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Research Article

The Vertex-Edge Resolvability of Some Wheel-Related Graphs

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A vertex $w \in V(H)$ distinguishes (or resolves) two elements (edges or vertices) $a, z \in V(H) \cup E(H)$ if $d(w,a) \neq d(w,z)$. A set W_m of vertices in a nontrivial connected graph H is said to be a mixed resolving set for H if every two different elements (edges and vertices) of H are distinguished by at least one vertex of W_m. The mixed resolving set with minimum cardinality in H is called the mixed metric dimension (vertex-edge resolvability) of H and denoted by $m\dim(H)$. The aim of this research is to determine the mixed metric dimension of some wheel graph subdivisions. We specifically analyze and compare the mixed metric, edge metric, and metric dimensions of the graphs obtained after the wheel graphs' spoke, cycle, and barycentric subdivisions. We also prove that the mixed resolving sets for some of these graphs are independent.

1. Introduction

Suppose $H = (V, E)$ is a nontrivial, simple, and connected graph, where E represents a set of edges and V represents a set of vertices. The distance between two vertices a and w in an undirected graph H, denoted by $d(a, w)$, is the length of a shortest $a - w$ path in H. In [1], Kelenc et al. introduced the concept of mixed metric dimension in graphs. This dimension of graph H is the mixture of metric and edge metric dimensions.

A vertex $w \in V$ is said to resolve two vertices v_1 and v_2 in H if $d(w, v_1) \neq d(w, v_2)$. Let w be a vertex and $W = \{v_1, v_2, v_3, \ldots, v_p\}$ be an ordered subset of vertices in H. The metric coordinate (or metric representation) $r(w|W)$ of w with respect to W is the p-tuple $(d(w, v_1), d(w, v_2), d(w, v_3), \ldots, d(w, v_p))$. Then, W is said to be a resolving set (or metric generator) for H if for every pair of vertices $v_1, v_2 \in V$ with $v_1 \neq v_2$, we have $r(v_1|W) \neq r(v_2|W)$. A resolving set with minimum cardinality is called the metric basis of H, and the cardinality of the metric basis set is the metric dimension $\dim(H)$ of H.

Slater introduced the idea of metric dimension in [2], where the metric generators were referred to as locating sets due to some relation with the problem of uniquely recognizing the location of intruders in networks. Harary and Melter, on the contrary, independently proposed the same concept of the metric dimension of a graph in [3], where metric generators were referred to as resolving sets. Several works on the applications and theoretical properties of this invariant have also been published. Metric dimension has various significant applications in computer science, mathematics, social sciences, chemical sciences, etc. [4–14]. There also exist some other variations of metric dimension in the literature: independent resolving sets [15], local metric dimension [16], solid metric dimension [11], fault-tolerant metric dimension [17], and so on.

The distance between an edge $e = ax$ and a vertex w is defined as $d(e, w) = d(ax, w) = \min\{d(a, w), d(x, w)\}$. A vertex $w \in V$ is said to resolve two edges e_1 and e_2 in H if $d(w, e_1) \neq d(w, e_2)$. Let e be an edge and $W_E = \{v_1, v_2, v_3, \ldots, v_p\}$ be an ordered subset of vertices in H. The edge metric codes $r_E(e|W_E)$ of e with respect to W_E are the

p-tuple $(d(e,v_1), d(e,v_2), d(e,v_3), \ldots, d(e,v_p))$. Then, W_E is said to be an edge resolving set for H if for every pair of edges $e_1, e_2 \in E$ with $e_1 \neq e_2$, we have $r_E(e_1|W_E) \neq r_E(e_2|W_E)$. An edge resolving set with minimum cardinality is called an edge metric basis for H, and the cardinality of this edge metric basis set is the edge metric dimension $e\dim(H)$ of H.

For a connected graph H, we see that every vertex of H is uniquely recognized by a resolving set W of H, and every edge of H is uniquely recognized by an edge resolving set W_E of H; the natural question is as follows: whether every resolving set W is also an edge resolving set W_E for H and vice versa? Kelenc et al. in [18] proved that there exist some families of graphs for which the resolving set W is also an edge resolving set W_E, but in general, this is not true for every graph H. Similarly, for every graph H, the edge resolving set is not necessarily a resolving set for H.

Let us define a set of elements as $V \cup E$, i.e., each element is an edge or a vertex. A vertex $w \in V$ is said to resolve two elements a and z from $V \cup E$ if $d(w,a) \neq d(w,z)$. Let a be an element and $W_m = \{v_1, v_2, v_3, \ldots, v_p\}$ be an ordered subset of vertices in H. The mixed metric codes $r_m(a|W_m)$ of a with respect to W_m are the p-tuple $(d(a,v_1), d(a,v_2), d(a,v_3), \ldots, d(a,v_p))$. Then, W_m is said to be a mixed resolving set for H if for every pair of distinct elements $a_1, a_2 \in V \cup E$, we have $r_m(a_1|W_m) \neq r_m(a_2|W_m)$. A mixed resolving set with minimum cardinality is called a mixed metric basis for H, and the cardinality of this mixed metric basis set is the mixed metric dimension $m\dim(H)$ of H. By the definition of the mixed metric dimension, it is clear that a mixed resolving set is both edge resolving set and a resolving set, so we have

$$m\dim(H) \geq \max\{e\dim(H), \dim(H)\}. \tag{1}$$

There are several studies [1, 19, 20] related to the mixed metric dimension of various graphs, for instance, cycle graphs, antiprism graphs, prism graphs, and convex polytopes, but there are many graphs for which the mixed metric dimension has not been found yet, such as the graphs obtained by some subdivisions of the wheel graph $W_{n,1}$. So, in this paper, we will compute the mixed metric dimension of the graphs obtained after the barycentric, spoke, and cycle subdivisions of the wheel graph $W_{n,1}$.

2. Preliminaries

In this section, we give the definition of a wheel and its related graphs, as well as recall some existing results on the edge metric dimension, and the metric dimension of wheel-related graphs.

2.1. Wheel Graph. A vertex u in an undirected graph G is said to be the universal vertex if it is adjacent to all other vertices of G. A wheel graph $W_{n,1}$ $(n \geq 3)$ is a graph with $n+1$ vertices obtained by joining a single universal vertex to all of the vertices of a cycle graph C_n. $W_{n,1}$ has a vertex set $V = \{v, k_1, k_2, k_3, \ldots, k_n\}$ and an edge set $E = \{vk_j, k_jk_{j+1} | 1 \leq j \leq n\}$, where all of the indices are taken to be modulo n. The edges k_jk_{j+1} are called the cycle edges of $W_{n,1}$, and the edges vk_j are called as the spokes of the wheel graph.

We state that a family F of nontrivial connected graphs has bounded mixed metric dimension if there exists a constant $L > 0$ for every graph H of F such that $m\dim(H) \leq L$; otherwise, F has an unbounded mixed metric dimension. If all of the graphs in F have the same mixed metric dimension, then F is referred to as a family with a constant mixed metric dimension. Cycles C_n and paths P_n for $n \geq 3$ are the graph families with a constant mixed metric dimension.

2.2. Independent Mixed Resolving Set. A set W_m of vertices from H is said to be an independent mixed resolving set for H if W_m is an independent as well as mixed resolving set.

Let $\mathrm{WSS}_{n,1}$, $\mathrm{WCS}_{n,1}$, and $\mathrm{WBS}_{n,1}$ be the graphs obtained from the wheel graph $W_{n,1}$ after spoke, cycle, and barycentric subdivisions of $W_{n,1}$, respectively. Recently, the metric and edge metric dimension for these three wheel-related graphs have been computed, and in [21], Raza and Bataineh made a comparison between the metric dimension and the edge metric dimension for these wheel-related graphs. The edge metric dimension and the metric dimension for these three graphs are as follows.

Proposition 1 (see [21]). $e\dim(WSS_{n,1}) = n-1$, *for* $n \geq 6$.

Proposition 2 (see [21]). *For* $n \geq 6$, *we have*

$$e\dim(\mathrm{WCS}_{n,1}) = e\dim(\mathrm{WBS}_{n,1}) = \begin{cases} 4h & \text{if } n = 6h \text{ or } n = 6h+1, \\ 4h+1 & \text{if } n = 6h+2, \\ 4h+2 & \text{if } n = 6h+3 \text{ or } n = 6h+4, \\ 4h+3 & \text{if } n = 6h+5. \end{cases} \tag{2}$$

Proposition 3 (see [22]). $\dim(WSS_{n,1}) = \lfloor 2n + 2/5 \rfloor$, *for* $n \geq 6$.

Proposition 4 (see [23, 22]). *For* $n \geq 6$, *we have*

$$\dim(\mathrm{WCS}_{n,1}) = \dim(\mathrm{WBS}_{n,1}) = \begin{cases} 4h & \text{if } n = 6h \text{ or } n = 6h+1, \\ 4h+1 & \text{if } n = 6h+2, \\ 4h+2 & \text{if } n = 6h+3 \text{ or } n = 6h+4, \\ 4h+3 & \text{if } n = 6h+5. \end{cases} \tag{3}$$

This article is organized as follows: in Section 3, we will study the mixed metric dimension of the spoke subdivision of the wheel graph $\mathrm{WSS}_{n,1}$. In Sections 4 and 5, we will study the mixed metric dimension of the cycle and barycentric subdivision of the wheel graph, i.e., $\mathrm{WCS}_{n,1}$ and $\mathrm{WBS}_{n,1}$, respectively. We also give the comparative analysis for the mixed metric, edge metric, and metric dimension of the graphs obtained after the spoke, cycle, and barycentric subdivisions of the wheel graph. In Section 6, we conclude the obtained results.

