_1', l_1', l_3']$, $[l_2', l_1', l_1', l_1', l_4', l_3']$, $[l_1', l_1', l_4', l_3', l_2', l_1']$, $[l_1', l_1', l_1', l_3', l_2', l_4']$, $[l_1', l_1', l_3', l_2', l_4', l_1']$, $[l_1', l_3', l_2', l_4', l_1', l_1']$, $[l_4', l_1', l_1', l_1', l_3', l_2']$, $[l_1', l_4', l_2', l_3', l_1', l_1']$, and so on.

In the aforementioned circuits, $[l_1', l_1', l_1', l_2', l_3', l_4']$, $[l_1', l_2', l_3', l_4', l_1', l_1']$, $[l_3', l_4', l_1', l_1', l_1', l_2']$, $[l_1', l_1', l_2', l_3', l_4', l_1']$, $[l_2', l_3', l_4', l_1', l_1', l_1']$, $[l_4', l_1', l_1', l_1', l_2', l_3']$, $[l_1', l_4', l_3', l_2', l_1', l_1']$, $[l_3', l_2', l_1', l_1', l_1', l_4']$, $[l_1', l_1', l_1', l_4', l_3', l_2']$, $[l_4', l_3', l_2', l_1', l_1', l_1']$, $[l_2', l_1', l_1', l_1', l_4', l_3']$, and $[l_1', l_1', l_4', l_3', l_2', l_1']$ are cyclically equivalent. By Lemma 1, $[l_1', l_1', l_1', l_2', l_3', l_4'] \sim_s [l_1', l_2', l_3', l_4', l_1', l_1'] \sim_s [l_3', l_4', l_1', l_1', l_1', l_2']$ is the circuit of α^G, $[l_1', l_1', l_2', l_3', l_4', l_1'] \sim_s [l_2', l_3', l_4', l_1', l_1', l_1'] \sim_s [l_4', l_1', l_1', l_1', l_2', l_3']$ is the circuit contained in $(-\alpha)^G$, $[l_1', l_4', l_3', l_2', l_1', l_1'] \sim_s [l_3', l_2', l_1', l_1', l_1', l_4'] \sim_s [l_1', l_1', l_1', l_4', l_3', l_2']$ is the circuit of $(-\overline{\alpha})^G$, $[l_4', l_3', l_2', l_1', l_1', l_1'] \sim_s [l_2', l_1', l_1', l_1', l_4', l_3'] \sim_s [l_1', l_1', l_4', l_3', l_2', l_1']$ is the circuit contained in $(\overline{\alpha})^G$, and so on.

Table 8 summarizes all the information. □

Corollary 12. *There are 180 equivalent circuits of length 6 in which 2 numbers are alike, 2 numbers are alike, and 2 numbers are different.*

Proof. By Lemma 4, the number of equivalent circuits of length 6 in which 2 numbers are alike, 2 numbers are alike, and 2 numbers are different is 6!/2!2!1!1! = 180. Some these circuits are $[l_1', l_1', l_2', l_2', l_3', l_4']$, $[l_2', l_2', l_3', l_4', l_1', l_1']$, $[l_3', l_4', l_1', l_1', l_2', l_2']$, $[l_1', l_2', l_2', l_3', l_4', l_1']$, $[l_2', l_3', l_4', l_1', l_1', l_2']$, $[l_4', l_1', l_1', l_2', l_2', l_3']$, $[l_1', l_4', l_3', l_2', l_2', l_1']$, $[l_3', l_2', l_2', l_1', l_1', l_4']$, $[l_2', l_1', l_1', l_4', l_3', l_2']$, $[l_4', l_3', l_2', l_2', l_1', l_1']$, $[l_2', l_2', l_1', l_1', l_4', l_3']$, $[l_1', l_1', l_4', l_3', l_2', l_2']$, $[l_1', l_1', l_2', l_3', l_4', l_2']$, $[l_1', l_1', l_2', l_4', l_2', l_3']$, $[l_1', l_1', l_2', l_4', l_3', l_2']$, $[l_1', l_1', l_3', l_2', l_2', l_4']$, $[l_1', l_1', l_3', l_2', l_4', l_2']$, $[l_1', l_1', l_3', l_4', l_2', l_2']$, $[l_1', l_1', l_4', l_2', l_2', l_3']$, and so on.