3. Mixed Metric Dimension of the Spoke Subdivision of $W_{n,1}$

In this section, we determine the mixed metric dimension of the spoke subdivision of a wheel graph.

3.1. Spoke Subdivision of $W_{n,1}$. Suppose $W_{n,1}$ is a wheel graph with the vertex set $V(W_{n,1}) = \{k_1, k_2, k_3, \ldots, k_n, v\}$ having a single universal vertex v. Now, each central spoke vk_j of $W_{n,1}$ is subdivided with a new vertex l_j. The resulting graph so obtained is known as the spoke subdivision wheel graph (SSWG) and is denoted by $\mathrm{WSS}_{n,1}$. SSWG has $3n$ edges, $E(W_{n,1}) = \{vl_j, l_jk_j, k_jk_{j+1} | 1 \le j \le n\}$, and $2n+1$ vertices, $V(W_{n,1}) = \{v, l_j, k_j | 1 \le j \le n\}$, where all indices are taken to be modulo n (see Figure 1). In this section, we obtain the mixed metric dimension of SSWG $\mathrm{WSS}_{n,1}$.

Theorem 1. *$m\dim(WSS_{n,1}) = n$, for $n \ge 6$.*

Proof. To prove that $m\dim(\mathrm{WSS}_{n,1}) \le n$, we construct a mixed resolving set for $\mathrm{WSS}_{n,1}$. Suppose $W_m = \{k_1, k_2, k_3, \ldots, k_n\} \subseteq V(\mathrm{WSS}_{n,1})$ having n cycle vertices from $\mathrm{WSS}_{n,1}$. We claim that W_m is a mixed resolving set for $\mathrm{WSS}_{n,1}$. Now, we can give mixed codes to each of the vertex and edge of $\mathrm{WSS}_{n,1}$ with respect to W_m.

The sets of mixed metric codes for the vertices $\{v, l_j, k_j | 1 \le j \le n\}$ of $\mathrm{WSS}_{n,1}$ are as follows:

$$\begin{aligned} A &= \left\{ r_m(v|W_m) = \underbrace{(2, 2, 2, \ldots, 2)}_{n\text{-times}} \right\}, \\ B &= \left\{ r_m(l_j|W_m) = \left(3, 3, \ldots, 3, 2, \underbrace{1}_{j^{\text{th}}}, 2, 3, \ldots, 3, 3 \right) | 1 \le j \le n \right\}, \\ C &= \left\{ r_m(k_j|W_m) = \left(4, 4, \ldots, 4, 3, 2, 1, \underbrace{0}_{j^{\text{th}}}, 1, 2, 3, 4, \ldots, 4, 4 \right) | 1 \le j \le n \right\}. \end{aligned} \tag{4}$$

Next, the sets of mixed metric codes for the edges $\{vl_j, l_jk_j, k_jk_{j+1} | 1 \le j \le n\}$ of $\mathrm{WSS}_{n,1}$ are as follows:

$$\begin{aligned} D &= \left\{ r_m(vl_j|W_m) = \left(2, 2, \ldots, 2, \underbrace{1}_{j^{\text{th}}}, 2, \ldots, 2, 2 \right) | 1 \le j \le n \right\}, \\ E &= \left\{ r_m(l_jk_j|W_m) = \left(3, 3, \ldots, 3, 2, 1, \underbrace{0}_{j^{\text{th}}}, 1, 2, 3, \ldots, 3, 3 \right) | 1 \le j \le n \right\}, \\ F &= \left\{ r_m(k_jk_{j+1}|W_m) = \left(4, 4, \ldots, 4, 3, 2, 1, \underbrace{0}_{j^{\text{th}}}, 0, 1, 2, 3, 4, \ldots, 4, 4 \right) | 1 \le j \le n \right\}. \end{aligned} \tag{5}$$

From these sets of mixed codes for $\mathrm{WSS}_{n,1}$, we obtain that $|A| = 1$, $|B| = |C| = |D| = |E| = |F| = n$, and $A \cap B \cap C \cap D \cap E \cap F = \varnothing$, implying W_m to be a mixed resolving set for $\mathrm{WSS}_{n,1}$, i.e., $m\dim(\mathrm{WSS}_{n,1}) \le n$. Conversely, suppose, on the contrary, that there exists a mixed resolving set $W_m \subseteq \mathrm{WSS}_{n,1}$ such that $|W_m| < n$. Then, we have the following cases to be considered:

Case (i): $v \notin W_m$. In this case, we further have two subcases:

Subcase (i): if $W_m \subset \{k_1, k_2, k_3, \ldots, k_n\}$, then there exists at least one vertex k_j such that $k_j \notin W_m$. Then, for an edge vl_j and the vertex v, we have $r_m(vl_j|W_m) = r_m(v|W_m) = (2, 2, 2, \ldots, 2)$, a contradiction. Therefore, the set W_m is not a mixed resolving set for $\mathrm{WSS}_{n,1}$.

Subcase (ii): if $W_m \nsubseteq \{k_1, k_2, k_3, \ldots, k_n\}$, then at least one vertex l_i belongs to the set W_m. Then, there exists one $k_j \notin W_m$, and the corresponding vertex $l_j \notin W_m$. Then, for an edge vl_j and the vertex v, we have

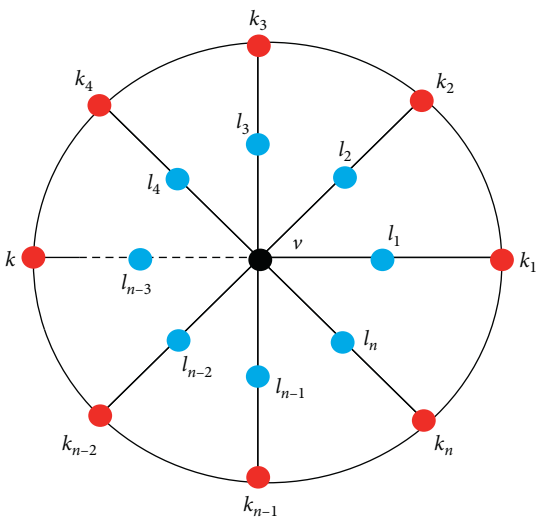

Figure 1: $\mathrm{WSS}_{n,1}$.

$r_m(vl_j|W_m) = r_m(v|W_m)$, a contradiction. Therefore, again, in this case, the set W_m is not a mixed resolving set for $\mathrm{WSS}_{n,1}$.

Case (ii): $v \in W_m$. In this case, we have two subcases:

Subcase (i): if $W_m \subset \{k_1, k_2, k_3, \ldots, k_n\} \cup \{v\}$, then there exists at least one vertex k_j such that $k_j \notin W_m$. Then, clearly, for an edge vl_j and the vertex v, we have $r_m(vl_j|W_m) = r_m(v|W_m)$, a contradiction. Therefore, the set W_m is not a mixed resolving set for $\mathrm{WSS}_{n,1}$.

Subcase (ii): if at least one l_j must belong to the set W_m, then there exists at least one vertex $k_j \notin W_m$, and the corresponding vertex $l_j \notin W_m$. Then, for an edge vl_j and a vertex v, we have $r_m(vl_j|W_m) = r_m(v|W_m)$, a contradiction. Therefore, again, in this case, the set W_m is not a mixed resolving set for $\mathrm{WSS}_{n,1}$. Thus, in all the cases, we have $|W_m| \geq n$, implying $m\dim(\mathrm{WSS}_{n,1}) = n$, which completes the proof of the theorem. □ □

Remark 1. For the spoke subdivision wheel graph $H = \mathrm{WSS}_{n,1}$, we find that $\dim(\mathrm{WSS}_{n,1}) < e\dim(\mathrm{WSS}_{n,1}) < m\dim(\mathrm{WSS}_{n,1})$ (using Propositions 1 and 3 and Theorem 1). The comparison between these three dimensions of $\mathrm{WSS}_{n,1}$ is clearly shown in Figure 2, and the value of each dimension depends on the number of vertices n in $\mathrm{WSS}_{n,1}$.

4. Mixed Metric Dimension of the Cycle Subdivision of $W_{n,1}$

In this section, we determine the mixed metric dimension of the cycle subdivision of a wheel graph.

4.1. Cycle Subdivision of $W_{n,1}$. Suppose $W_{n,1}$ is a wheel graph with the vertex set $V(W_{n,1}) = \{k_1, k_2, k_3, \ldots, k_n, v\}$ having a single universal vertex v. Now, each cycle edge k_jk_{j+1} of $W_{n,1}$ is subdivided with a new vertex l_j. The resulting graph so obtained is known as the cycle subdivision wheel graph (CSWG) and is denoted by $\mathrm{WCS}_{n,1}$. CSWG has $3n$ edges, $E(\mathrm{WCS}_{n,1}) = \{vk_j, k_jl_j, l_jk_{j+1} | 1 \leq j \leq n\}$, and $2n+1$ vertices, $V(\mathrm{WCS}_{n,1}) = \{v, l_j, k_j | 1 \leq j \leq n\}$, where all indices are taken to be modulo n (see Figure 3). In this section, we obtain the mixed metric dimension of CSWG $\mathrm{WCS}_{n,1}$.

Theorem 2. *For $n \geq 6$, we have*

$$m\dim(\mathrm{WCS}_{n,1}) = \begin{cases} 4h & \text{if } n = 6h, \\ 4h+1 & \text{if } n = 6h+1, \\ 4h+2 & \text{if } n = 6h+2, \\ 4h+2 & \text{if } n = 6h+3, \\ 4h+3 & \text{if } n = 6h+4, \\ 4h+4 & \text{if } n = 6h+5. \end{cases} \tag{6}$$

Proof. To prove this, we first generate the mixed resolving sets for all the cases, obtaining the upper bounds depending on the positive integer n. Then, in the end, we show that the lower bound (or reverse inequality) is the same as the upper bound to conclude the theorem.

Case (I): $n \equiv 0 (\mathrm{mod} 6)$. In this case, we have $n = 6h$, where $h \geq 2$ and $h \in \mathbb{N}$. Suppose an ordered subset $W_m = \{l_1, l_2, l_4, l_5, \ldots, l_{n-2}, l_{n-1}\} = \{l_{3i+1}, l_{3i+2} | 0 \leq i \leq 2h-1\}$ of vertices in $\mathrm{WCS}_{n,1}$ with $|W_m| = 4h$. Next, we claim that W_m is the mixed resolving set for $\mathrm{WCS}_{n,1}$. Now, we can give mixed codes to every vertex and edge of $\mathrm{WCS}_{n,1}$ with respect to W_m. The sets of mixed metric codes for the vertices $\{u = v, l_j, k_j | 1 \leq j \leq n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$A = \left\{ r_m(v|W_m) = \underbrace{(2, 2, 2, \ldots, 2)}_{4h-\text{times}} \right\},$$

$$B = \left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+2}, k_{3i+3}) = 1, 3, \ldots, 3) | \\ j \equiv 0 (\mathrm{mod} 3) 0 \leq i \leq 2h-1 \end{array} \right\} \cup$$

$$\left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+1}) = 1, 3, \ldots, 3) | \\ j \equiv 1 (\mathrm{mod} 3) 0 \leq i \leq 2h-1 \end{array} \right\} \cup$$

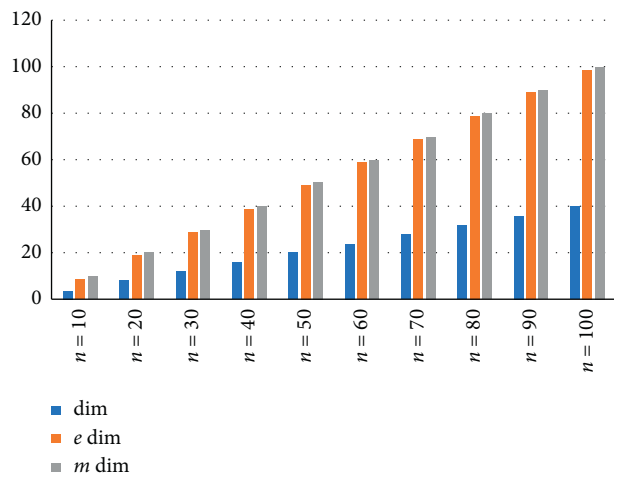

Figure 2: Comparison between $\dim(H)$, $e\dim(H)$, and $m\dim(H)$.

$$C = \left\{ \begin{array}{c} r_m\left(k_j|W_m\right) = \left(3,3,3,\ldots,3,d\left(l_{3i+1},k_{3i+2}\right)=1,d\left(l_{3i+2},k_{3i+2}\right)=1,3,\ldots,3\right)| \\ j \equiv 2\,(\mathrm{mod}3)0 \le i \le 2h-1 \end{array} \right\}, \\ \left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4,,4,\ldots,4,d\left(l_{3i+2},l_{3i+3}\right)=2,d\left(l_{3i+4},l_{3i+3}\right)=2,4,\ldots,4\right)| \\ j \equiv 0\,(\mathrm{mod}3)0 \le i \le 2h-1 \end{array} \right\} \cup \\ \left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4,4,4,\ldots,4,d\left(l_{3i+1},l_{3i+1}\right)=0,d\left(l_{3i+1},l_{3i+2}\right)=2,4,\ldots,4\right)| \\ j \equiv 1\,(\mathrm{mod}3)0 \le i \le 2h-1 \end{array} \right\} \cup \\ \left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4,4,4,\ldots,4,d\left(l_{3i+1},l_{3i+1}\right)=2,d\left(l_{3i+1},l_{3i+2}\right)=0,4,\ldots,4\right)| \\ j \equiv 2\,(\mathrm{mod}3)0 \le i \le 2h-1 \end{array} \right\}. \tag{7}$$

Next, the sets of mixed metric codes for the edges $\left\{vk_j, k_jl_j, l_jk_{j+1} | 1 \le j \le n\right\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$\begin{aligned} D = &\left\{r_m\left(vk_j \mid W_m\right) = \left(2,2,2,\ldots,2,d\left(l_{3i+2},vk_{3i+3}\right)=1,2,\ldots,2\right) \mid j \equiv 0\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\} \\ &\cup \left\{r_m\left(vk_j \mid W_m\right) = \left(2,2,2,\ldots,2,d\left(l_{3i+1},vk_{3i+2}\right)=1,2,\ldots,2\right) \mid j \equiv 1\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\} \\ &\cup \left\{r_m\left(vk_j \mid W_m\right) = \left(2,2,2,\ldots,2,d\left(l_{3i+1},vk_{3i+2}\right)=1,d\left(l_{3i+2},vk_{3i+2}\right)=1,2,\ldots,2\right) \mid j \equiv 2\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\}; \\ E = &\left\{r_m\left(k_jl_j \mid W_m\right) = \left(3,3,3,\ldots,3,d\left(l_{3i+2},k_{3i+3}l_{3i+3}\right)=1,d\left(l_{3i+4},k_{3i+3}l_{3i+3}\right)=2,3,\ldots,3\right) \mid j \equiv 0\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\} \\ &\cup \left\{r_m\left(k_jl_j \mid W_m\right) = \left(3,3,3,\ldots,3,d\left(l_{3i+1},k_{3i+1}l_{3i+1}\right)=0,d\left(l_{3i+2},k_{3i+1}l_{3i+1}\right)=2,3,\ldots,3\right) \mid j \equiv 1\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\} \\ &\cup \left\{r_m\left(k_jl_j \mid W_m\right) = \left(3,3,3,\ldots,3,d\left(l_{3i+1},k_{3i+2}l_{3i+2}\right)=1,d\left(l_{3i+2},k_{3i+2}l_{3i+2}\right)=0,3,\ldots,3\right) \mid j \equiv 2\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\}; \\ F = &\left\{r_m\left(l_jk_{j+1} \mid W_m\right) = \left(3,3,3,\ldots,3,d\left(l_{3i+2},l_{3i+3}k_{3i+4}\right)=2,d\left(l_{3i+4},l_{3i+3}k_{3i+4}\right)=1,3,\ldots,3\right) \mid j \equiv 0\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\} \\ &\cup \left\{r_m\left(l_jk_{j+1} \mid W_m\right) = \left(3,3,3,\ldots,3,d\left(l_{3i+1},l_{3i+1}k_{3i+2}\right)=0,d\left(l_{3i+2},l_{3i+1}k_{3i+2}\right)=1,3,\ldots,3\right) \mid j \equiv 1\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\} \\ &\cup \left\{r_m\left(l_jk_{j+1} \mid W_m\right) = \left(3,3,3,\ldots,3,d\left(l_{3i+1},l_{3i+2}k_{3i+3}\right)=2,d\left(l_{3i+2},l_{3i+2}k_{3i+3}\right)=0,3,\ldots,3\right) \mid j \equiv 2\,(\mathrm{mod}3)\&0 \le i \le 2h-1\right\}. \end{aligned} \tag{8}$$

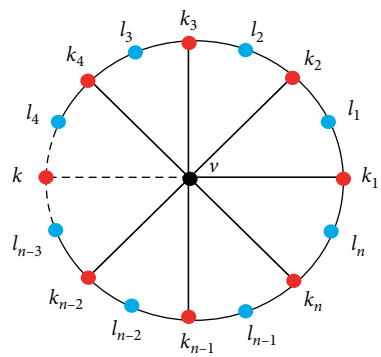

Figure 3: $\mathrm{WCS}_{n,1}$.

From these sets of mixed codes for $\mathrm{WCS}_{n,1}$, we obtain that $|A| = 1$, $|B| = |C| = |D| = |E| = |F| = n$, and $A \cap B \cap C \cap D \cap E \cap F = \varnothing$, implying W_m to be a mixed resolving set for $\mathrm{WCS}_{n,1}$, i.e., $m\dim(\mathrm{WCS}_{n,1}) \leq 4h$. Next, using equation (1) and Proposition 2, we find that $m\dim(\mathrm{WCS}_{n,1}) = 4h$, in this case.