In the aforementioned circuits, $[l_1', l_1', l_2', l_2', l_3', l_4']$, $[l_2', l_2', l_3', l_4', l_1', l_1']$, $[l_3', l_4', l_1', l_1', l_2', l_2']$, $[l_1', l_2', l_2', l_3', l_4', l_1']$, $[l_2', l_3', l_4', l_1', l_1', l_2']$, $[l_4', l_1', l_1', l_2', l_2', l_3']$, $[l_1', l_4', l_3', l_2', l_2', l_1']$, $[l_3', l_2', l_2', l_1', l_1', l_4']$, $[l_2', l_1', l_1', l_4', l_3', l_2']$, $[l_4', l_3', l_2', l_2', l_1', l_1']$, $[l_2', l_1', l_1', l_4', l_3']$, and $[l_1', l_1', l_4', l_3', l_2', l_2']$ are cyclically equivalent. Moreover, by Lemma 1, $[l_1', l_1', l_2', l_2', l_3', l_4'] \sim_s [l_2', l_2', l_3', l_4', l_1', l_1'] \sim_s [l_3', l_4', l_1', l_1', l_2', l_2']$ is the circuit of α^G, $[l_1', l_2', l_2', l_3', l_4', l_1'] \sim_s [l_2', l_3', l_4', l_1', l_1', l_2'] \sim_s [l_4', l_1', l_1', l_2', l_2', l_3']$ is the circuit of $(-\alpha)^G$, $[l_1', l_4', l_3', l_2', l_2', l_1'] \sim_s [l_3', l_2', l_2', l_1', l_1', l_4'] \sim_s [l_2', l_1', l_1', l_4', l_3', l_2']$ is the circuit contained in $(-\overline{\alpha})^G$, $[l_4', l_3', l_2', l_2', l_1', l_1'] \sim_s [l_2', l_2', l_1', l_1', l_4', l_3'] \sim_s [l_1', l_1', l_4', l_3', l_2', l_2']$ is the circuit contained in $(\overline{\alpha})^G$, and so on.

Table 9 summarizes all the information. □

Corollary 13. *There are 360 equivalent circuits of length 6 in which 2 numbers are alike and 4 numbers are different.*

Proof. By Lemma 4, the number of equivalent circuits of length 6 in which 2 numbers are alike and 4 numbers are different is 6!/2!1!1!1! = 360. Some of these circuits are $[l_1', l_1', l_2', l_3', l_4', l_5']$, $[l_2', l_3', l_4', l_5', l_1', l_1']$, $[l_4', l_5', l_1', l_1', l_2', l_3']$, $[l_1', l_2', l_3', l_4', l_5', l_1']$, $[l_3', l_4', l_5', l_1', l_1', l_2']$, $[l_3', l_2', l_1', l_1', l_5', l_4']$, $[l_5', l_1', l_1', l_2', l_3', l_4']$, $[l_1', l_5', l_4', l_3', l_2', l_1']$, $[l_4', l_3', l_2', l_1', l_1', l_5']$, $[l_2', l_1', l_1', l_5', l_4', l_3']$, $[l_5', l_4', l_3', l_2', l_1', l_1']$, $[l_1', l_1', l_5', l_4', l_3', l_2']$, and so on.

In the aforementioned circuits, $[l_1', l_1', l_2', l_3', l_4', l_5']$, $[l_1', l_2', l_3', l_4', l_5', l_1']$, $[l_2', l_3', l_4', l_5', l_1', l_1']$, $[l_3', l_4', l_5', l_1', l_1', l_2']$, $[l_4', l_5', l_1', l_1', l_2', l_3']$, $[l_5', l_1', l_1', l_2', l_3', l_4']$, $[l_1', l_5', l_4', l_3', l_2', l_1']$, $[l_5', l_4', l_3', l_2', l_1', l_1']$, $[l_4', l_3', l_2', l_1', l_1', l_5']$, $[l_3', l_2', l_1', l_1', l_5', l_4']$, $[l_2', l_1', l_1', l_5', l_4', l_3']$ and $[l_1', l_1', l_5', l_4', l_3', l_2']$ are cyclically equivalent.