Case (II): $n \equiv 1(\mathrm{mod}6)$. In this case, we have $n = 6h + 1$, where $h \geq 2$ and $h \in \mathbb{N}$. Suppose an ordered subset $W_m = \{l_1, l_2, l_4, l_5, \ldots, l_{n-3}, l_{n-2}, l_n\} = \{l_{3i+1}, l_{3i+2} | 0 \leq i \leq 2h - 1\} \cup \{l_n\}$ of vertices in $\mathrm{WCS}_{n,1}$ with $|W_m| = 4h + 1$. Next, we claim that W_m is the mixed resolving set for $\mathrm{WCS}_{n,1}$. Now, we can give mixed codes to every vertex and edge of $\mathrm{WCS}_{n,1}$ with respect to W_m. The sets of mixed metric codes for the vertices $\{u = v, l_j, k_j | 1 \leq j \leq n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$A = \left\{ r_m(v|W_m) = \underbrace{(2, 2, 2, \ldots, 2)}_{(4h+1)-\text{times}} \right\};$$

$$B = \left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+2}, k_{3i+3}) = 1, 3, \ldots, 3)| \\ j \equiv 0(\mathrm{mod}3) 0 \leq i \leq 2h - 1 \end{array} \right\} \cup$$

$$\left\{ r_m(k_1|W_m) = \left(1, \underbrace{3, 3, \ldots, 3}_{(4h-1)-\text{times}}, 1 \right) \right\} \cup$$

$$\left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+1}) = 1, 3, \ldots, 3)| \\ j \equiv 1(\mathrm{mod}3) 1 \leq i \leq 2h \end{array} \right\} \cup$$

$$\left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+2}) = 1, d(l_{3i+2}, k_{3i+2}) = 1, 3, \ldots, 3)| \\ j \equiv 2(\mathrm{mod}3) 0 \leq i \leq 2h - 1 \end{array} \right\},$$

$$C = \left\{ \begin{array}{c} r_m(l_j|W_m) = (4, , 4, \ldots, 4, d(l_{3i+2}, l_{3i+3}) = 2, d(l_{3i+4}, l_{3i+3}) = 2, 4, \ldots, 4)| \\ j \equiv 0(\mathrm{mod}3) 0 \leq i \leq 2h - 1 \end{array} \right\} \cup$$

$$\left\{ r_m\left(l_1|W_m\right) = \left(0, 2, \underbrace{4, 4, 4, \ldots, 4}_{(4h-2)-\text{times}}, 2\right)\right\} \cup$$
$$\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4, 4, 4, \ldots, 4, d\left(l_{3i+1}, l_{3i+1}\right) = 0, d\left(l_{3i+1}, l_{3i+2}\right) = 2, 4, \ldots, 4\right)| \\ j \equiv 1\,(\text{mod}3)1 \le i \le 2h \end{array} \right\} \cup$$
$$\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4, 4, 4, \ldots, 4, d\left(l_{3i+1}, l_{3i+1}\right) = 2, d\left(l_{3i+1}, l_{3i+2}\right) = 0, 4, \ldots, 4\right)| \\ j \equiv 2\,(\text{mod}3)0 \le i \le 2h-1 \end{array} \right\}. \tag{9}$$

Next, the sets of mixed metric codes for the edges $\left\{vk_j, k_jl_j, l_jk_{j+1}|1 \le j \le n\right\}$ of $\text{WCS}_{n,1}$ are as follows:

$$D = \left\{ \begin{array}{c} r_m\left(vk_j|W_m\right) = \left(2, 2, 2, \ldots, 2, d\left(l_{3i+2}, vk_{3i+3}\right) = 1, 2, \ldots, 2\right)| \\ j \equiv 0\,(\text{mod}3)0 \le i \le 2h-1 \end{array} \right\} \cup$$
$$\left\{ r_m\left(vk_1|W_m\right) = \left(1, \underbrace{2, 2, 2, \ldots, 2}_{(4h-1)-\text{times}}, 1\right)\right\} \cup$$
$$\left\{ \begin{array}{c} r_m\left(vk_j|W_m\right) = \left(2, 2, 2, \ldots, 2, d\left(l_{3i+1}, vk_{3i+1}\right) = 1, 2, \ldots, 2\right)| \\ j \equiv 1\,(\text{mod}3)1 \le i \le 2h \end{array} \right\} \cup$$
$$\left\{ \begin{array}{c} r_m\left(vk_j|W_m\right) = \left(2, 2, 2, \ldots, 2, d\left(l_{3i+1}, vk_{3i+2}\right) = 1, d\left(l_{3i+2}, vk_{3i+2}\right) = 1, 2, \ldots, 2\right)| \\ j \equiv 2\,(\text{mod}3)0 \le i \le 2h-1 \end{array} \right\},$$
$$E = \left\{ \begin{array}{c} r_m\left(k_jl_j|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+2}, k_{3i+3}l_{3i+3}\right) = 1, d\left(l_{3i+4}, k_{3i+3}l_{3i+3}\right) = 2, 3, \ldots, 3\right)| \\ j \equiv 0\,(\text{mod}3)0 \le i \le 2h-1 \end{array} \right\} \cup$$
$$\left\{ r_m\left(k_1l_1|W_m\right) = \left(0, 2, \underbrace{3, 3, 3, \ldots, 3}_{(4h-2)-\text{times}}, 1\right)\right\} \cup$$
$$\left\{ \begin{array}{c} r_m\left(k_jl_j|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, k_{3i+1}l_{3i+1}\right) = 0, d\left(l_{3i+2}, k_{3i+1}l_{3i+1}\right) = 2, 3, \ldots, 3\right)| \\ j \equiv 1\,(\text{mod}3)1 \le i \le 2h \end{array} \right\} \cup \tag{10}$$
$$\left\{ \begin{array}{c} r_m\left(k_jl_j|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, k_{3i+2}l_{3i+2}\right) = 1, d\left(l_{3i+2}, k_{3i+2}l_{3i+2}\right) = 0, 3, \ldots, 3\right)| \\ j \equiv 2\,(\text{mod}3)0 \le i \le 2h-1 \end{array} \right\},$$
$$F = \left\{ \begin{array}{c} r_m\left(l_jk_{j+1}|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+2}, l_{3i+3}k_{3i+4}\right) = 2, d\left(l_{3i+4}, l_{3i+3}k_{3i+4}\right) = 1, 3, \ldots, 3\right)| \\ j \equiv 0\,(\text{mod}3)0 \le i \le 2h-1 \end{array} \right\} \cup$$
$$\left\{ r_m\left(l_1k_2|W_m\right) = \left(0, 1, \underbrace{3, 3, 3, \ldots, 3}_{(4h-2)-\text{times}}, 1\right)\right\} \cup$$
$$\left\{ \begin{array}{c} r_m\left(l_jk_{j+1}|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, l_{3i+1}k_{3i+2}\right) = 0, d\left(l_{3i+2}, l_{3i+1}k_{3i+2}\right) = 1, 3, \ldots, 3\right)| \\ j \equiv 1\,(\text{mod}3)1 \le i \le 2h \end{array} \right\} \cup$$
$$\left\{ \begin{array}{c} r_m\left(l_jk_{j+1}|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, l_{3i+2}k_{3i+3}\right) = 2, d\left(l_{3i+2}, l_{3i+2}k_{3i+3}\right) = 0, 3, \ldots, 3\right)| \\ j \equiv 2\,(\text{mod}3)0 \le i \le 2h-1 \end{array} \right\}.$$

From these sets of mixed codes for $\mathrm{WCS}_{n,1}$, we obtain that $|A| = 1$, $|B| = |C| = |D| = |E| = |F| = n$, and $A \cap B \cap C \cap D \cap E \cap F = \varnothing$, implying W_m to be a mixed resolving set for $\mathrm{WCS}_{n,1}$, i.e., $m\dim(\mathrm{WCS}_{n,1}) \leq 4h + 1$.

Case (III): $n \equiv 2\,(\mathrm{mod}6)$. In this case, we have $n = 6h + 2$, where $h \geq 2$ and $h \in \mathbb{N}$. Suppose an ordered subset $W_m = \{l_1, l_2, l_4, l_5, l_7 \ldots, l_{n-1}, l_n\} = \{l_{3i+1}, l_{3i+2} | 0 \leq i \leq 2h\}$ of vertices in $\mathrm{WCS}_{n,1}$ with $|W_m| = 4h + 2$. Next, we claim that W_m is the mixed resolving set for $\mathrm{WCS}_{n,1}$. Now, we can give mixed codes to every vertex and edge of $\mathrm{WCS}_{n,1}$ with respect to W_m. The sets of mixed metric codes for the vertices $\{u = v, l_j, k_j | 1 \leq j \leq n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$
\begin{aligned}
A &= \left\{ r_m(v|W_m) = \underbrace{(2, 2, 2, \ldots, 2)}_{(4h+2)-\text{times}} \right\}, \\
B &= \left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+2}, k_{3i+3}) = 1, 3, \ldots, 3)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \leq i \leq 2h - 1 \end{array} \right\} \cup \\
&\left\{ r_m(k_1|W_m) = \left(1, \underbrace{3, 3, \ldots, 3}_{(4h)-\text{times}}, 1 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+1}) = 1, 3, \ldots, 3)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \leq i \leq 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+2}) = 1, d(l_{3i+2}, k_{3i+2}) = 1, 3, \ldots, 3)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \leq i \leq 2h \end{array} \right\}, \\
C &= \left\{ \begin{array}{c} r_m(l_j|W_m) = (4, , 4, \ldots, 4, d(l_{3i+2}, l_{3i+3}) = 2, d(l_{3i+4}, l_{3i+3}) = 2, 4, \ldots, 4)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \leq i \leq 2h - 1 \end{array} \right\} \cup \\
&\left\{ r_m(l_1|W_m) = \left(0, 2, \underbrace{4, 4, 4, \ldots, 4}_{(4h-1)-\text{times}}, 2 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_j|W_m) = (4, 4, 4, \ldots, 4, d(l_{3i+1}, l_{3i+1}) = 0, d(l_{3i+1}, l_{3i+2}) = 2, 4, \ldots, 4)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \leq i \leq 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_j|W_m) = (4, 4, 4, \ldots, 4, d(l_{3i+1}, l_{3i+2}) = 2, d(l_{3i+2}, l_{3i+2}) = 0, 4, \ldots, 4)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \leq i \leq 2h \end{array} \right\}.
\end{aligned}
\tag{11}
$$

Next, the sets of mixed metric codes for the edges $\{vk_j, k_jl_j, l_jk_{j+1} | 1 \leq j \leq n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$
\begin{aligned}
D &= \left\{ \begin{array}{c} r_m(vk_j|W_m) = (2, 2, 2, \ldots, 2, d(l_{3i+2}, vk_{3i+3}) = 1, 2, \ldots, 2)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \leq i \leq 2h - 1 \end{array} \right\} \cup \\
&\left\{ r_m(vk_1|W_m) = \left(1, \underbrace{2, 2, 2, \ldots, 2}_{(4h)-\text{times}}, 1 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(vk_j|W_m) = (2, 2, 2, \ldots, 2, d(l_{3i+1}, vk_{3i+1}) = 1, 2, \ldots, 2)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \leq i \leq 2h \end{array} \right\} \cup
\end{aligned}
$$

$$
\begin{aligned}
E = &\left\{ \begin{array}{c} r_m\left(vk_j|W_m\right) = \left(2,2,2,\ldots,2, d\left(l_{3i+1}, vk_{3i+2}\right) = 1, d\left(l_{3i+2}, vk_{3i+2}\right) = 1,2,\ldots,2\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\}, \\
&\left\{ \begin{array}{c} r_m\left(k_j l_j|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+2}, k_{3i+3} l_{3i+3}\right) = 1, d\left(l_{3i+4}, k_{3i+3} l_{3i+3}\right) = 2,3,\ldots,3\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ r_m\left(k_1 l_1|W_m\right) = \left(0, 2, \underbrace{3,3,3,\ldots,3}_{(4h-1)-times}, 1\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(k_j l_j|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+1}, k_{3i+1} l_{3i+1}\right) = 0, d\left(l_{3i+2}, k_{3i+1} l_{3i+1}\right) = 2,3,\ldots,3\right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(k_j l_j|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+1}, k_{3i+2} l_{3i+2}\right) = 1, d\left(l_{3i+2}, k_{3i+2} l_{3i+2}\right) = 0,3,\ldots,3\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\}, \\
F = &\left\{ \begin{array}{c} r_m\left(l_j k_{j+1}|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+2}, l_{3i+3} k_{3i+4}\right) = 2, d\left(l_{3i+4}, l_{3i+3} k_{3i+4}\right) = 1,3,\ldots,3\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ r_m\left(l_1 k_2|W_m\right) = \left(0, 1, \underbrace{3,3,3,\ldots,3}_{(4h-1)-\mathrm{times}}, 1\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_j k_{j+1}|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+1}, l_{3i+1} k_{3i+2}\right) = 0, d\left(l_{3i+2}, l_{3i+1} k_{3i+2}\right) = 1,3,\ldots,3\right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_j k_{j+1}|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+1}, l_{3i+2} k_{3i+3}\right) = 2, d\left(l_{3i+2}, l_{3i+2} k_{3i+3}\right) = 0,3,\ldots,3\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\}.
\end{aligned} \tag{12}
$$

From these sets of mixed codes for $\mathrm{WCS}_{n,1}$, we obtain that $|A| = 1$, $|B| = |C| = |D| = |E| = |F| = n$, and $A \cap B \cap C \cap D \cap E \cap F = \varnothing$, implying W_m to be a mixed resolving set for $\mathrm{WCS}_{n,1}$, i.e., $m\dim(\mathrm{WCS}_{n,1}) \le 4h+2$.

Case (IV): $n \equiv 3\,(\mathrm{mod}6)$. In this case, we have $n = 6h+3$, where $h \ge 2$ and $h \in \mathbb{N}$. Suppose an ordered subset $W_m = \{l_1, l_2, l_4, l_5, l_7 \ldots, l_{n-2}, l_{n-1}\} = \{l_{3i+1}, l_{3i+2}| 0 \le i \le 2h\}$ of vertices in $\mathrm{WCS}_{n,1}$ with $|W_m| = 4h+2$. Next, we claim that W_m is the mixed resolving set for $\mathrm{WCS}_{n,1}$. Now, we can give mixed codes to every vertex and edge of $\mathrm{WCS}_{n,1}$ with respect to W_m. The sets of mixed metric codes for the vertices $\left\{u = v, l_j, k_j | 1 \le j \le n\right\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$
\begin{aligned}
A = &\left\{ r_m\left(v|W_m\right) = \underbrace{(2,2,2,\ldots,2)}_{(4h+2)-\mathrm{times}} \right\}, \\
B = &\left\{ \begin{array}{c} r_m\left(k_j|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+2}, k_{3i+3}\right) = 1,3,\ldots,3\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(k_j|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+1}, k_{3i+1}\right) = 1,3,\ldots,3\right)| \\ j \equiv 1\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(k_j|W_m\right) = \left(3,3,3,\ldots,3, d\left(l_{3i+1}, k_{3i+2}\right) = 1, d\left(l_{3i+2}, k_{3i+2}\right) = 1,3,\ldots,3\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\}, \\
C = &\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4,,4,\ldots,4, d\left(l_{3i+2}, l_{3i+3}\right) = 2, d\left(l_{3i+4}, l_{3i+3}\right) = 2,4,\ldots,4\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4,4,4,\ldots,4, d\left(l_{3i+1}, l_{3i+1}\right) = 0, d\left(l_{3i+1}, l_{3i+2}\right) = 2,4,\ldots,4\right)| \\ j \equiv 1\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4,4,4,\ldots,4, d\left(l_{3i+1}, l_{3i+2}\right) = 2, d\left(l_{3i+2}, l_{3i+2}\right) = 0,4,\ldots,4\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\}.
\end{aligned} \tag{13}
$$

Next, the sets of mixed metric codes for the edges $\{vk_j, k_jl_j, l_jk_{j+1} | 1 \le j \le n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$\begin{aligned}
D = &\{r_m(vk_j \mid W_m) = (2,2,2,\ldots,2, d(l_{3i+2}, vk_{3i+3}) = 1,2,\ldots,2) \mid j \equiv 0\,(\mathrm{mod}3)\&0 \le i \le 2h\} \\
&\cup \{r_m(vk_j \mid W_m) = (2,2,2,\ldots,2, d(l_{3i+1}, vk_{3i+2}) = 1,2,\ldots,2) \mid j \equiv 1\,(\mathrm{mod}3)\&0 \le i \le 2h\} \\
&\cup \{r_m(vk_j \mid W_m) = (2,2,2,\ldots,2, d(l_{3i+1}, vk_{3i+2}) = 1, d(l_{3i+2}, vk_{3i+2}) = 1,2,\ldots,2) \mid j \equiv 2\,(\mathrm{mod}3)\&0 \le i \le 2h\}; \\
E = &\{r_m(k_jl_j \mid W_m) = (3,3,3,\ldots,3, d(l_{3i+2}, k_{3i+3}l_{3i+3}) = 1, d(l_{3i+4}, k_{3i+3}l_{3i+3}) = 2,3,\ldots,3) \mid j \equiv 0\,(\mathrm{mod}3)\&0 \le i \le 2h\} \\
&\cup \{r_m(k_jl_j \mid W_m) = (3,3,3,\ldots,3, d(l_{3i+1}, k_{3i+1}l_{3i+1}) = 0, d(l_{3i+2}, k_{3i+1}l_{3i+1}) = 2,3,\ldots,3) \mid j \equiv 1\,(\mathrm{mod}3)\&0 \le i \le 2h\} \\
&\cup \{r_m(k_jl_j \mid W_m) = (3,3,3,\ldots,3, d(l_{3i+1}, k_{3i+2}l_{3i+2}) = 1, d(l_{3i+2}, k_{3i+2}l_{3i+2}) = 0,3,\ldots,3) \mid j \equiv 2\,(\mathrm{mod}3)\&0 \le i \le 2h\}; \\
F = &\{r_m(l_jk_{j+1} \mid W_m) = (3,3,3,\ldots,3, d(l_{3i+2}, l_{3i+3}k_{3i+4}) = 2, d(l_{3i+4}, l_{3i+3}k_{3i+4}) = 1,3,\ldots,3) \mid j \equiv 0\,(\mathrm{mod}3)\&0 \le i \le 2h\} \\
&\cup \{r_m(l_jk_{j+1} \mid W_m) = (3,3,3,\ldots,3, d(l_{3i+1}, l_{3i+1}k_{3i+2}) = 0, d(l_{3i+2}, l_{3i+1}k_{3i+2}) = 1,3,\ldots,3) \mid j \equiv 1\,(\mathrm{mod}3)\&0 \le i \le 2h\}. \\
&\cup \{r_m(l_jk_{j+1} \mid W_m) = (3,3,3,\ldots,3, d(l_{3i+1}, l_{3i+2}k_{3i+3}) = 2, d(l_{3i+2}, l_{3i+2}k_{3i+3}) = 0,3,\ldots,3) \mid j \equiv 2\,(\mathrm{mod}3)\&0 \le i \le 2h\};
\end{aligned} \tag{14}$$

From these sets of mixed codes for $\mathrm{WCS}_{n,1}$, we obtain that $|A| = 1$, $|B| = |C| = |D| = |E| = |F| = n$, and $A \cap B \cap C \cap D \cap E \cap F = \varnothing$, implying W_m to be a mixed resolving set for $\mathrm{WCS}_{n,1}$, i.e., $m\dim(\mathrm{WCS}_{n,1}) \le 4h + 2$. Next, using equation (1) and Proposition 2, we find that $m\dim(\mathrm{WCS}_{n,1}) = 4h + 2$, in this case.