Table 10 summarizes all the information.

It was proved in [8] that any circuit $[l_1', l_2', l_3', l_4', l_5', l_6']$ of length 6 corresponds to the orbit contained in $Q^*(\sqrt{f(l_1', l_2', l_3', l_4', l_5', l_6')})$. □

Theorem 2. *All the circuits in $E^c_{[l_1', l_2', l_3', l_4', l_5', l_6']}$ correspond to the orbits contained in $Q^*(\sqrt{n_1})$.*

Proof. To prove this result, it is enough to show that $f(l_1', l_2', l_3', l_4', l_5', l_6')$ is unchanged for $\theta \in D_6$, where $f(l_1', l_2', l_3', l_4', l_5', l_6') = (l_1'(l_6' + l_4' + l_2') + l_3'(l_6' + l_4') + l_5'l_6' + l_2'l_1'(l_4'l_3' + l_6'l_5') + l_4'l_5'l_6'(l_3' + l_1') + l_1'l_6'l_3'l_2'(1 + l_4'l_5') - l_2'(l_5' + l_3') - l_5'l_4'(1 + l_2'l_3'))^2 +$

Table 5: Cyclically equivalent classes of T_7.

$E^c_{T_7}$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_7}\rvert$	Structure of orbits
$E^c_{[l'_1,l'_1,l'_2,l'_1,l'_1,l'_3]}$	1	2	$\gamma^G = (-\overline{\gamma})^G,\ (-\gamma)^G = (\overline{\gamma})^G$
$E^c_{[l'_1,l'_1,l'_1,l'_1,l'_2,l'_3]}, E^c_{[l'_1,l'_2,l'_1,l'_3,l'_1,l'_1]}$	2	8	$\gamma^G,\ (\overline{\gamma})^G,\ (-\gamma)^G,\ (-\overline{\gamma})^G$

Table 6: Cyclically equivalent classes of T_6.

$E^c_{T_6}$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_6}\rvert$	Structure of orbits
$E^c_{[l'_1,l'_3,l'_2,l'_2,l'_1,l'_1]}, E^c_{[l'_1,l'_1,l'_2,l'_1,l'_2,l'_3]}, E^c_{[l'_1,l'_2,l'_2,l'_1,l'_3,l'_1]}, E^c_{[l'_1,l'_2,l'_1,l'_1,l'_2,l'_3]}$	4	16	$\gamma^G, (\overline{\gamma})^G, (-\gamma)^G, (-\overline{\gamma})^G$
$E^c_{[l'_1,l'_1,l'_1,l'_2,l'_3,l'_2]}, E^c_{[l'_1,l'_2,l'_1,l'_2,l'_1,l'_3]}$	2	4	$\gamma^G = (-\overline{\gamma})^G,\ (-\gamma)^G = (\overline{\gamma})^G$

Table 7: Cyclically equivalent classes of T_5.