Case (V): $n \equiv 4\,(\mathrm{mod}6)$. In this case, we have $n = 6h + 4$, where $h \ge 2$ and $h \in \mathbb{N}$. Suppose an ordered subset $W_m = \{l_1, l_2, l_4, l_5, \ldots, l_{n-3}, l_{n-2}, l_n\} = \{l_{3i+1}, l_{3i+2} | 0 \le i \le 2h\} \cup \{l_n\}$ of vertices in $\mathrm{WCS}_{n,1}$ with $|W_m| = 4h + 3$. Next, we claim that W_m is the mixed resolving set for $\mathrm{WCS}_{n,1}$. Now, we can give mixed codes to every vertex and edge of $\mathrm{WCS}_{n,1}$ with respect to W_m. The sets of mixed metric codes for the vertices $\{u = v, l_j, k_j | 1 \le j \le n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$A = \left\{ r_m(v|W_m) = \underbrace{(2,2,2,\ldots,2)}_{(4h+3)-\text{times}} \right\},$$

$$B = \left\{ \begin{array}{c} r_m(k_j|W_m) = (3,3,3,\ldots,3, d(l_{3i+2}, k_{3i+3}) = 1,3,\ldots,3)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup$$

$$\left\{ r_m(k_1|W_m) = \left(1, \underbrace{3,3,\ldots,3}_{(4h+1)-\text{times}}, 1\right) \right\} \cup$$

$$\{r_m(k_j|W_m) = (3,3,3,\ldots,3, d(l_{3i+1}, k_{3i+1}) = 1,3,\ldots,3)| j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h+1\} \cup$$

$$\left\{ \begin{array}{c} r_m(k_j|W_m) = (3,3,3,\ldots,3, d(l_{3i+1}, k_{3i+2}) = 1, d(l_{3i+2}, k_{3i+2}) = 1,3,\ldots,3)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h+1 \end{array} \right\},$$

$$
\begin{aligned}
C = &\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4,,4,\ldots,4, d\left(l_{3i+2}, l_{3i+3}\right) = 2, d\left(l_{3i+4}, l_{3i+3}\right) = 2, 4, \ldots, 4\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \\
&\cup \left\{ r_m\left(l_1|W_m\right) = \left(0, 2, \underbrace{4, 4, 4, \ldots, 4}_{(4h)-\text{times}}, 2\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4, 4, 4, \ldots, 4, d\left(l_{3i+1}, l_{3i+1}\right) = 0, d\left(l_{3i+1}, l_{3i+2}\right) = 2, 4, \ldots, 4\right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h+1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_j|W_m\right) = \left(4, 4, 4, \ldots, 4, d\left(l_{3i+1}, l_{3i+1}\right) = 2, d\left(l_{3i+1}, l_{3i+2}\right) = 0, 4, \ldots, 4\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h+1 \end{array} \right\}.
\end{aligned}
\tag{15}
$$

Next, the sets of mixed metric codes for the edges $\left\{vk_j, k_jl_j, l_jk_{j+1} | 1 \le j \le n\right\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$
\begin{aligned}
D = &\left\{ \begin{array}{c} r_m\left(vk_j|W_m\right) = \left(2, 2, 2, \ldots, 2, d\left(l_{3i+2}, vk_{3i+3}\right) = 1, 2, \ldots, 2\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ r_m\left(vk_1|W_m\right) = \left(1, \underbrace{2, 2, 2, \ldots, 2}_{(4h+1)-\text{times}}, 1\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(vk_j|W_m\right) = \left(2, 2, 2, \ldots, 2, d\left(l_{3i+1}, vk_{3i+1}\right) = 1, 2, \ldots, 2\right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h+1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(vk_j|W_m\right) = \left(2, 2, 2, \ldots, 2, d\left(l_{3i+1}, vk_{3i+2}\right) = 1, d\left(l_{3i+2}, vk_{3i+2}\right) = 1, 2, \ldots, 2\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\}, \\
E = &\left\{ \begin{array}{c} r_m\left(k_jl_j|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+2}, k_{3i+3}l_{3i+3}\right) = 1, d\left(l_{3i+4}, k_{3i+3}l_{3i+3}\right) = 2, 3, \ldots, 3\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ r_m\left(k_1l_1|W_m\right) = \left(0, 2, \underbrace{3, 3, 3, \ldots, 3}_{(4h)-\text{times}}, 1\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(k_jl_j|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, k_{3i+1}l_{3i+1}\right) = 0, d\left(l_{3i+2}, k_{3i+1}l_{3i+1}\right) = 2, 3, \ldots, 3\right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h+1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(k_jl_j|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, k_{3i+2}l_{3i+2}\right) = 1, d\left(l_{3i+2}, k_{3i+2}l_{3i+2}\right) = 0, 3, \ldots, 3\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h+1 \end{array} \right\}, \\
F = &\left\{ \begin{array}{c} r_m\left(l_jk_{j+1}|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+2}, l_{3i+3}k_{3i+4}\right) = 2, d\left(l_{3i+4}, l_{3i+3}k_{3i+4}\right) = 1, 3, \ldots, 3\right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ r_m\left(l_1k_2|W_m\right) = \left(0, 1, \underbrace{3, 3, 3, \ldots, 3}_{(4h)-\text{times}}, 1\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_jk_{j+1}|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, l_{3i+1}k_{3i+2}\right) = 0, d\left(l_{3i+2}, l_{3i+1}k_{3i+2}\right) = 1, 3, \ldots, 3\right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h+1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m\left(l_jk_{j+1}|W_m\right) = \left(3, 3, 3, \ldots, 3, d\left(l_{3i+1}, l_{3i+2}k_{3i+3}\right) = 2, d\left(l_{3i+2}, l_{3i+2}k_{3i+3}\right) = 0, 3, \ldots, 3\right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h \end{array} \right\}.
\end{aligned}
\tag{16}
$$

From these sets of mixed codes for $\mathrm{WCS}_{n,1}$, we obtain that $|A| = 1, |B| = |C| = |D| = |E| = |F| = n$, and $A \cap B \cap C \cap D \cap E \cap F = \varnothing$, implying W_m to be a mixed resolving set for $\mathrm{WCS}_{n,1}$, i.e., $m\dim(\mathrm{WCS}_{n,1}) \leq 4h + 3$.

Case (VI): $n \equiv 5 (\mathrm{mod} 6)$. In this case, we have $n = 6h + 5$, where $h \geq 1$ and $h \in \mathbb{N}$. Suppose an ordered subset $W_m = \{l_1, l_2, l_4, l_5, \ldots, l_{n-1}, l_n\} = \{l_{3i+1}, l_{3i+2} | 0 \leq i \leq 2h + 1\}$ of vertices in $\mathrm{WCS}_{n,1}$ with $|W_m| = 4h + 4$. Next, we claim that W_m is the mixed resolving set for $\mathrm{WCS}_{n,1}$. Now, we can give mixed codes to every vertex and edge of $\mathrm{WCS}_{n,1}$ with respect to W_m. The sets of mixed metric codes for the vertices $\{u = v, l_j, k_j | 1 \leq j \leq n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$
\begin{aligned}
A &= \left\{ r_m(v|W_m) = \underbrace{(2, 2, 2, \ldots, 2)}_{(4h+4)-\text{times}} \right\}, \\
B &= \left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+2}, k_{3i+3}) = 1, 3, \ldots, 3)| \\ j \equiv 0 (\mathrm{mod} 3) 0 \leq i \leq 2h \end{array} \right\} \cup \\
&\left\{ r_m(k_1|W_m) = \left(1, \underbrace{3, 3, \ldots, 3}_{(4h+2)-\text{times}}, 1 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+1}) = 1, 3, \ldots, 3)| \\ j \equiv 1 (\mathrm{mod} 3) 1 \leq i \leq 2h + 1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+2}) = 1, d(l_{3i+2}, k_{3i+2}) = 1, 3, \ldots, 3)| \\ j \equiv 2 (\mathrm{mod} 3) 0 \leq i \leq 2h + 1 \end{array} \right\}, \\
C &= \left\{ \begin{array}{c} r_m(l_j|W_m) = (4, , 4, \ldots, 4, d(l_{3i+2}, l_{3i+3}) = 2, d(l_{3i+4}, l_{3i+3}) = 2, 4, \ldots, 4)| \\ j \equiv 0 (\mathrm{mod} 3) 0 \leq i \leq 2h \end{array} \right\} \cup \\
&\left\{ r_m(l_1|W_m) = \left(0, 2, \underbrace{4, 4, 4, \ldots, 4}_{(4h+1)-\text{times}}, 2 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_j|W_m) = (4, 4, 4, \ldots, 4, d(l_{3i+1}, l_{3i+1}) = 0, d(l_{3i+1}, l_{3i+2}) = 2, 4, \ldots, 4)| \\ j \equiv 1 (\mathrm{mod} 3) 1 \leq i \leq 2h + 1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_j|W_m) = (4, 4, 4, \ldots, 4, d(l_{3i+1}, l_{3i+2}) = 2, d(l_{3i+2}, l_{3i+2}) = 0, 4, \ldots, 4)| \\ j \equiv 2 (\mathrm{mod} 3) 0 \leq i \leq 2h + 1 \end{array} \right\}.
\end{aligned}
\tag{17}
$$

Next, the sets of mixed metric codes for the edges $\{vk_j, k_j l_j, l_j k_{j+1} | 1 \leq j \leq n\}$ of $\mathrm{WCS}_{n,1}$ are as follows:

$$
\begin{aligned}
D &= \left\{ \begin{array}{c} r_m(vk_j|W_m) = (2, 2, 2, \ldots, 2, d(l_{3i+2}, vk_{3i+3}) = 1, 2, \ldots, 2)| \\ j \equiv 0 (\mathrm{mod} 3) 0 \leq i \leq 2h \end{array} \right\} \cup \\
&\left\{ r_m(vk_1|W_m) = \left(1, \underbrace{2, 2, 2, \ldots, 2}_{(4h+2)-\text{times}}, 1 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(vk_j|W_m) = (2, 2, 2, \ldots, 2, d(l_{3i+1}, vk_{3i+1}) = 1, 2, \ldots, 2)| \\ j \equiv 1 (\mathrm{mod} 3) 1 \leq i \leq 2h + 1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(vk_j|W_m) = (2, 2, 2, \ldots, 2, d(l_{3i+1}, vk_{3i+2}) = 1, d(l_{3i+2}, vk_{3i+2}) = 1, 2, \ldots, 2)| \\ j \equiv 2 (\mathrm{mod} 3) 0 \leq i \leq 2h + 1 \end{array} \right\}, \\
E &= \left\{ \begin{array}{c} r_m(k_j l_j|W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+2}, k_{3i+3} l_{3i+3}) = 1, d(l_{3i+4}, k_{3i+3} l_{3i+3}) = 2, 3, \ldots, 3)| \\ j \equiv 0 (\mathrm{mod} 3) 0 \leq i \leq 2h \end{array} \right\} \cup
\end{aligned}
$$

$$
\begin{aligned}
F = &\left\{ r_m(k_1 l_1 | W_m) = \left(0, 2, \underbrace{3, 3, 3, \ldots, 3}_{(4h+1)-\text{times}}, 1\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_j l_j | W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+1} l_{3i+1}) = 0, d(l_{3i+2}, k_{3i+1} l_{3i+1}) = 2, 3, \ldots, 3) | \\ j \equiv 1 \,(\text{mod}3) 1 \le i \le 2h+1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_j l_j | W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, k_{3i+2} l_{3i+2}) = 1, d(l_{3i+2}, k_{3i+2} l_{3i+2}) = 0, 3, \ldots, 3) | \\ j \equiv 2 \,(\text{mod}3) 0 \le i \le 2h+1 \end{array} \right\}, \\
&\left\{ \begin{array}{c} r_m(l_j k_{j+1} | W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+2}, l_{3i+3} k_{3i+4}) = 2, d(l_{3i+4}, l_{3i+3} k_{3i+4}) = 1, 3, \ldots, 3) | \\ j \equiv 0 \,(\text{mod}3) 0 \le i \le 2h \end{array} \right\} \cup \\
&\left\{ r_m(l_1 k_2 | W_m) = \left(0, 1, \underbrace{3, 3, 3, \ldots, 3}_{(4h+1)-\text{times}}, 2\right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_j k_{j+1} | W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, l_{3i+1} k_{3i+2}) = 0, d(l_{3i+2}, l_{3i+1} k_{3i+2}) = 1, 3, \ldots, 3) | \\ j \equiv 1 \,(\text{mod}3) 1 \le i \le 2h+1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_j k_{j+1} | W_m) = (3, 3, 3, \ldots, 3, d(l_{3i+1}, l_{3i+2} k_{3i+3}) = 2, d(l_{3i+2}, l_{3i+2} k_{3i+3}) = 0, 3, \ldots, 3) | \\ j \equiv 2 \,(\text{mod}3) 0 \le i \le 2h+1 \end{array} \right\}.
\end{aligned} \tag{18}
$$

From these sets of mixed codes for $\text{WCS}_{n,1}$, we obtain that $|A| = 1$, $|B| = |C| = |D| = |E| = |F| = n$, and $A \cap B \cap C \cap D \cap E \cap F = \varnothing$, implying W_m to be a mixed resolving set for $\text{WCS}_{n,1}$, i.e., $m\dim(\text{WCS}_{n,1}) \le 4h + 4$. Now, for the second, third, fifth, and sixth case, we obtain their lower bounds as follows.

For the second case, suppose that $W_m \subset V(\text{WCS}_{n,1})$ with $|W_m| < 4h + 1$ is a mixed resolving set for $\text{WCS}_{n,1}$. We have the following two cases to be considered:

Subcase (i): if $W_m \nsubseteq \{k_1, k_2, k_3, \ldots, k_n\}$, then there must exist a vertex l_j such that $l_j \in W_m$. Then, there exists at least one vertex $l_i \in W_m$ such that $k_{i-1}, k_{i+1} \notin W_m$. Then, for the corresponding edges vk_{i-1} and vk_{i+1}, we have $r_m(vk_{i+1}|W_m) = r_m(vk_{i-1}|W_m)$, a contradiction. Therefore, W_m is not a mixed resolving set for $\text{WCS}_{n,1}$ in this case.

Subcase (ii): if $W_m \subset \{k_1, k_2, k_3, \ldots, k_n\}$, then there exist at least two vertices k_i and k_j such that $k_i, k_j \notin W_m$. Then, for the edges vk_i and vk_j, we have $r_m(vk_i|W_m) = r_m(vk_j|W_m)$, a contradiction. Therefore, W_m is not a mixed resolving set for $\text{WCS}_{n,1}$ in this case as well. Thus, $|W_m| \ge 4h + 1$. This completes the proof for the second case.

For rest of the cases, the pattern is the same as that in Case (II). □

5. Mixed Metric Dimension of the Barycentric Subdivision of $W_{n,1}$

In this section, we determine the mixed metric dimension of the barycentric subdivision of a wheel graph.

5.1. Barycentric Subdivision of $W_{n,1}$. Suppose $W_{n,1}$ is a wheel graph with the vertex set $V(W_{n,1}) = \{k_1, k_2, k_3, \ldots, k_n, v\}$ having a single universal vertex v. Now, each of the edges $k_j k_{j+1}$ and vk_j $(1 \le j \le n)$ of $W_{n,1}$ is subdivided with a new vertex. The resulting graph so obtained is known as the barycentric subdivision wheel graph (BSWG) and is denoted by $\text{WBS}_{n,1}$. BSWG has $4n$ edges, $E(\text{WBS}_{n,1}) = \{vl_j, l_j k_j, k_j m_j, m_j k_{j+1} | 1 \le j \le n\}$, and $3n + 1$ vertices, $V(\text{WBS}_{n,1}) = \{v, l_j, k_j, m_j | 1 \le j \le n\}$, where all indices are taken to be modulo n (see Figure 4). In this section, we obtain the mixed metric dimension of BSWG $\text{WBS}_{n,1}$.

Theorem 3. *For $n \ge 6$, we have*

$$
m\dim(\text{WBS}_{n,1}) = \begin{cases} 4h & \text{if } n = 6h, \\ 4h+1 & \text{if } n = 6h+1, \\ 4h+2 & \text{if } n = 6h+2, \\ 4h+2 & \text{if } n = 6h+3, \\ 4h+3 & \text{if } n = 6h+4, \\ 4h+4 & \text{if } n = 6h+5. \end{cases} \tag{19}
$$

Proof. To prove this, we first generate the mixed resolving sets for all the cases, obtaining the upper bounds depending on the positive integer n. Then, in the end, we show that the lower bound (or reverse inequality) is the same as the upper bound to conclude the theorem.

Case (I): $n \equiv 0\,(\text{mod}6)$. In this case, we have $n = 6h$, where $h \ge 2$ and $h \in \mathbb{N}$. Suppose an ordered subset $W_m = \{m_1, m_2, m_4, m_5, \ldots, m_{n-2}, m_{n-1}\} = \{m_{3i+1}, m_{3i+2} | 0 \le i \le 2h - 1\}$ of vertices in $\text{WBS}_{n,1}$ with $|W_m| = 4h$. Next, we claim that W_m is the mixed resolving set for $\text{WBS}_{n,1}$. Now, we can give mixed codes to every vertex

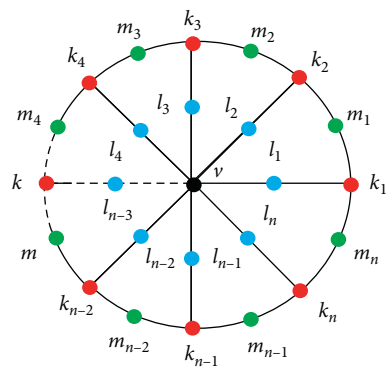

FIGURE 4: $\mathrm{WBS}_{n,1}$.