$E^c_{T_5}$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_5}\rvert$	Structure of orbits
$E^c_{[l'_3,l'_3,l'_2,l'_2,l'_1,l'_1]}, E^c_{[l'_1,l'_3,l'_2,l'_3,l'_2,l'_1]}, E^c_{[l'_1,l'_2,l'_1,l'_3,l'_3,l'_2]}, E^c_{[l'_1,l'_3,l'_2,l'_2,l'_1,l'_3]}$	4	16	$\gamma^G, (\overline{\gamma})^G, (-\gamma)^G, (-\overline{\gamma})^G$
$E^c_{[l'_3,l'_2,l'_1,l'_2,l'_3,l'_1]}, E^c_{[l'_1,l'_3,l'_2,l'_3,l'_1,l'_2]}, E^c_{[l'_1,l'_3,l'_1,l'_2,l'_3,l'_2]}$	3	12	$\gamma^G = (-\overline{\gamma})^G, (-\gamma)^G = (\overline{\gamma})^G$
$E^c_{[l'_1,l'_3,l'_2,l'_2,l'_3,l'_1]}, E^c_{[l'_1,l'_2,l'_2,l'_1,l'_3,l'_3]}, E^c_{[l'_1,l'_1,l'_2,l'_3,l'_3,l'_2]}$	3	12	$\gamma^G = (\overline{\gamma})^G, (-\gamma)^G = (-\overline{\gamma})^G$
$E^c_{[l'_1,l'_2,l'_3,l'_1,l'_2,l'_3]}$	1	2	$\gamma^G = (-\gamma)^G, (\overline{\gamma})^G = (-\overline{\gamma})^G$

Table 8: Cyclically equivalent classes of T_4.

$E^c_{T_4}$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_4}\rvert$	Structure of orbits
$E^c_{[l'_1,l'_4,l'_3,l'_2,l'_1,l'_1]}, E^c_{[l'_1,l'_1,l'_1,l'_2,l'_4,l'_3]}, E^c_{[l'_1,l'_2,l'_4,l'_1,l'_3,l'_1]}, E^c_{[l'_1,l'_3,l'_4,l'_1,l'_2,l'_1]}, E^c_{[l'_1,l'_2,l'_4,l'_1,l'_1,l'_3]}, E^c_{[l'_1,l'_4,l'_2,l'_3,l'_1,l'_1]},$ $E^c_{[l'_1,l'_2,l'_1,l'_1,l'_4,l'_3]}, E^c_{[l'_1,l'_1,l'_4,l'_1,l'_2,l'_3]}, E^c_{[l'_1,l'_3,l'_2,l'_1,l'_1,l'_4]}, E^c_{[l'_1,l'_4,l'_1,l'_2,l'_1,l'_3]}$	10	40	$\gamma^G, (\overline{\gamma})^G, (-\gamma)^G, (-\overline{\gamma})^G$

Table 9: Cyclically equivalent classes of T_3.

$E^c_{T_3}$	Number of classes of cyclically equivalent circuits	$\lvert O_{T_3}\rvert$	Structure of orbits
$E^c_{[l'_1,l'_4,l'_3,l'_2,l'_2,l'_1]}, E^c_{[l'_1,l'_1,l'_2,l'_4,l'_4,l'_3]}, E^c_{[l'_1,l'_4,l'_2,l'_3,l'_2,l'_1]}, E^c_{[l'_1,l'_3,l'_2,l'_4,l'_2,l'_1]}, E^c_{[l'_1,l'_1,l'_3,l'_2,l'_2,l'_4]}, E^c_{[l'_1,l'_1,l'_2,l'_3,l'_4,l'_2]},$ $E^c_{[l'_1,l'_4,l'_3,l'_2,l'_1,l'_2]}, E^c_{[l'_1,l'_2,l'_1,l'_2,l'_4,l'_3]}, E^c_{[l'_1,l'_3,l'_2,l'_2,l'_1,l'_4]}, E^c_{[l'_1,l'_2,l'_1,l'_3,l'_2,l'_4]}, E^c_{[l'_1,l'_3,l'_1,l'_2,l'_2,l'_4]}, E^c_{[l'_1,l'_4,l'_2,l'_1,l'_3,l'_2]},$ $E^c_{[l'_1,l'_2,l'_2,l'_1,l'_3,l'_4]}, E^c_{[l'_1,l'_3,l'_2,l'_1,l'_2,l'_4]}$	14	56	$\gamma^G, (\overline{\gamma})^G, (-\gamma)^G, (-\overline{\gamma})^G$
$E^c_{[l'_1,l'_2,l'_4,l'_2,l'_1,l'_3]}, E^c_{[l'_2,l'_3,l'_2,l'_1,l'_4,l'_1]}$	2	4	$\gamma^G = (-\overline{\gamma})^G, (-\gamma)^G = (\overline{\gamma})^G$

Table 10: Cyclically equivalent classes of T_2.