and edge of $\mathrm{WBS}_{n,1}$ with respect to W_m. The sets of mixed metric codes for the vertices $\{u = v, k_j, l_j, m_j | 1 \le j \le n\}$ of $\mathrm{WBS}_{n,1}$ are as follows:

$$
\begin{aligned}
A &= \left\{ r_m(v|W_m) = \underbrace{(3,3,3,\ldots,3)}_{4h-\text{times}} \right\},\\
B &= \left\{ r_m(k_j|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+1},k_{3i+3}) = 3, d(m_{3i+2},k_{3i+3}) = 1, \\ d(m_{3i+4},k_{3i+3}) = 5,5,\ldots,5 \end{array} \right) | \begin{array}{c} j \equiv 0 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup\\
&\left\{ r_m(k_1|W_m) = \left(1,3, \underbrace{5,\ldots,5}_{(4h-3)-\text{times}}, 3 \right) \right\} \cup\\
&\left\{ r_m(k_j|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+2},k_{3i+1}) = 3, d(m_{3i+4},k_{3i+1}) = 1, \\ d(m_{3i+5},k_{3i+1}) = 3,5,\ldots,5 \end{array} \right) | \begin{array}{c} j \equiv 1 \,(\mathrm{mod}3) 1 \le i \le 2h-1 \end{array} \right\} \cup\\
&\left\{ r_m(k_j|W_m) = (5,5,5,\ldots,5, d(m_{3i+1},k_{3i+2}) = 1, d(m_{3i+2},k_{3i+2}) = 1,5,\ldots,5) | \begin{array}{c} j \equiv 2 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\},\\
C &= \left\{ r_m(l_j|W_m) = (4,,4,\ldots,4, d(m_{3i+2},l_{3i+3}) = 2,4,\ldots,4) | \begin{array}{c} j \equiv 0 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup\\
&\left\{ r_m(l_j|W_m) = (4,4,4,\ldots,4, d(m_{3i+1},l_{3i+1}) = 2,4,\ldots,4) | \begin{array}{c} j \equiv 1 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup\\
&\left\{ r_m(l_j|W_m) = (4,4,4,\ldots,4, d(m_{3i+1},l_{3i+2}) = 2, d(m_{3i+2},l_{3i+2}) = 2,4,\ldots,4) | \begin{array}{c} j \equiv 2 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\},\\
D &= \left\{ r_m(m_j|W_m) = \left(\begin{array}{c} 6,,6,\ldots,6, d(m_{3i+1},m_{3i+3}) = 4, d(m_{3i+2},m_{3i+3}) = 2, \\ d(m_{3i+4},m_{3i+3}) = 2, d(m_{3i+5},m_{3i+3}) = 4,6,\ldots,6 \end{array} \right) | \begin{array}{c} j \equiv 0 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup\\
&\left\{ r_m(m_1|W_m) = \left(0,2, \underbrace{6,6,\ldots,6}_{(4h-3)-\text{times}}, 4 \right) \right\} \cup\\
&\left\{ r_m(m_j|W_m) = \left(\begin{array}{c} 6,6,6,\ldots,6, d(m_{3i+2},m_{3i+1}) = 4, d(m_{3i+4},m_{3i+1}) = 0, \\ d(m_{3i+5},m_{3i+1}) = 2,6,\ldots,6 \end{array} \right) | \begin{array}{c} j \equiv 1 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup\\
&\left\{ r_m(l_j|W_m) = \left(\begin{array}{c} 6,6,6,\ldots,6, d(m_{3i+1},m_{3i+2}) = 2, d(m_{3i+2},m_{3i+2}) = 0, \\ d(m_{3i+4},m_{3i+2}) = 4,6,\ldots,6 \end{array} \right) | \begin{array}{c} j \equiv 2 \,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\}.
\end{aligned}
\tag{20}
$$

Next, the sets of mixed metric codes for the edges $\{vl_j, l_jk_j, k_jm_j, m_jk_{j+1} | 1 \le j \le n\}$ of $\mathrm{WBS}_{n,1}$ are as follows:

$$
\begin{aligned}
E = &\left\{ \begin{array}{c} r_m(vl_j|W_m) = (3,3,3,\ldots,3, d(m_{3i+2}, vl_{3i+3}) = 2,3,\ldots,3)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(vl_j|W_m) = (3,3,3,\ldots,3, d(m_{3i+1}, vl_{3i+1}) = 2,3,\ldots,3)| \\ j \equiv 1\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(vl_j|W_m) = (3,3,3,\ldots,3, d(m_{3i+1}, vl_{3i+2}) = 2, d(m_{3i+2}, vl_{3i+2}) = 2,3,\ldots,3)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\}, \\
F = &\left\{ \begin{array}{c} r_m(l_jk_j|W_m) = \left(\begin{array}{c} 4,4,4,\ldots,4, d(m_{3i+1}, l_{3i+3}k_{3i+3}) = 3, d(m_{3i+2}, l_{3i+3}k_{3i+3}) = 1, \\ d(m_{3i+4}, l_{3i+3}k_{3i+3}) = 3,4,\ldots,4 \end{array} \right)| \\ j \equiv 0\,(\mathrm{mod}3)\, 0 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ r_m(l_1k_1|W_m) = \left(1,3,\underbrace{4,4,\ldots,4}_{(4h-3)-\mathrm{times}},3 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_jk_j|W_m) = \left(\begin{array}{c} 4,4,4,\ldots,4, d(m_{3i+2}, l_{3i+1}k_{3i+1}) = 3, d(m_{3i+4}, l_{3i+1}k_{3i+1}) = 1, \\ d(m_{3i+5}, l_{3i+1}k_{3i+1}) = 3,4,\ldots,4 \end{array} \right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(l_jk_j|W_m) = (4,4,4,\ldots,4, d(m_{3i+1}, l_{3i+2}k_{3i+2}) = 1, d(m_{3i+2}, l_{3i+2}l_{3i+2}) = 1,4,\ldots,4)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\}, \\
G = &\left\{ \begin{array}{c} r_m(k_jm_j|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+1}, k_{3i+3}m_{3i+3}) = 3, d(m_{3i+2}, k_{3i+3}m_{3i+3}) = 1, \\ d(m_{3i+4}, k_{3i+3}m_{3i+3}) = 2, d(m_{3i+5}, k_{3i+3}m_{3i+3}) = 4,5,\ldots,5 \end{array} \right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ r_m(k_1m_1|W_m) = \left(0,2,\underbrace{5,5,\ldots,5}_{(4h-3)-\mathrm{times}},3 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_jm_j|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+2}, k_{3i+1}m_{3i+1}) = 3, d(m_{3i+4}, k_{3i+1}m_{3i+1}) = 0, \\ d(m_{3i+5}, k_{3i+1}m_{3i+1}) = 2,5,\ldots,5 \end{array} \right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(k_jm_j|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+1}, k_{3i+2}m_{3i+2}) = 1, d(m_{3i+2}, k_{3i+2}m_{3i+2}) = 0, \\ d(m_{3i+4}, k_{3i+2}m_{3i+2}) = 4,5,\ldots,5 \end{array} \right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\}, \\
H = &\left\{ \begin{array}{c} r_m(m_jk_{j+1}|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+1}, m_{3i+3}k_{3i+4}) = 4, d(m_{3i+2}, m_{3i+3}k_{3i+4}) = 2, \\ d(m_{3i+4}, m_{3i+3}k_{3i+4}) = 1, d(m_{3i+5}, m_{3i+3}k_{3i+4}) = 3,5,\ldots,5 \end{array} \right)| \\ j \equiv 0\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ r_m(m_1k_2|W_m) = \left(0,1,\underbrace{5,5,\ldots,5}_{(4h-3)-\mathrm{times}},4 \right) \right\} \cup \\
&\left\{ \begin{array}{c} r_m(m_jk_{j+1}|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+2}, m_{3i+1}k_{3i+2}) = 4, d(m_{3i+4}, m_{3i+1}k_{3i+2}) = 0, \\ d(m_{3i+5}, m_{3i+1}k_{3i+2}) = 1,5,\ldots,5 \end{array} \right)| \\ j \equiv 1\,(\mathrm{mod}3) 1 \le i \le 2h-1 \end{array} \right\} \cup \\
&\left\{ \begin{array}{c} r_m(m_jk_{j+1}|W_m) = \left(\begin{array}{c} 5,5,5,\ldots,5, d(m_{3i+1}, m_{3i+2}k_{3i+3}) = 2, d(m_{3i+2}, m_{3i+2}k_{3i+3}) = 0, \\ d(m_{3i+4}, m_{3i+2}k_{3i+3}) = 3,5,\ldots,5 \end{array} \right)| \\ j \equiv 2\,(\mathrm{mod}3) 0 \le i \le 2h-1 \end{array} \right\}.
\end{aligned}
\tag{21}
$$

From these sets of mixed codes for $\mathrm{WBS}_{n,1}$, we obtain that $|A| = 1$, $|B| = |C| = |D| = |E| = |F| = |G| = |H| = n$, and $A \cap B \cap C \cap D \cap E \cap F \cap G \cap H = \varnothing$, implying W_m to be a mixed resolving set for $\mathrm{WBS}_{n,1}$, i.e., $m\dim(\mathrm{WBS}_{n,1}) \leq 4h$. Next, using equation (1) and Proposition 2, we find that $m\dim(\mathrm{WBS}_{n,1}) = 4h$, in this case.

Like the first case, the rest of the proof is similar to that of Theorem 2. □

Remark 2. For the cycle and barycentric subdivision wheel graph, i.e., $H = \mathrm{WCS}_{n,1}$ and $H = \mathrm{WBS}_{n,1}$, we find that $\dim(H) = e\dim(H) = m\dim(H)$ when $n = 6h$ and $n = 6h + 3$. For the rest of the values of the positive integer n, we have $\dim(H) = e\dim(H) < m\dim(H)$ (using Propositions 2 and 4 and Theorems 2 and 3).

6. Conclusion

In this article, we have computed the mixed metric dimension for three families of graphs, namely, $\mathrm{WBS}_{n,1}$, $\mathrm{WCS}_{n,1}$, and $\mathrm{WSS}_{n,1}$, obtained after the barycentric, cycle, and spoke subdivisions of the wheel graph $W_{n,1}$, respectively. We also observed that the mixed resolving sets for $\mathrm{WBS}_{n,1}$ and $\mathrm{WCS}_{n,1}$ are independent. For $\mathrm{WSS}_{n,1}$, we found that $\dim(\mathrm{WSS}_{n,1}) < e\dim(\mathrm{WSS}_{n,1}) < m\dim(\mathrm{WSS}_{n,1})$, and for $H = \mathrm{WBS}_{n,1}$ and $H = \mathrm{WCS}_{n,1}$, we obtained the following relation: $\dim(H) = e\dim(H) \leq m\dim(H)$ (partial answers to the questions raised in [1, 18]).

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

All the authors contributed equally to the final manuscript.

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