$E^c_{T_3}$	Number of classes of the cyclically equivalent circuits	$\lvert O_{T_3}\rvert$	Structure of orbits
$E^c_{[l'_1,l'_1,l'_2,l'_3,l'_4,l'_5]}, E^c_{[l'_1,l'_1,l'_2,l'_3,l'_5,l'_4]}, E^c_{[l'_1,l'_1,l'_2,l'_4,l'_3,l'_5]}, E^c_{[l'_1,l'_1,l'_2,l'_4,l'_5,l'_3]}, E^c_{[l'_1,l'_1,l'_2,l'_5,l'_3,l'_4]}, E^c_{[l'_1,l'_1,l'_2,l'_5,l'_4,l'_3]},$ $E^c_{[l'_1,l'_1,l'_3,l'_2,l'_5,l'_4]}, E^c_{[l'_1,l'_1,l'_3,l'_2,l'_4,l'_5]}, E^c_{[l'_1,l'_1,l'_3,l'_4,l'_2,l'_5]}, E^c_{[l'_1,l'_1,l'_3,l'_5,l'_2,l'_4]}, E^c_{[l'_1,l'_1,l'_4,l'_2,l'_3,l'_5]}, E^c_{[l'_1,l'_1,l'_4,l'_3,l'_2,l'_5]},$ $E^c_{[l'_1,l'_4,l'_1,l'_2,l'_3,l'_5]}, E^c_{[l'_2,l'_1,l'_4,l'_3,l'_5,l'_1]}, E^c_{[l'_2,l'_1,l'_5,l'_4,l'_3,l'_1]}, E^c_{[l'_4,l'_3,l'_2,l'_1,l'_5,l'_1]}, E^c_{[l'_3,l'_4,l'_1,l'_2,l'_5,l'_1]}, E^c_{[l'_4,l'_2,l'_5,l'_1,l'_3,l'_1]},$ $E^c_{[l'_2,l'_1,l'_3,l'_5,l'_4,l'_1]}, E^c_{[l'_3,l'_2,l'_1,l'_5,l'_4,l'_1]}, E^c_{[l'_3,l'_5,l'_1,l'_2,l'_4,l'_1]}, E^c_{[l'_5,l'_1,l'_4,l'_2,l'_3,l'_1]}, E^c_{[l'_3,l'_5,l'_2,l'_1,l'_4,l'_1]}, E^c_{[l'_5,l'_4,l'_2,l'_1,l'_3,l'_1]},$ $E^c_{[l'_2,l'_5,l'_4,l'_1,l'_3,l'_1]}, E^c_{[l'_5,l'_2,l'_3,l'_1,l'_4,l'_1]}, E^c_{[l'_5,l'_1,l'_3,l'_4,l'_2,l'_1]}, E^c_{[l'_5,l'_2,l'_1,l'_3,l'_4,l'_1]}, E^c_{[l'_5,l'_3,l'_1,l'_4,l'_2,l'_1]}, E^c_{[l'_5,l'_4,l'_1,l'_2,l'_3,l'_1]}$	30	120	$\gamma^G,\ (\overline{\gamma})^G,\ (-\gamma)^G,\ (-\overline{\gamma})^G$

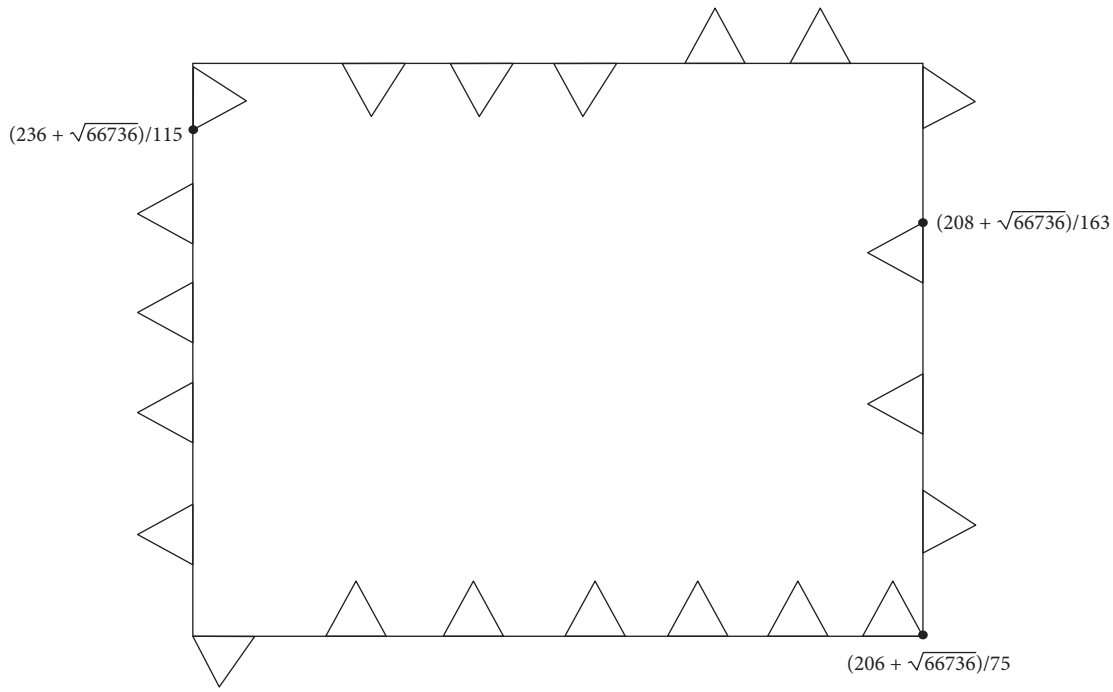

Figure 1: Shows the location of reduced numbers and describes that $[1,2,3,4,5,6] \sim_s [3,4,5,6,1,2] \sim_s [5,6,1,2,3,4]$.

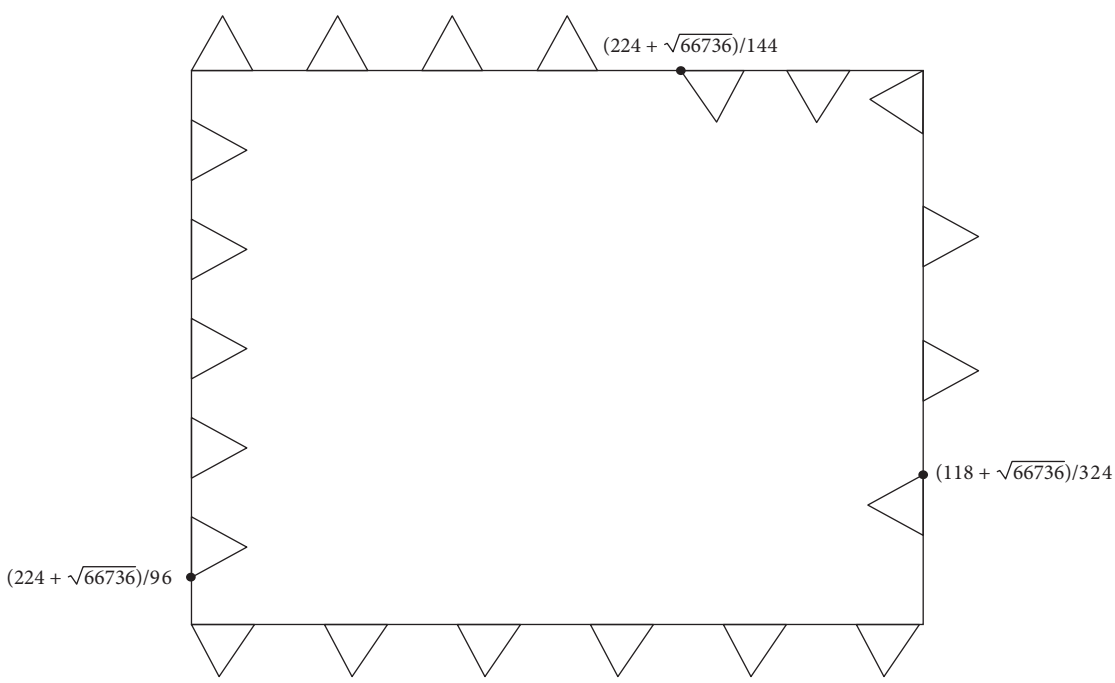

Figure 2: Shows the location of reduced numbers and describes that $[2,3,4,5,6,1] \sim_s [4,5,6,1,2,3] \sim_s [6,1,2,3,4,5]$.

$4(l_5'(1+l_1'l_2'+l_3'l_4'+l_1'l_4'+l_1'l_2'l_3'l_4') + l_3'+l_1'+l_3'l_2'l_1')((1+l_6'l_5')(l_4'+l_2'+l_2'l_3'l_4)+l_6'(1+l_2'l_3'))$.

It is easy to see that after applying $\theta \in D_6$ and simplifying, the expression for $f(l_1', l_2', l_3', l_4', l_5', l_6')$ is unaltered. Since there are 60 cyclically equivalent classes, we have sixty $n's$, namely, $n_1, n_2, n_3, \cdots, n_{60}$. □

Example 4. We verify the above results by considering the circuits $[1,2,3,4,5,6]$, $[2,3,4,5,6,1]$, $[1,6,5,4,3,2]$, and $[6,5,4,3,2,1]$. These four circuits correspond to the orbits γ^G, $(-\gamma)^G$, $(-\overline{\gamma})^G$, and $(\overline{\gamma})^G$, and these four circuits are shown in Figures 1–4, respectively.

Now, in all figures containing 3 reduced numbers, $E^c_{[1,2,3,4,5,6]}$ contains 12 reduced numbers.

4. Conclusion

The idea of G-circuits of G-orbits on the real quadratic field by the modular group, which is given in this paper, is new and

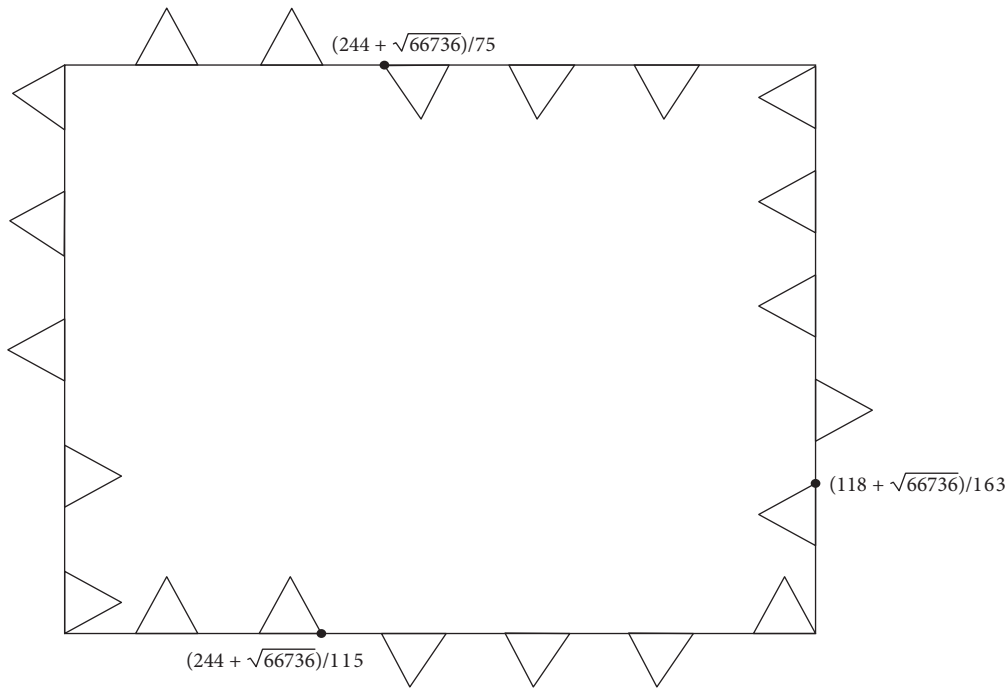

Figure 3: Shows the location of reduced numbers and describes that $[1, 6, 5, 4, 3, 2] \sim_s [5, 4, 3, 2, 1, 6] \sim_s [3, 2, 1, 6, 5, 4]$.

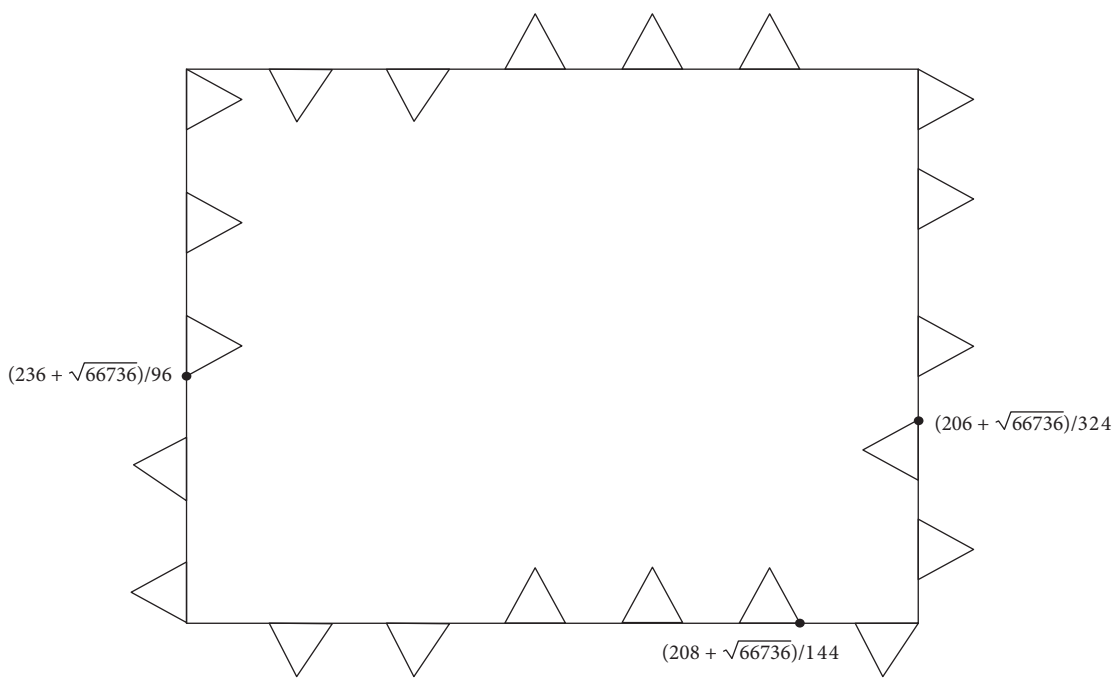

Figure 4: Shows the location of reduced numbers and describes that $[6, 5, 4, 3, 2, 1] \sim_s [4, 3, 2, 1, 6, 5] \sim_s [2, 1, 6, 5, 4, 3]$.

original. We have classified G-circuits into the distinct equivalence classes of equivalent circuits, and they are precisely $p(q) - 1$ in number; particularly, for circuits of length 6, we have $p(6) - 1 = 10$ equivalence classes of equivalent circuits, i.e., $E_6 = \cup_{i=1}^{10} E_{T_i}$. We further classify equivalence classes of equivalent circuits into cyclically equivalence classes and

determine G-orbits corresponding to each cyclically equivalence class. All results are verified by suitable examples.

Data Availability

All the necessary informaion/data for this article are already included. So, no additional data are available or required for this article.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

Muhammad Nadeem Bari and Muhammad Aslam Malik contributed to conceptualization, and Hafiz Muhammad Afzal Siddiqui developed the methodology. Jia-Bao Liu contributed to writing of the original draft. Tianlan Chen provided data correction and revision.

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